Let $f: X \to Y$ be a morphism of smooth algebraic varieties over a field of characteristic 0. Define $K \subset X \times Y$ by

$$K := \{(x, \xi) \mid x \in X, \xi \in T^*_f(x) Y, \xi(\text{Im } df_x) = 0\}.$$

Let $\pi: X \times Y \to Y$ denote the projection. Then $\pi(K) \subset T^* Y$ is a constructible subset. The proposition below gives an upper and lower bound for $\pi(K)$.

Set $X_r := \{x \in X \mid \text{rank } (dx_f) = r\}$. Let $X_{r, \alpha}, \alpha \in \text{Irr}(X_r)$, be the irreducible components of $X_r$. Set $Y_{r, \alpha} := f(X_{r, \alpha})$. Set

$$\text{Irr}_{\text{ess}}(X_r) := \{x \in \text{Irr}(X_r) \mid \dim X_{r, \alpha} \geq r\}$$

where "ess" stands for "essential".

**Proposition.** (i) $\pi(K)$ is contained in the union of the conormal bundles of the subvarieties $Y_{r, \alpha}, r \in \mathbb{Z}_+, \alpha \in \text{Irr}_{\text{ess}}(X_r)$.

(ii) If $\dim Y_{r, \alpha} = r$ then $\pi(K)$ contains the conormal bundle of $Y_{r, \alpha}$.

**Remarks.** (a) The characteristic 0 assumption implies that $\dim f(X_r) \leq r$, so $\dim Y_{r, \alpha} \leq r$.

(b) The conormal bundle of a singular subvariety is defined to be the closure of the conormal bundle of its smooth locus.

The proposition follows immediately from lemmas 1 and 2 below.
Lemma 1. If \( \dim X_{\Gamma, \omega} < r \) then \( X_{\Gamma, \omega} \times X \) is nowhere dense in \( K \).

Proof. We have

\[
\dim (X_{\Gamma, \omega} \times K) = \dim X_{\Gamma, \omega} + \dim Y - 1 < \dim Y.
\]

On the other hand, \( K \subseteq X \times Y \) is locally defined by \( m \) equations, \( m := \dim X \). So the dimension of each irreducible component of \( K \) is not less than \( \dim (X \times Y) - m = \dim Y \).

Lemma 2. Let \( X'_{\Gamma, \omega} \subset X_{\Gamma, \omega} \) denote the open subset of all \( x \in X_{\Gamma, \omega} \) such that

(a) \( x \) is a nonsingular point of \( X_{\Gamma, \omega} \);
(b) \( f(x) \) is a nonsingular point of \( Y_{\Gamma, \omega} \);
(c) the map \( T_x X_{\Gamma, \omega} \xrightarrow{df_x} T_{f(x)} Y_{\Gamma, \omega} \) is surjective.

Then \( X'_{\Gamma, \omega} \neq \emptyset \) and

\[ \forall x \in X'_{\Gamma, \omega}, \quad \text{Im} (df_x : T_x X \rightarrow T_{f(x)} Y) = T_{f(x)} Y_{\Gamma, \omega}. \]

Moreover, if \( \dim Y_{\Gamma, \omega} = r \) then the inclusion in (\( \star \)) is an equality.

Proof. \( X'_{\Gamma, \omega} \neq \emptyset \) by the characteristic \( D \) assumption. The inclusion (\( \star \)) follows from (c). If \( \dim Y_{\Gamma, \omega} = r \) then the inclusion (\( \star \)) has to be an equality because the l.h.s. of (\( \star \)) has dimension \( r \).

Example \((\text{M. Kashiwara})\). \( X = Y = \mathbb{A}^2 \), \( f(t, x) = (t, t^n x) \), \( n \geq 1 \).

Then \( X_0 = \emptyset \), \( X_1 = \{(t, x)| t = 0\} \), \( X_2 = X \setminus X_1 \). If \( n = 1 \) then \( \pi(K) \) is the union of the zero section and \( T_y^* Y \), where \( y_0 := (0, 0) \in Y \). But if \( n > 1 \) then \( \pi(K) \) is the union of the zero section and a 1-dimensional subspace of \( T_y^* Y \).
Proposition 2. If \( r \geq \dim Y - 1 \) then \( \text{Irr}_{\text{ess}}(X_r) = \text{Irr}(X_r) \).

Proof. If \( r = \dim Y \) then \( X_r \) is open in \( X \), so if \( X_r \neq \emptyset \) then \( \dim X_r = \dim X \geq r \). If \( r = \dim Y - 1 \) then \( X_r \subset X \) is locally defined by \( \dim X - r \) equations, so each irreducible component of \( X_r \) has dimension \( \geq r \). \( \blacksquare \)

In general, it may happen that \( \text{Irr}_{\text{ess}}(X_r) \neq \text{Irr}(X_r) \).

Example. \( X = \mathbb{A}^{n+1}, n \geq 1, Y = \mathbb{A}^3 \), \( f(t, x_1, \ldots, x_n) = (t, \sum_{i=1}^{n} x_i^2, t x_1) \).

Then \( X_1 = \{0\} \), so \( \dim X_1 = 0 \) and \( \text{Irr}_{\text{ess}}(X_1) = \emptyset \).