The ω-Vaught's Conjecture

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- There is a "purely structural" strengthening of Vaught's conjecture named the ω-Vaught's conjecture (ω-VC).
- Linear orders satisfy the ω-Vaught's conjecture.

Summary of the talk

1. Vaught's conjecture and the Morley analysis
2. ω-VC
3. Selected points from the proof for linear orders

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- $\mathcal{L}_{\omega_1,\omega}$ is infinitary logic; it extends first order logic by allowing countable conjunctions and disjunctions.
- Under CH the conjecture trivially holds. You can replace "continuum" with "perfectly many" to get a statement independent of set theoretic considerations.

Selected Variations on Vaught's Conjecture

Conjecture: [Martin] Given a complete, consistent first order theory T over a countable vocabulary, add a predicate for every type to create T_1. If T has fewer than $2^{\aleph_0}$ many models, then any model of T is $\aleph_0-$categorical in its T_1 theory.

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Theorem: [Becker] One of the following holds for any complete, left invariant Polish G-space X:

- X has perfectly many orbits.
- Every orbit of X is $\mathbf{\Pi}^0_\omega$.

Complexity of $\mathcal{L}_{\omega_1,\omega}$ formulas

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- For $\alpha \in \omega_1$, φ is Σ_α^{in} if $\varphi = \bigvee\!\!\!\!\bigvee_i \exists(\bar{x})\psi_i(\bar{x})$ for $\psi_i \in \Pi_\beta^{in}$ with $\beta < \alpha$.
- For $\alpha \in \omega_1$, φ is Π_α^{in} if $\varphi = \bigwedge\!\!\!\!\bigwedge_i \forall(\bar{x})\psi_i(\bar{x})$ for $\psi_i \in \Sigma_\beta^{in}$ with $\beta < \alpha$.

Complexity of $\mathcal{L}_{\omega_1,\omega}$ formulas

- For two models M, N we say $M \leq_\alpha N$ if $\Pi^{in}_\alpha - \mathrm{Th}(M) \subseteq \Pi^{in}_\alpha - \mathrm{Th}(N)$.
- Note that $M \geq_\alpha N$ if and only if $\Sigma^{in}_\alpha - \mathrm{Th}(M) \subseteq \Sigma^{in}_\alpha - \mathrm{Th}(N)$.
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Fact: The $\equiv_\alpha$ are Borel equivalence relations.
Theorem: [Silver 80] Borel equivalence relations have either countable or continuum many equivalence classes.

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Definition: A φ as in the theorem statement is called a *Scott sentence*.
Definition: [Montalbán] The (parametrized) *Scott rank* of M is the least $\alpha \in \omega_1$ such that M has a $\Sigma^{in}_{\alpha+2}$ Scott sentence. We write $\mathrm{SR}(M) = \alpha$.

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3. $SS(\varphi)$ is cofinal in ω_1 and for all $\beta < \omega_1$ there are countably many $\equiv_\beta$ classes.This means there are $\aleph_1$ many models.

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The Vaught ordinal

Given a $\varphi \in \mathcal{L}_{\omega_1,\omega}$ we define the **Vaught ordinal**, written $\mathrm{vo}(\varphi)$ as the least β such that either

- there are continuum many models of φ up to $\equiv_\beta$ equivalence,
- or there are only countably many models of φ and they all have Scott rank less than β.

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Vaught's conjecture holds if and only if $\mathrm{vo}(\varphi)$ is well defined for all $\mathcal{L}_{\omega_1,\omega}$ sentences φ.

Vaught ordinal examples

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- If $\psi \in \Sigma_{\alpha+2}^{in}$ is a Scott sentence then $\mathrm{vo}(\psi) = \alpha + 1$.
- Both $\mathbb{Q}$-vector spaces and algebraically closed fields: a Π_2^{in} sentence with $\mathrm{vo}(\chi) = 3$ as they always have $\mathrm{SR}(M) < 3$.
- Boolean algebras: a Π_2^{in} sentence with $\mathrm{vo}(\theta) = \omega$ as there are uncountably many $\equiv_\omega$ classes but only countably many $\equiv_n$ classes for $n \in \omega$.

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- It is unknown if one implies the other.

Linear orders

Theorem: [Steel 78] For any $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over $\{\leq\}$ that implies all models are linear orders,

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- is really, really big,
- is dependant on more than the complexity of the formula φ,
- uses notions from higher recursion theory.

