Maxima for $\leq_{tc}$ with respect to $\sim_{\alpha}^{c}$

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Given classes of structures, can we determine which has a more difficult classification problem?
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Knight, et al. considered computable languages, and structures with universes subsets of $\omega$, in order to formulate an effective analogue of the Borel embedding.

Definition

A *Turing computable embedding* of $(K, E)$ into $(K', E')$ is an operator $\Phi = \phi_e$ such that

- for each $A \in K$ there exists $B \in K'$ such that $\Phi(A) = \phi_e^{D(A)} = \chi_{D(B)}$, and
- if $A, A' \in K$, then $AE A' \iff \Phi(A) E' \Phi(A')$.

This induces a preordering, denoted by $(K, E) \leq_{tc} (K', E')$. If we do not mention the equivalence relation, then it is assumed to be isomorphism.
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Examples

- **UG, LO, RCF** are universal (every class $K$ of countable structures has $K \leq_{tc} UG$).
- If the isomorphism classes of $K$ are distinguished by computable infinitary sentences, then $K$ cannot be universal.
- Isomorphism for abelian $p$-groups is not distinguished by sentences at a single level, yet they are not universal.
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- Isomorphism for abelian $p$-groups is not distinguished by sentences at a single level, yet they are not universal.
Motivated by the importance of the $\Sigma^c_\alpha$ sentences in the previous work, we define the following equivalence relation.

**Definition**

- We say that two structures $A$ and $B$ in the same language are $\Sigma^c_\alpha$ *equivalent* if and only if $A$ and $B$ satisfy the same $\Sigma^c_\alpha$ sentences (denoted $A \sim^c_\alpha B$).

- If for all $\alpha < \omega_1^{CK}$, $A \sim^c_\alpha B$, then we say that $A$ and $B$ are *computably infinitarily equivalent* ($A \sim^c_{\omega_1^{CK}} B$).

Can we find classes which are universal for $\sim^c_\alpha$, but not for $\cong$?
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- To each group, we associate a sequence from \( (\omega + 1)^{<\omega_1} \) (its Ulm sequence).
- For \( \alpha < \omega_1^{CK} \) we can say \( U_\alpha(G) \geq k \) with a computable infinitary sentence.

**Theorem (Ulm, 1933)**

Two countable, reduced Abelian \( p \)-groups are isomorphic if and only if they have the same Ulm sequences.
Abelian $p$-groups

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Maxima for $\sim^c_{\alpha}$

Analysis of results of Sara Quinn yields the following.

Lemma

For any class $K$, $(K, \sim^c_{2}) \leq_{tc} (ApG_{\omega}, \sim^c_{2})$.

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For any class $K$, $(K, \sim^c_{2n}) \leq_{tc} (ApG_{\omega \cdot n}, \sim^c_{2n})$.

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For any class $K$, $(K, \sim^c_{\omega}) \leq_{tc} (ApG_{\omega^2}, \sim^c_{\omega})$.

Lemma

For any class $K$, $(K, \sim^c_{\alpha}) \leq_{tc} (ApG_{\beta}, \sim^c_{\alpha})$ if and only if $\beta = \omega \cdot \gamma$ and $\alpha < \gamma$. 

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Uniformity of Embeddings

- In each case, we are essentially coding a set (the set of $\Sigma^c_\alpha$ sentences true in the input structure).
- For bounded $\alpha < \omega_1^{CK}$, there is a certain length $\beta$ such that $(K, \sim^c_\alpha) \leq tc (ApG_\beta, \sim^c_\alpha)$.
- Finding the appropriate $\beta$, and a $\Phi = \phi_e$ to witness the embedding seems very uniform.
- Can we ‘lace together’ all of these embeddings into a ‘master procedure,’ showing that

$$(K, \sim^c_{\omega_1^{CK}}) \leq tc (ApG, \sim^c_{\omega_1^{CK}})?$$
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$$(K, \sim^c_{\omega_1^{CK}}) \leq_{tc} (\text{ApG}, \sim^c_{\omega_1^{CK}})?$$
Theorem (V.)

There is a partial computable function \( f : \omega \to \omega^2 \), such that if \( a \in O \), then \( f(a) = \langle e, b \rangle \), where \( \phi_e \) is the operator witnessing that \( (K, \sim|^c_a) \leq_{tc} (ApG|b|, \sim|^c_a) \).

The proof is by transfinite induction on ordinal notation, and we will need a few lemmas.
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Lemma (H. Rogers)

There is a partial computable function \( \cdot : \mathcal{O} \times \mathcal{O} \to \mathcal{O} \) such that, for all \( a, b \in \mathcal{O} \),

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|a| \cdot |b| = |a \cdot b|.
\]

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There is a computable operator \( \Phi = \phi_e \) such that, given an indexed family of groups \( (G_i)_{i \in \omega} = \{ \langle i, \psi \rangle : \psi \in D(G_i) \} \),

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\Phi((G_i)_{i \in \omega}) = \phi_e^{(G_i)_{i \in \omega}} = \chi_G,
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where \( G \cong \bigoplus_{i \in \omega} G_i \).
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There is a partial computable function \( \cdot : O \times O \rightarrow O \) such that, for all \( a, b \in O \),

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Lemma and Proof

Lemma (V.)

