Math 27800 / CS 27800, Winter 2024: Problem Session 1

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Exercise 1. Prove that the following are well-defined operations in **ZFC**. First, write down precisely what this means in each case.

(a) $A \cup B$,

(b) $A \cap B$,

(c) $A \setminus B$.

Solution: We will take note of the axioms we are applying in the following solution. In all, we use extensionality to obtain that the resulting set is uniquely well-defined.

(a) Given two sets A and B, $A \cup B$ is the (unique by extensionality) set such that
$$\forall_x (x \in (A \cup B) \leftrightarrow (x \in A \vee x \in B)). \tag{1}$$
It exists by applying the axiom of union to the set obtained by applying the axiom of pairing to A and B. In other words, $A \cup B := \bigcup\{A, B\}$. Indeed, under this definition, an arbitrary x is in $A \cup B$ iff $\exists_y (y \in \{A, B\} \wedge x \in y)$ iff $\exists_y ((y = A \vee y = B) \wedge x \in y)$, and first-order logic proves that the latter is equivalent to $x \in A \vee x \in B$.

(b) Given A and B, $A \cap B$ is the unique set such that
$$\forall_x (x \in (A \cap B) \leftrightarrow (x \in A \wedge x \in B)). \tag{2}$$
It can be shown to exist by applying the axiom schema of comprehension to create the set $\{x \in A \mid \varphi(x, B)\}$ with $\varphi(x, B) \equiv x \in B$.

(c) Given A and B, $A \setminus B$ is the unique set such that
$$\forall_x (x \in (A \setminus B) \leftrightarrow (x \in A \wedge x \notin B)). \tag{3}$$
It can be shown to exist by applying the axiom schema of comprehension to create the set $\{x \in A \mid \varphi(x, B)\}$ with $\varphi(x, B) \equiv x \notin B$. ■

Exercise 2. Recall Kuratowski's definition of ordered pair. Denote a pair as $\langle x, y\rangle$. Prove in **ZFC** that if $\langle x, y\rangle = \langle x', y'\rangle$ then $x = x'$ and $y = y'$. Take note of the axioms that you need to use to make this definition work.

Solution: Kuratowski defines $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$.

Suppose that $\langle x, y\rangle = \langle x', y'\rangle$. We show $x = x'$ and $y = y'$, but first we make some observations about $\langle x, y\rangle$.

First, note that $x \in z$ for every $z \in \langle x, y\rangle$, and indeed the only set satisfying this property is x itself. From this, we immediately conclude $x = x'$.

Second, note that $y \in z$ for *some* $z \in \langle x, y \rangle$, and only x and y satisfy this property. Thus, we conclude $y = x'$ or $y = y'$. Likewise, $y' = x$ or $y' = y$. From this we conclude that $y = y'$: the only way for this not to directly be the case would have $y = x' = x = y'$ anyway.

The only axiom needed to make this definition work is the axiom of pairing, to ensure that the pair $\langle x, y \rangle$ exists. Otherwise, no axioms were used. ∎

Exercise 3. John's professor erased the definition from the board too quickly for him to write it down, so he had to jot it from memory. Instead of Kuratowski's definition, he wrote down: $\langle x, y \rangle = \{x, y\}$. What is wrong with this definition?

Rose suffered a similar issue, but instead she wrote: $\langle x, y \rangle = \{x, \{x, y\}\}$. Is there anything wrong with this definition?

Dave missed the class entirely, and came up with the following definition on his own: $\langle x, y \rangle = \{\{0, x\}, \{1, y\}\}$. Is there anything wrong with this definition?

Finally, Jade tried to simplify Dave's definition, and defined $\langle x, y \rangle = \{x, \{y\}\}$. What is wrong with this definition?

Bonus question: Can you come up with any interesting alternate definitions of your own?

Solution: John's definition fails to distinguish the pair $\langle x, y \rangle$ from $\langle y, x \rangle$.

Rose's definition works, but, unlike in Kuratowski's definition, we will require the axiom of foundation to do so. We prove now that Rose's definition works.

First, note that x may be recovered from $\langle x, y \rangle_R$ as the $\in$-minimal element of $\langle x, y \rangle_R$ (there must be some by regularity, and x is the only possibility). Then, we may also recover the set $\langle x, y \rangle$ as the non-$\in$-minimal element of $\langle x, y \rangle_R$, and either this is a singleton set, in which case $y = x$ is recovered, or it is a set with two distinct elements, in which case y is the unique element which is not x.

This proof requires foundation in an essential way. To understand why, the reader will have to accept that, without the axiom of foundation, it is consistent that there exists a set x such that $x = \{\{x, 0\}, 1\}$. (The choice of 0 and 1 are irrelevant; any two distinct sets of the reader's choice would suffice.) Then, if $x' = \{x, 0\}$, it is easy to check that $\langle x, 0 \rangle = \langle x', 1 \rangle$, so this is not a good definition of pair in this case!

Dave's definition works, but unlike in Kuratowski's definition, one requires enough axioms of **ZFC** to prove that 0 and 1 exist. It suffices to know that 0 exists (a very reasonable demand...), as in this case 1 exists by pairing, which we will need to use anyway.

To prove that Dave's definition works, we begin by proving a lemma.

Lemma X. If $\{a, b\} = \{a, c\}$ then $b = c$.

Proof of Lemma. In this event, we know that $b \in \{a, c\}$, hence either $b = c$ (in which case we're done) or $b = a$. In the latter case, likewise, we have either $c = b$, and so we're done, or $c = a = b$, and so we're also done. □

Now, suppose that $\{\{0, x\}, \{1, y\}\} = \{\{0, x'\}, \{1, y'\}\}$. If $\{0, x\} = \{0, x'\}$, then three applications of Lemma X immediately give us $x = x'$ and $y = y'$, so let us suppose that this is not the case.

Then, $\{0, x\} = \{1, y'\}$, and so we conclude $1 \in \{0, x\}$. Since $1 \neq 0$, we have $x = 1$. Likewise, $y' = 0$. Now, an application of Lemma X gives us that $\{1, y\} = \{0, x'\}$, and the same argument again will yield $x' = 1 = x$ and $y = 0 = y'$, and the proof that Dave's definition works is complete.

Jade's definition fails to distinguish $\langle \{x\}, y \rangle$ from $\langle \{y\}, x \rangle$. ■

Exercise 4. Given two sets A, B, define the cartesian product $A \times B$ and prove in **ZFC** that it exists.

Solution: The cartesian product $A \times B$ is the (unique by extensionality) set satisfying the condition

$$\forall_x (x \in (A \times B) \leftrightarrow \exists_a \exists_b (a \in A \wedge b \in B \wedge \langle a, b \rangle = x)). \tag{4}$$

To prove that this set exists, we apply the pairing and union axioms to take $A \cup B$, then apply the power set axiom twice, finally followed by the comprehension axiom. Indeed, we construct by comprehension:

$$A \times B = \{\, x \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists_a \exists_b (a \in A \wedge b \in B \wedge \langle a, b \rangle = x) \,\}. \tag{5}$$

The only thing that we need to show is that every pair is in $A \times B$. This is just a matter of noticing that, for $a \in A$ and $b \in B$, $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$ is in $\mathcal{P}(\mathcal{P}(A \cup B))$, and thus:

$$\begin{aligned} x \in \mathcal{P}(\mathcal{P}(A \cup B)) \wedge \exists_a \exists_b (a \in A \wedge b \in B \wedge \langle a, b \rangle = x) \iff \\ \iff \exists_a \exists_b (a \in A \wedge b \in B \wedge \langle a, b \rangle = x). \end{aligned} \tag{6}$$

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