A characterization of the strongly η-representable many-one degrees

Josiah Jacobsen-Grocott

University of Wisconsin—Madison
Partially supported by NSF Grant No. DMS-2053848

Midwest Computability Seminar, Sep 13, 2021

η-representations

Recall that η is the order type of the rationals.

Definition

For a set A a linear order L is said to be an *η-representation* of A if there is a surjective function $F : \omega \to A$ such that L has order type

$$\sum_{n \in \omega} \eta + F(n)$$

We say L is a *strong* η-representation if the function F is strictly increasing and a *increasing* η-representation if F is non-decreasing. If a set A has a computable (strong, increasing) η-representation then we say A is *(strongly, increasingly) η-representable*.

Successor and block relations

Definition

For any linear order L, S_L the successor relation on L is defined by

$$S_L(x,y) \iff x < y \wedge (x,y) = \emptyset$$

The block relation B_L is given by

$$B_L(x,y) \iff (x,y),(y,x) \text{ are finite}$$

A block of size n in L is a collection $x_0 <_L \cdots <_L x_{n-1}$ such that $B(y,x_0) \to \bigvee_{i<n} y = x_i$

It can be seen that S_L is $\Pi^0_1(L)$ and B_L is $\Sigma^0_2(L)$.

Upper bound of η-representable sets

Theorem (Feiner, 1970)

For a linear order L the set $\{n : L \text{ has a block of size } n\}$ is Σ^0_3 in L.

For an η-representation L of a set A we have that $A = \{n : L \text{ has a block of size } n\}$. This gives us the following.

Corollary

If a set A has a computable η-representation then A is Σ^0_3.

Upper bound of strongly η representable sets

In his thesis, Fellner proved

Theorem (Fellner, 1976)

If A has a computable strong η-representation then A is Δ^0_3.

Fellner was able to prove that every Σ^0_2 and Π^0_2 set has a strong η-representation. This lead him to make the make the following conjecture:

Conjecture (Fellner, 1976)

Every Δ^0_3 set has a strong η-representation.

Fellner's conjecture is false

Theorem (Lerman, 1981)

There is a Δ^0_3 set with no computable η-representation.

Lerman was able to find a subclass which always has an η-representation.

Theorem (Lerman, 1981)

If A is Σ^0_3 then $A \oplus \omega$ has a computable η-representation.

This gives a characterization of the *m*-degrees with η-representations.

Corollary (Lerman, 1981)

Every Σ^0_3 m-degree has a set with a computable η-representation.

Degrees without computable strong η-representations

In the case of strong η-representations Harris showed that Fellner's conjecture does not hold even for Turing degrees.

Theorem (Harris, 2008)

There is a Δ^0_3 degree that does not compute any non-computable set with a computable strong η-representation.

Questions

- What are the (strongly) η-representable sets?
- What are the degrees with computable strong η-representations?

Limitwise monotonic functions

Harris came up with a characterization of the η-representable sets. To motivate our definitions we will consider this characterization.

Definition

A function, $F : \omega \to \omega$, is *limitwise monotonic* if there is a computable function, $f : \omega^2 \to \omega$ such that $F(n) = \lim_s f(n, s)$ and for all n, s $f(n, s) \leq f(n, s + 1)$.

So we can computably approximate the values of F from below.

Characterization of η-representable sets

By the Limit Lemma if F is limitwise monotonic then F is Δ^0_2 and hence if $A = \mathrm{rang}(F)$ then A is Σ^0_2.

Theorem (Harris, 2008)

A set A is η-representable if and only if A is the range of a $0'$-limitwise monotonic function.

The construction of the η-representation, L, is done uniformly, constructing linear orders $L_n \cong \eta + F(n)$ and taking $L = \sum_n L_n$. From this it can be seen that if A is the range of a strictly increasing $0'$-limitwise monotonic function then A is strongly η-representable. However Harris showed that this is not a characterization of the strongly η-representable sets.

Support increasing limitwise monotonic

Katch and Turetsky modified the notion of limitwise monotonic as follows:

Definition

A function $F : \mathbb{Q} \to \omega$ is *support (strictly) increasing limitwise monotonic function on* $\mathbb{Q}$ if there is computable $f : \mathbb{Q} \times \omega \to \omega$ such that

- $F(q) = \lim_s f(q, s)$.
- For all q, s $f(q, s) \leq f(q, s + 1)$.
- The set $S := \{q \in \mathbb{Q} : F(q) \neq 0\}$ has order type ω.
- $F \upharpoonright S$ is a (strictly) increasing.

The intuition is that members of S represent blocks of an increasing η-representation. When we discover a new block in that η-representation we pick a new rational representative that is between the representatives for the blocks to the left and right.

Characterization of increasingly η-representable sets

We define $\mathsf{SILM}^{0'}(\mathbb{Q})$ to be the set of all A such that A is the range of a $0'$-support increasing limitwise monotonic function on $\mathbb{Q}$ and $\mathsf{SSILM}^{0'}(\mathbb{Q})$ to be the set of all A such that A is the range of a $0'$-support strictly increasing limitwise monotonic function on $\mathbb{Q}$.

Theorem (Katch, Turetsky, 2010)

1. $A \in \mathsf{SILM}^{0'}(\mathbb{Q}) \iff$ *A has a computable increasing η-representation.*
2. *Every Δ^0_3 m-degree has a computable increasing η-representation.*
3. $A \in \mathsf{SSILM}^{0'}(\mathbb{Q}) \implies$ *A has a computable strong η-representation.*

The converse of 3 does not hold.

