True Stages and Descriptive Set Theory

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Baire Space

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This is a (Polish) topological space with basic clopen sets

$$[\sigma] = \{\tau \in \omega^{<\omega} : \tau \succeq \sigma\}.$$

Closed sets correspond to paths through trees.

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- $\mathbf{\Pi}^0_\alpha$: Countable intersections of $\mathbf{\Sigma}^0_\beta$ sets for $\beta < \alpha$.

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- A set is $D_\eta(\mathbf{\Sigma}^0_\alpha)$ if it is a difference of η-many $\mathbf{\Sigma}^0_\alpha$ sets. E.g., if η even, of the form

$$\bigcup_{\gamma < \eta \text{ odd}} \left(U_\gamma - \bigcup_{\gamma' < \gamma} U_{\gamma'} \right)$$

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  where each U_γ is $\mathbf{\Sigma}^0_\alpha$.

For example, a $D_2(\mathbf{\Sigma}^0_\alpha)$ set is of the form

$$U_1 - U_0$$

and a $D_3(\mathbf{\Sigma}^0_\alpha)$ set is of the form

$$U_2 - (U_1 - U_0).$$

The Hausdorff-Kuratowski Theorem

Theorem (Hausdorff-Kuratowski)

$$\mathbf{\Delta}^0_2 = \bigcup_\eta D_\eta(\mathbf{\Sigma}^0_1).$$

Proof.

See blackboard. □

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$$\mathbf{\Delta}^0_{\alpha+1} = \bigcup_\eta D_\eta(\mathbf{\Sigma}^0_\alpha).$$

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Let A be $\mathbf{\Delta}^0_{\alpha+1}$.

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Each $\mathbf{\Sigma}^0_1$ sets in the new topology is $\mathbf{\Sigma}^0_\alpha$ in the old topology. □

Change-of-Topology

Change-of-topology is a useful tool in descriptive set theory.

Theorem

Let $(X, \mathcal{T})$ be a Polish space with topology $\mathcal{T}$.

Let $B_1, B_2, \ldots$ be any countable collection of Borel sets in $(X, \mathcal{T})$.

There is a finer Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ such that $B_1, B_2, \ldots$ are open.

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There is a finer Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ such that $B_1, B_2, \ldots$ are open.

Often you can also say something about the open sets in the new topology. Before, we needed that the open sets in the new topology are $\boldsymbol{\Sigma}^0_\alpha$ in the old topology.

What is this talk about?

This talk will be about a way of understanding change-of-topology in descriptive set theory using iterated true stages from computability theory.

The Effective Borel Hierarchy

The effective Borel hierarchy allows only computable unions and intersections. For α a computable ordinal:

- Σ^0_1: Effectively open sets, i.e., sets of the form $\bigcup_{\sigma \in W}[\sigma]$ for W c.e.
- Π^0_1: Effectively closed sets, i.e., paths through a computable tree.
- Σ^0_α: Unions of c.e. collections of (names for) Π^0_β sets for $\beta < \alpha$.
- Π^0_α: Intersections of c.e. collections of (names for) Σ^0_β sets for $\beta < \alpha$.

We can also define Δ^0_α, $D_\eta(\Sigma^0_\alpha)$, etc.

These hierarchies also relativize to an oracle.

Effective Descriptive Set Theory

Any $\mathbf{\Sigma}^0_\alpha$ set is $\Sigma^0_\alpha(X)$ (relative to X) for some set X. Thus it can be useful to apply effective methods even if we are not initially interested in computability.

Theorem (Hausdorff-Kuratowski, Selivanov)

$$\Delta^0_2 = \bigcup_{\eta<\omega_1^{CK}} D_\eta(\Sigma^0_1).$$

The Turing Jump

The key connection is that there is a way of thinking about $\Sigma^0_{\alpha+1}$ sets using the αth jump.

Fact

A set $A \subseteq \omega^\omega$ is $\Sigma^0_{\alpha+1}$ if and only if there is a Σ^0_1 set $V \subseteq \omega^\omega$ such that $A = \{x : x^{(\alpha)} \in V\}$.

We will use true stage constructions to approximate the jumps.

Iterated True Stage Constructions

The idea is to think of $\varnothing^{(\alpha)}$ as an iteration of the limit lemma. Each jump is a simple step, and we just need a good way to organize how they fit together.