Suppose that \( \alpha \) is a computable ordinal of the form \( \omega \cdot \beta + n \), and let the function \( g \) be defined as

\[
g(\omega \cdot \beta + n) = \begin{cases} 
\omega^2 \cdot \beta + \omega \cdot \frac{n}{2} & \text{if } n \text{ is even} \\
\omega^2 \cdot \beta + \omega \cdot \frac{n+1}{2} & \text{if } n \text{ is odd}
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Then, for any \( K, (K, \sim^c_\alpha) \leq_{tc} (\text{ApG}_{g(\alpha)}, \sim^c_\alpha) \).

Now we can prove the theorem.
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Barwise Compactness

An another approach is to use Barwise-Kreisel compactness.

Theorem (Barwise; Kreisel)

Let $\Gamma$ be a $\Pi^1_1$ set of computable infinitary sentences. If every $\Delta^1_1$ subset of $\Gamma$ has a model, then $\Gamma$ also has a model.

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For $X$ such that $\omega_1^X = \omega_1^{CK}$, ApG lies on top for $\sim_{\omega_1^{CK}}^{c}$ amongst classes of $X$-computable structures.

Sketch: We let $\Gamma_\alpha$ be a set of sentences axiomatizing an appropriate operator that works for $\sim_{\alpha}^{c}$. Namely, sentences like

$$\forall A, j[(A \in K) \rightarrow (A \models \psi_j \leftrightarrow U_{h(\beta,j)}(\Phi(A)) = \omega)]$$

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But how do we quantify over structures?
Let $\Gamma = \bigcup_{\alpha<\omega_1^{CK}} \Gamma_\alpha$, this is a $\Pi^1_1$ set.

Take some $\Delta^1_1$ subset $\Gamma' \subseteq \Gamma$.

Kleene Bounding theorem tells us that the ordinals mentioned in $\Gamma'$ have a computable ordinal bound, say $\beta$.

By the earlier work, we can produce an appropriate model (operator) witnessing $(K, \sim^c_\beta) \leq tc (ApG_{g(\beta)}, \sim^c_\beta)$.

So Barwise-Kreisel compactness gives us a model.

Barwise compactness works for any $X$ such that $\omega_1^X = \omega_1^{CK}$. 
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Toward Consistency Result

That is still not \textit{quite} what we want, so we use new tools.

Definition
A structure $\mathcal{A}$ is \textit{hyperarithmetically saturated} if it satisfies the following conditions

- for all tuples $\bar{a} \in \mathcal{A}$, and all $\Pi^1_1$ sets $\Gamma(\bar{a}, x)$ of computable infinitary formulas with parameters $\bar{a}$, if every $\Delta^1_1$ set $\Gamma'(\bar{a}, x) \subseteq \Gamma(\bar{a}, x)$ is satisfied in $\mathcal{A}$, then $\Gamma(\bar{a}, x)$ is also satisfied in $\mathcal{A}$,

- a clause on infinitary disjunctions (technical, needed for following theorem).
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The Workhorse

Why investigate hyperarithmetical saturation?

Theorem (Ressayre)

Suppose that $A$ is a hyperarithmetically saturated structure, and $\Gamma$ is a $\Pi^1_1$ set of sentences involving symbols from the language of $A$, along with a new symbol. If the consequences of $\Gamma$ (sentences true in all models of $\Gamma$) in the language of $A$ are all true in $A$, then $A$ can be expanded to a model $A'$ of $\Gamma$. 
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Theorem (V.)

If ZFC has an $\omega$-model, then it has an $\omega$-model, $\mathcal{Z}$ having the following property. There is an $e \in \omega$, such that for all $\alpha < \omega_1^{CK}$ (real world $\omega_1^{CK}$), $\mathcal{Z} \models \psi_\alpha(e)$, where $\psi_\alpha(e)$ says $\phi_e$ is an embedding witnessing that ApG is on top for $\sim^c_\alpha$.

Subtle point:
There are some weird ordinals in this model, and its interpretation of $\omega_1^{CK}$ is bizarre.
Theorem (V.)

If ZFC has an $\omega$-model, then it has an $\omega$-model, $\mathcal{Z}$ having the following property. There is an $e \in \omega$, such that for all $\alpha < \omega_1^{\text{CK}}$ (real world $\omega_1^{\text{CK}}$), $\mathcal{Z} \models \psi_\alpha(e)$, where $\psi_\alpha(e)$ says ‘$\phi_e$ is an embedding witnessing that ApG is on top for $\sim_\alpha^c$.’

