Scott Complexity and Finitely α-generated Structures

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What's a Scott Sentence?

Everything in this talk is motivated by a theorem of Scott's:

Scott's Isomorphism Theorem

Every countable structure can be described up to isomorphism (among countable structures) by a sentence φ of $L_{\omega_1\omega}$.

Such a sentence is called a **Scott sentence** for A.

This is exactly the kind of categoricity result which is not possible in the finitary first-order context.

Complexity of formulas in $L_{\omega_1\omega}$

Every formula of $L_{\omega_1\omega}$ has a normal form.

- A $\Sigma_0 = \Pi_0$ formula is a finitary quantifier-free formula of L.
- A Σ_α formula is a formula of the form $\bigvee_{i\in\omega} \exists\bar{x}\phi_i(\bar{x})$ where each ϕ_i is Π_β for $\beta < \alpha$.
- A Π_α formula is the negation of a Σ_α formula. Equivalently, a formula of the form $\bigwedge_{i\in\omega} \forall\bar{x}\phi_i(\bar{x})$ where each ϕ_i is Σ_β for $\beta < \alpha$.

A d-Σ_α formula is a finite conjunction of Π_α and Σ_α formulas

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A Measure of Internal Complexity

The standard proof of Scott's isomorphism theorem uses the following fact:

Fact

For any structure A, there is some ordinal α such that whenever two finite tuples agree on all Π_α formulas, they must be automorphic.

The least such α, denoted $\mathbf{r(A)}$, is one definition of the *Scott Rank* of A, and is thought be an "internal" measure of A's descriptive complexity.

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Disagreement in the Literature

Unfortunately, many non-equivalent definitions of Scott Rank exist in the literature. Antonio Montalban in "A Robuster Scott Rank" argued to standardize the following definition:

Definition (A. Montalban)

The **(Categoricity) Scott Rank** of A is the least α such that A has a $\Pi_{\alpha+1}$ Scott sentence.

Note briefly that the complexity of a Scott sentence gives an "external" measure of the structure's complexity.

The List

Montalban believed this notion was most robust, having many other conditions equivalent to it.

Theorem

The following are equivalent:

1. A has a $\Pi_{\alpha+1}$ Scott sentence.
2. The automorphism orbit of any tuple can be defined by a Σ_α formula (without parameters).
3. The set $Iso(A)$ of presentations of A is $\Pi_{\alpha+1}$ in the Borel hierarchy.
4. A is uniformly boldface Δ_α-categorical.
5. And so on...

In other words, Scott Sentences are also related to notions in computability theory and descriptive set theory.

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Why just consider $\Pi_{\alpha+1}$ Scott sentences?

Fact: A structure has a $\Sigma_{\alpha+1}$ Scott sentence iff there is some finite tuple $\bar{c}$ such that $(A, \bar{c})$ has a Π_α Scott sentence.

Theorem (A. Miller)

For $\alpha \geq 1$, A has a Scott sentence that is d-$\Sigma_{<\alpha}$ iff it has one that is Π_α and one that is Σ_α.

Miller's result implies a unique least-complexity Scott sentence for the structure $(\Pi_\alpha, \Sigma_\alpha, d\text{-}\Sigma_\alpha)$.

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Scott Complexity

Definition (R.A*, M. Harrison-Trainor, D. Turetsky, N. Greenberg)

The **Scott Complexity** of a structure A is the least complexity of a Scott sentence for A.

Scott Complexity is finer than Scott Rank, and just as robust.

Finitely α-generated Structures: Motivation

In previous work with Dino Rossegger, we gave sharp upper bounds on the Scott Complexity of an arbitrary scattered linear order. To give a $\Sigma_{\alpha+1}$ Scott sentence for a scattered linear order A, we had to identify the tuple $\bar{c}$ such that $(A, \bar{c})$ has a Π_α Scott sentence.

In doing so, we noticed that such a tuple acted in many ways like the generating tuple of the structure, even though the structure was not finitely generated. Call such a tuple an α-generator for A.