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Many computability-theoretic frameworks have been introduced to help organize this:

- Harrington worker arguments
- Lempp and Lerman's tree of strategies
- Ash and Knight's α-systems
- Montalbán's η-systems
- Greenberg and Turetsky's variation on the η-systems

Approximating the First Jump

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We could think of approximating the infinite binary string K by the finite binary strings $K_s \upharpoonright s$. But it might be that every $K_s \upharpoonright s$ makes some incorrect guess.

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Suppose that at stage s, we guess that $K_s \upharpoonright n_s$ is an initial segment of K. At non-deficiency stages, our guess is correct.

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- There are infinitely many 1-true stages.
- If s is a 1-true stage, then it appears 1-true at every stage $t > s$.
- If s is not 1-true, then there might be stages $t > s$ which do not have enough information to see this, i.e.,

$$K_s \restriction n_s \prec K_t \restriction n_t.$$

We say that s appears 1-true at stage t. Such t are also not 1-true.

Approximating More Jumps

Montalbán: Iterate this through the hyperarithmetic hierarchy:

- Having approximated $\varnothing^{(\alpha)}$ at stage s by a finite string ∇_s^α, use this finite string as an oracle to approximate $\varnothing^{(\alpha+1)}$ by a finite string ∇_{s+1}^α.

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- Say that s appears α-true at stage t, and write $s \leq_{\alpha} t$, if $\nabla^{\beta}_{s} \preceq \nabla^{\beta}_{t}$ for $\beta \leq \alpha$.
- The ∇^{α}_{s} and the relations $\leq_{\alpha}$ are all computable.

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- The ∇_s^α and the relations $\leq_\alpha$ are all computable.

Disclaimer: This is all morally correct, but needs some adjustment for technical reasons.

References

For the technical details, see:

- Ash and Knight's book *Computable Structures and the Hyperarithmetical Hierarchy*
- Montalban, η-systems, in *Priority Arguments via True Stages* and *Computable Structure Theory: Beyond the arithmetic*
- Day, Greenberg, HT, Turetsky, *An effective classification of Borel Wadge classes* and *Iterated priority arguments in descriptive set theory*

Relativizing True Stages

In the true stage constructions before, we approximated $\varnothing$, $\varnothing'$, $\varnothing''$, ...

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In fact, given $x \in \omega^\omega$, we can make it so that the approximation to $x^{(\alpha)}$ at stage s only depends on $x \upharpoonright s$:

- For each finite string σ and computable ordinal α, define $\sigma^{(\alpha)}$, the approximation to $x^{(\alpha)}$ for x extending σ at stage $|\sigma|$.

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- Define $\sigma \leq_\alpha \tau$ if $\sigma^{(\beta)} \leq \tau^{(\beta)}$ for $\beta \leq \alpha$. We say σ appears α-true at τ.

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- Define $\sigma \leq_\alpha \tau$ if $\sigma^{(\beta)} \leq \tau^{(\beta)}$ for $\beta \leq \alpha$. We say σ appears α-true at τ.
- Say that σ is α-true for $x \in 2^\omega$, and write $\sigma \leq_\alpha x$, if $\sigma^{(\beta)} \leq x^{(\beta)}$ for $\beta \leq \alpha$.

Note that being true is now relative to the extension x.

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- $\sigma \leq_0 \tau \Leftrightarrow \sigma \preceq \tau$.
- $\sigma \leq_\alpha \tau \Rightarrow \sigma \leq_\beta \tau$ for $\beta < \alpha$.
- for each $x \in \omega^\omega$, there infinitely many strings which are α-true for x:

$$\sigma_0 \leq_\alpha \sigma_1 \leq_\alpha \sigma_2 \leq_\alpha \cdots \leq_\alpha x.$$

- $(\omega^{<\omega}, \leq_\alpha)$ is a tree/forest.

True Stages and Topology

Some additional properties of our true stages:

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  for some c.e. set W.

Taking $[\sigma]_\alpha = \{\bar{x} : \sigma \leq_\alpha \bar{x}\}$ as a basis, we get a Polish topology $\mathcal{T}'$ extending the standard topology where the open sets are exactly those generated by the Σ^0_α sets.

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We can adjust our proof of the Hausdorff-Kuratowski theorem to get a proof for $\Delta^0_{\alpha+1}$ by replacing the standard tree $(\omega^{<\omega}, \leq)$ by the tree $(\omega^{<\omega}, \leq_\alpha)$.

Theorem (Hausdorff-Kuratowski, Selivanov)

For all computable α,

$$\Delta^0_{\alpha+1} = \bigcup_{\eta < \omega_1^{ck}} D_\eta(\Sigma^0_\alpha).$$

Proof.