Linear orders continued

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Proof uses Σ_1^1 bounding; is not true at non-admissible ordinals (e.g your ordering is itself an ordinal).

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A better bound requires a finer combinatorial analysis of $L/\sim_\alpha$ for smaller α.

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- Version 3: $\mathrm{vo}(\varphi) \leq (\alpha + \omega) \cdot 5 + \omega \cdot 5$

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Theorem:[G., Montalbán] For any $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over $\{\leq\}$ that implies all models are linear orders, φ satisfies ω-VC.

The main lemma

Definition: A structure M is $(\beta, \beta+\omega)$-small if for all $n \in \omega$

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Lemma: The following are equivalent for $\varphi \in \Pi^{in}_\alpha$:

1. Every ψ that implies φ satisfies $\omega - \mathrm{VC}$.
2. For every $\beta \geq \alpha$ and $(\beta, \beta+\omega)$-small A with $A \models \varphi$ and $\mathrm{SR}(A) \geq \beta + \omega$, there is a $B \equiv_\beta A$ with $\mathrm{SR}(B) \geq \beta + \omega$ and $B \not\equiv_{\beta+\omega} A$.

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Proof idea for (2) implies (1): Assume there is some $(\alpha, \alpha+\omega)$-small model of φ with a large Scott rank. Build a perfect binary tree of $\equiv_\alpha$ structures that are not $\equiv_{\alpha+\omega}$ at a given height. The set of limit structures at each path witness distinct $\equiv_{\alpha+\omega}$ classes.

What this gets us

The objective: Given a $(\beta, \beta+\omega)$-small A with $\mathrm{SR}(A) \geq \beta+\omega$, explore the space of B that have $B \equiv_\beta A$. Try to find a transformation of A into a B that satisfies the two competing goals:

1. The Scott rank of B stays at at least $\beta+\omega$,
2. B disagrees with A on some $\Pi^{in}_{\beta+n}$ formula.

The replacement lemma

Lemma: There is a non-decreasing function $f : \omega \to \omega$ which, given an $(\alpha, \alpha + \omega)$-small structure L with $\mathrm{SR}(L) \geq \alpha + n$, guarantees that there is a structure P with

$$L \equiv_{\alpha+n} P \text{ and } \alpha + n \leq \mathrm{SR}(P) \leq \alpha + f(n).$$

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Idea: Apply this lemma to intervals inside of a linear ordering to control the Scott ranks of end segments.

A splitting formula

Lemma: For a fixed vocabulary, given any ordinal α, there is a $\Pi^{in}_{2\alpha+3}$ sentence ρ_α such that

$$\mathcal{A} \models \rho_\alpha \iff \mathsf{SR}(\mathcal{A}) \geq \alpha.$$

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In nearly all cases considered we construct models that disagree on some Boolean combination of the $\psi_{\leq,i}$ and $\psi_{\geq,i}$.

Fine Scott rank analysis of linear orderings

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Lemma: For any linear ordering A with $\mathrm{SR}(A_{\leq x}) \leq \beta$ for all $x \in A$,

$$\mathrm{SR}(A) \leq \beta + 4.$$

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One big idea: Steel used that $L/\sim_{\omega_1^\varphi}$ is a dense linear order. We can reduce to the case that for some n, $L/\sim_{\alpha+n}$ with a suitable application of the replacement lemma.

While ordinals are *descriptively complicated* they are actually quite *combinatorially simple*; this is quite an important reduction.

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- A purely structural proof of Vaught's conjecture for other structures may be possible via ω-VC.
- Vaught's conjecture is only the beginning.
- If you think this is a straw-man, please tear it down!

Thank you!

References

D. Gonzalez and A. Montalbán
The ω-Vaught's conjecture
To appear in *Trans. Amer. Math. Soc.*
https://arxiv.org/abs/2211.02156

R. L. Vaught.
Denumerable models of complete theories.
In *Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959)*, pages 303–321. Pergamon, Oxford, 1961.

J. R. Steel.
On Vaught's conjecture.
In *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pages 193–208. Springer, Berlin, 1978.

A. Montalbán.
A robuster scott rank.
Proc. Amer. Math. Soc., 143(12):5427–5436, 2015.