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This tuple is important and has several equivalent characterizations.

Observation

The following are equivalent:

- The structure $(A, \bar{c})$ has a $\Pi_{\alpha+1}$ Scott sentence.
- The tuple $\bar{c}$ is a tuple over which no other tuple is α-free.
- The structure A has a Scott family of Σ_α sentences with parameters from $\bar{c}$.

Aside: α-freeness

A tuple α-free over $\bar{c}$ is just a "witness" to the fact that a relation does not have a Σ_α definition with parameters $\bar{c}$. It happens to have a combinatorial characterization which can be useful in practice.

Theorem

A relation R has a Σ_α definition over a tuple of parameters $\bar{c}$ iff there is no tuple $\bar{a}$ which is α-free for R over $\bar{c}$.

For a family of relations, each has a Σ_α definition with parameters $\bar{c}$ iff no tuple is α-free for the family over $\bar{c}$.

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While one could call the desired tuple $\bar{c}$ "a tuple over which no other tuple of the structure is α-free," this is cumbersome.

Definition

A tuple $\bar{c}$ is said to be an α-**generator** for a structure A if:

1. the automorphism orbit of each finite tuple of A is Σ_α-definable over $\bar{c}$.
2. The ordinal α is the least such that (1) holds.

A structure A with an α-generator is called an α-**generated structure.** These are exactly the structures with Scott complexity $\Sigma_{\alpha+2}$, d-$\Sigma_{\alpha+1}$, or $\Sigma_{\alpha+1}$ for limit α.

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Example: Finitely α-generated Structures

The structure $\mathbb{Z} + \mathbb{Z}$ is finitely 2-generated and has Scott complexity d-Σ_3.

It is not finitely generated in the language of linear orders, but is finitely generated in the language with the ordering, the predecessor, and the successor relations.

The generating tuples for $\mathbb{Z} + \mathbb{Z}$ in this expanded language are precisely the tuples which are 2-generators for $\mathbb{Z} + \mathbb{Z}$ as a linear order.

Every finitely generated structure is almost rigid.

In the case where A is almost rigid, being finitely α-generated and being finitely generated (after some alterations) coincide.

Lemma (R.A.*)

Suppose that A is finitely α-generated by $\bar{c}$ and almost rigid, witnessed by $\bar{d}$. Let $\{\phi_{\bar{a}}(\bar{x}, \bar{c}, \bar{d}) : \bar{a} \in A\}$ be the family of $\Sigma_{<\alpha}$-formulas defining the automorphism orbits of $(A, \bar{c}\bar{d})$. In the definitional expansion which includes a relation predicate for each $\phi_{\bar{a}}$, A is finitely generated by $\bar{c}\bar{d}$.

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Theorem (R.A.*)

A structure A has a d-$\Sigma_{\alpha+1}$ Scott sentence iff some α-generator has a Π_α automorphism orbit.

This generalizes a theorem about finitely generated groups obtained with Julia Knight and Charlie McCoy.

In fact, more is true.

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Connecting Old Notions of Scott Rank

Recall that $r(A)$ is the least ordinal α such that whenever two finite tuples in A agree on all Π_α formulas, they must be automorphic.

Corollary (R.A.*)

For a structure A, $r(A) = \alpha$ iff α is the least ordinal such that A has a $\Pi_{\alpha+2}$ Scott sentence.

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Proof Sketch: Note first that $r(A) = \alpha$ iff α is the least ordinal such that the automorphism orbits of A are Π_α-definable. Then the result follows from the fact that A has a $\Pi_{\alpha+1}$ Scott sentence iff the automorphism orbits of A are $\Pi_{<\alpha}$-definable.

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Open Question

Matthew Harrison-Trainor and Turbo Ho showed that a finitely generated group has Scott complexity Σ_3 iff it contains a proper Σ_1 elementary substructure isomorphic to itself.

Conjecture

A finitely α-generated structure has Scott complexity $\Sigma_{\alpha+2}$ iff it contains a proper Σ_α elementary substructure isomorphic to itself.

Thank You!

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