See blackboard. □

This sounds great, but can we do anything new?

Wadge Reducibility

Definition (Wadge)

Let A and B be subsets of Baire space ω^ω.

We say that A is *Wadge reducible* to B, and write $A \leq_W B$, if there is a continuous function f on ω^ω with $A = f^{-1}[B]$, i.e.

$$x \in A \Longleftrightarrow f(x) \in B.$$

Structure of Wadge Degrees

Theorem (Martin and Monk, AD)

The Wadge order is well-founded.

Theorem (Wadge's Lemma, AD)

Given $A, B \subseteq \omega^\omega$, either $A \leq_W B$ or $B \leq_W \omega^\omega - A$.

These theorems are proved by playing a game. For Borel sets, we have Borel Determinacy without having to assume AD, and so these are always true for Borel sets.

Wadge Degrees in Second-order Arithmetic

Borel determinacy requires iterations of power-set.

Theorem (Friedman)

Borel determinacy requires ω_1 iterations of the Power Set Axiom.

Martin showed that Σ^0_4 Determinacy is not provable in second-order arithmetic.

One the other hand, one can prove that Borel Wadge games are determined in second-order arithmetic.

Theorem (Louveau and Saint-Raymond)

Borel Wadge determinacy is provable in second-order arithmetic.

Description of Wadge Degrees

There are also many comprehensive descriptions of the Borel Wadge classes:

- Louveau (1983)
- Duparc (2001)
- Selivanov, for k-partitions (2007, 2017)
- Kihara and Montalbán, for functions into a countable BQO (2019)

We use our true stage machinery to give a new description of the Borel Wadge classes, and use them to prove Borel Wadge determinacy in a reasonable fragment of second-order arithmetic.

Wadge Degrees and Reverse Math

Theorem (Day, Greenberg, HT, Turetsky)

Borel Wadge determinacy is provable in $ATR_0 + \Pi^1_1$–Ind, *and there is a complete description of the Borel Wadge classes.*

Thus the Borel Wadge degrees are semilinearly ordered and well-founded.

This simplifies Louveau and Saint-Raymond's proof in second-order arithmetic and uses a weaker subsystem. Our descriptions of the classes are inherently dynamic, and naturally lightface.

- Make a list of described classes. These are non-self-dual. Our descriptions are dynamic in nature.
- Prove a Louveau-Saint Raymond separation result for each described class Γ, which implies that if A is universal for Γ, and B is Borel, then either $A \leq_W B$ or $B \in \check{\Gamma}$, in which case $B \leq_W A^c$.
- Prove that the intersection of a described class and its dual is either a union of described classes of lower Wadge degree, like

$$\Delta^0_{\xi+1} = \bigcup_\eta D_\eta(\Sigma^0_\xi),$$

  or is a Wadge class in its own right like Δ^0_1.
- Given a Borel set, take the least described class (or dual of a described class, or $\Delta(\Gamma)$) containing it. Prove that it is complete for that class.

Theorem (Loueveau, Saint Raymond)

Suppose that Γ is a described class. Let $A \in \Gamma$. Let B_0 and B_1 be two disjoint Σ^1_1 sets. Then either:

- *There is a continuous reduction of (A, A^c) into (B_0, B_1), or*
- *There is a $\check{\Gamma}$ separator of B_0 from B_1.*

If A is universal for Γ, and B is Borel, then either $A \leq_W B$ or $B \in \check{\Gamma}$, in which case $B \leq_W A^c$.

The direct way to prove this would be to use Borel determinacy for a naturally associated game.

Louveau and Saint Raymond show by an unravelling process that there is an associated closed game.

Using true stages, we get a relatively simple description of such a game.

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Take $\Gamma = \Sigma^0_\xi$. Let T_i be a tree whose projection is B_i.

- Player 1 plays x in A or A^c.
- Player 2 attempts to play y in B_0 (if $x \in A$) or B_1 (if $x \notin A$), with a corresponding witness f in $[T_0]$ or $[T_1]$.
- Player 2 guesses, using the true stage machinery, at whether x is in A or not. At each stage, they play an attempt at extending y and f. But they are only committed to which f they play at true stages.

Theorem (Day, Greenberg, HT, Turetsky)

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References

Day, Greenberg, Harrison-Trainor, Turetsky:

- *Iterated priority arguments in descriptive set theory*
- *An effective classification of Borel Wadge classes*

Thanks!