Theorem (Turetsky, 2011)

There is a set $A \notin \mathsf{SSILM}^{0'}(\mathbb{Q})$ with a computable strong η-representation.

Observations

- Notice that both the existing characterizations involve relativizing something to $0'$. In fact they can be seen as relativizations of characterizations of the sets with computable (increasing) η-representation with computable successor relation.
- Another observation is that in the proofs of the above characterizations, the construction of the η-representation creates each block separately and does not take advantage of the fact that blocks may merge. Essentially $0'$ can compute B_L for the η-representation.

η-s-representations

From the first observation we come up with the following definition.

Definition

A (strong) η-s-representation of a set A is a computable (strong) η-representation, L, where the successor relation, S_L, is also computable.

Since all existing characterizations can be seen as relativizations of characterizations of η-s-representations the hope is that we can come up with a characterization of the strongly η-s-representable sets and turn that into a characterization of the strongly η-representable sets.

Something new is needed for strong η-representations

We have a characterization of $\mathsf{SSILM}^{0'}(\mathbb{Q})$ in terms of strong η-representations.

Theorem

A set A is in $\mathsf{SSILM}^{0'}(\mathbb{Q})$ if and only if there is a computable strong η-representation with $0'$-computable block relation.

This and Turetsky's result mean that any characterization of the strongly η-representable sets must involve constructions that allow for blocks to merge.

Connected approximations

The second observation leads us to the following definition.

Definition

A *connected approximation* to a set A is a sequence of finite functions $(c_n)_n$ with associated sequences of finite sets $(A_{n,m})_m$ that satisfy the following:

1. $\mathrm{rang}(c_n) \subseteq \mathrm{dom}(c_{n+1})$ for all n.
2. $A_{n,0} := \mathrm{dom}(c_n)$, $A_{n,m+1} := c_{n+m}(A_{n,m})$.
3. The limit $A_{n,\omega} := \lim_m A_{n,m}$ always exists.
4. $A = \cup_n A_{n,\omega}$.

We call a connected approximation $(c_n)_n$ *monotonic* if $c_n(x) \geq x$ for each n and $x \in \mathrm{dom}(c_n)$ and order preserving if each c_n preserves $\leq$.

The idea is that the elements of $\mathrm{dom}(c_n)$ represent the sizes of blocks at stage n of creating a strong η-representation. If $c_n(x) = c_n(y)$ there the corresponding blocks of x and y have merged at stage $n+1$.

Characterizations with connected approximations

Theorem (J-G, 2019)

For a set A we have the following characterizations.

- *A has a computable connected approximation if and only if A is Σ^0_2.*
- *A has a computable monotonic connected approximation if and only if A is the range of a computable limitwise monotonic function.*
- *A has a computable MOP connected approximation if and only if $A \in \mathsf{SILM}(\mathbb{Q})$.*
- *A has a computable MOP connected approximation where each c_n is injective if and only if $A \in \mathsf{SSILM}(\mathbb{Q})$.*

Relativizing the middle two results we get new characterizations of η-representable and increasingly η-representable sets.

Characterization of sets with strong η-s-representations

Theorem (J-G, 2019)

A set A has a computable strong η-s-representation if and only if it has a computable MOP connected approximation where each c_n satisfies

$$\psi(n) = \forall x \in \mathrm{rang}(c_n)[\sum_{m \in c_n^{-1}(\{x\})} m + n \leq x + n]$$

What $\psi(n)$ says is that blocks can merge, but the new block must have the size larger than the sum of the merged blocks and the points between them. The later the stage where blocks are merged, the more points there are between blocks.

Not a characterization of sets with strong η-representations

Relativizing give the following corollary.

Corollary

If a set A has a strong η-representation then A has a $0'$-computable MOP connected approximation where each c_n satisfies

$$\psi(n) = \forall x \in \mathrm{rang}(c_n)[\sum_{m \in c_n^{-1}(\{x\})} m + n \leq x + n]$$

Unfortunately we only have one direction. The other is open.

Question

Is there a set with a $0'$-computable strong η-s-representation but no computable strong η-representation?

Density result

Theorem (J-G, 2019)

Suppose $g : \omega \to \omega$ is a computable increasing function. If a set A has a strong η-s-representation and satisfies $|A \cap g(n)| \geq n$ for all n then $A \in \mathrm{SSILM}(\mathbb{Q})$.

This can be relativised to give the following result characterization for dense enough sets.

Theorem (J-G, 2019)

For a set A suppose that there is $0'$-computable increasing function g such that $|A \cap g(n)| \geq n$ for all n. The following are equivalent.

- *A has a computable strong η-representation.*
- *$A \in \mathrm{SSILM}^{0'}(\mathbb{Q})$.*
- *A has a $0'$-computable strong η-s-representation.*

Characterization of degrees with strong η-representations

Lemma

If A is a set with a strong η-s-representation then $A \oplus \omega$ also has a strong η-s-representation.

Relativizing this lemma and putting it together with the density result gives us the following characterization of the degrees of sets with strong η-representations.

Theorem (J-G, 2019)

The following coincide.

- *The m-degrees of sets with computable strong η-representations.*
- *The m-degrees of sets in $\mathrm{SSILM}^{0'}(\mathbb{Q})$.*
- *The m-degrees of sets with $0'$-computable strong η-s-representations.*

Thank You