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Ressayre’s results imply that if $ZFC$ has an $\omega$-model, then it has a hyperarithmetically saturated $\omega$-model.

We take the language of set theory, and add a single new constant, $e$, to be interpreted as an index for an operator.

So, let $\Gamma$ say statements similar to those in the previous proof, but now add $e \in \omega$.

Since we are in set theory, the following sentence is now acceptable

$$\forall \mathcal{A}, j[(\mathcal{A} \in K) \rightarrow (\mathcal{A} \models \psi_j \leftrightarrow U_{h(\beta,j)}(\Phi(\mathcal{A})) = \omega)]$$

Apply the theorem.
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Questions

- Is it true that $ApG$ is universal for $\leq_{tc}$ with respect to $\sim_{\omega_1^{CK}}^c$ in any model of $ZFC$?
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Thank You!
(References available upon request)
Abelian $p$-groups

Fix any prime, $p$. Recall that an **Abelian $p$-group** is an Abelian group where each element has order a power of $p$. Let $G$ be a countable Abelian $p$-group.

- Define inductively: $G_0 = G$, $G_{\beta+1} = pG_\beta$, and $G_\lambda = \cap_{\gamma < \beta} G_\gamma$.
- $G$ is **divisible** if every $x \in G$ is divisible by $p^n$ for all $n$.
- For each $G$, there is a length, $\lambda$ such that $G_\lambda = G_{\lambda+1}$.
- If $G_\lambda = \{0\}$, we call $G$ **reduced**.
- Each element obtains a **height** in this way. The height of $x$ is the unique $\beta$ such that $x \in G_\beta$ but not in $G_{\beta+1}$.

These can be visualized as trees—this is very useful.
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To each (reduced) Abelian $p$-group, we associate an *Ulm sequence*, an element of $(\omega + 1)^{<\omega_1}$.

The $\alpha$th element of the sequence associated to $\mathcal{G}$ is denoted $U_\alpha(\mathcal{G})$.

$P_\alpha(\mathcal{G}) = \{ x \in \mathcal{G}_\alpha : px = 0 \}$, and

$U_\alpha(\mathcal{G}) = \dim(P_\alpha(\mathcal{G})/P_{\alpha+1}(\mathcal{G}))$.

We can say $U_\alpha(\mathcal{G}) \geq k$ with a computable infinitary sentence.

Theorem (Ulm, 1933)

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Let $ApG\alpha$ be the class of countable reduced abelian $p$-groups of length $\alpha$.

**Theorem (S. Quinn)**

For any $K$, $K \leq_{tc} ApG_\omega$ if and only if there is a computable sequence of $\Sigma^c_2$ sentences $(\psi_n)_{n \in \omega}$ such that for $A \not\cong B \in K$, there is an $n$ such that $\psi_n$ is true in exactly one of $A$ and $B$.

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For any $K$, $K \leq_{tc} ApG_\omega \cdot m$ if and only if there is a computable sequence $(\psi_n)_{n \in \omega}$ of $\Sigma^c_2m$ sentences in the language of $K$ such that for $A \not\cong B \in K$, there is an $n$ such that $\psi_n$ is true in exactly one of $A$ and $B$. 
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Suppose that $\alpha = \omega \cdot \beta$, and $(\beta_n)_{n \in \omega}$ has limit $\beta$. For any $K$, $K \leq^t c \text{ ApG}_\alpha$ if and only if there is a computable sequence $(\psi_{\beta_n})_{n \in \omega}$ of $\Sigma^c_{\beta_n}$ sentences in the language of $K$ such that for $A \ncong B \in K$, there is an $n$ such that $\psi_n$ is true in exactly one of $A$ and $B$.

Despite these theorems, not all $K$ embed into ApG (daisy graphs, linear orders, etc.).

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Toward Consistency Result

Definition

A structure $\mathcal{A}$ is \textit{hyperarithmetically saturated} if it satisfies the following conditions

- for all tuples $\bar{a} \in \mathcal{A}$, and all $\Pi^1_1$ sets $\Gamma(\bar{a}, x)$ of computable infinitary formulas with parameters $\bar{a}$, if every $\Delta^1_1$ set $\Gamma'(\bar{a}, x) \subseteq \Gamma(\bar{a}, x)$ is satisfied in $\mathcal{A}$, then $\Gamma(\bar{a}, x)$ is also satisfied in $\mathcal{A}$,

- for all tuples $\bar{a} \in \mathcal{A}$ and all $\Pi^1_1$ sets $\Lambda$ of pairs $(i, \gamma(\bar{a}))$, where $i \in \omega$ and $\gamma(\bar{a})$ is a computable infinitary sentence with parameters $\bar{a}$, if for every $\Delta^1_1$ set $\Lambda' \subseteq \Lambda$,

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\mathcal{A} \models \bigvee \bigwedge_{i \in \omega, (i, \gamma(\bar{a}))} \gamma(\bar{a}),
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