Introducing K3 Surfaces: Kummer Surfaces

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At this point we've declared victory on the classification theorem and we're moving forward to discuss K3 surfaces. Our goal is to prove the Torelli theorem for K3 surfaces, which says roughly that one may recover a K3 surface up to isomorphism from the Hodge structure on its second cohomology. In this lecture we'll go over some basic facts about K3 surfaces, and indicate the proof of the Torelli theorem for a special class of these- the Kummer surfaces.

(1) Just for kicks, I'd like to point out how a topologist might view the construction of a K3 surface. We begin with a 4-torus:

$$\mathbb{T}^4 = S^1 \times S^1 \times S^1 \times S^1$$

This has an involution ι given by complex conjugation which has $2^4 = 16$ fixed points. We may consider the topological space

$$X = \mathbb{T}^4/\iota$$

Away from the images of the fixed points, this space has a natural structure of a smooth manifold coming from the double-cover of the punctured torus. On the other hand, let B_ϵ be a small ball around a fixed point $x \in \mathbb{T}^4$. Since complex conjugation preserves distance, the involution acts freely on the boundary ∂B_ϵ which is a 3-sphere. There aren't two many such actions, and one checks that the quotient of the boundary is a copy of $\mathbb{R}P^3$.

In other words, the space X is a manifold away from 16 points, and the neighborhoods of each singularity look like cones on $\mathbb{R}P^3$. We can remove these singularities by finding come manifold with boundary $\mathbb{R}P^3$ and replacing these cones with said manifold. It turns out that the disk bundle associated to the cotangent bundle on S^2 does the trick. Gluing these in gives a 4-manifold $\widetilde{X}$ which topologists call 'the' K3 surface (since they are all diffeomorphic in any case.)

Since the Euler class of the tangent bundle of S^2 is 2, the Euler class of the cotangent bundle is -2, so we see that this manifold $\widetilde{X}$ comes with 16 elements in $H_2(X, \mathbb{Z})$ with self-intersection -2. One can play around a lot here, just using topology, and prove many of the results that we are about to list below in a more pedestrian way. For example, you should amuse yourself by computing the Euler characteristic directly from the description above.

Anyway, here is what we will need to know about K3 surfaces:

Theorem 1.1. *Let X be a K3-surface. Then we have the following:*

1. $c_1(X) = 0$, $c_2(X) = 24$, *and* $\mathrm{sign}(X) = -16$
2. $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$
3. $H^2(X, \mathbb{Z})$ *is torsion-free of rank* 22.
4. *The Hodge numbers vanish except for* $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$ *and* $h^{1,1} = 20$.

Proof. By the definition of K3 surface, $h^1(\mathcal{O}_X) = 0$, so $\chi(\mathcal{O}_X) = 2$. We also know that $K = 0$, so Noether's formula becomes

$$2 = \frac{K^2 + c_2}{12}$$

thus $c_2(X) = 24$. Since the cotangent bundle is trivial, so is the tangent bundle, thus $c_1(X) = 0$. The Hirzebruch signature theorem reads:

$$\mathrm{sign}(X) = \frac{p_1}{3}$$

For a complex vector bundle, the first Pontrjagin class is $-2c_2$, which completes the verification of (1).

For (2) it suffices to show that $H_1(X, \mathbb{Z})$ is torsion free (and then use the Hodge decomposition for $H^1(X, \mathbb{C})$). Any n-torsion would appear in the fundamental group, and so correspond to an n-fold covering of X

$$Y \to X$$

In this case, we can take Y to be a variety and so $\chi(\mathcal{O}_Y) = 2 - 2h^1(\mathcal{O}_X)$. On the other hand,

$$\chi(\mathcal{O}_Y) = n \cdot \chi(\mathcal{O}_X) = 2n$$

which forces $n = 1$, so $H_1(X, \mathbb{Z})$ is torsion-free. Now (3) follows from a dimension count since we know the Euler characteristic. Finally, (3) follows from the Hodge symmetries and Serre duality. $\square$

Corollary 1.2. *The intersection pairing on $H^2(X, \mathbb{Z})$ is given by $L = -2E_8 \oplus 3H$, i.e. it may be written in a basis as*

$$-\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \oplus - \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Proof. I don't know a conceptual proof, but this follows from the classification of indefinite unimodular forms: one computes the rank and signature of the above form and sees that it agrees with the data for $H^2(X, \mathbb{Z})$. $\square$

Finally, we will use the following at a crucial point in our proof of the Torelli theorem later on:

Proposition 1.3. *The cup product map*

$$H^1(X, T_X) \to \mathrm{Hom}_{\mathbb{C}}(H^1(X, \Omega_X^1), H^2(X, \mathcal{O}_X))$$

is an isomorphism.

Proof. Serre duality gives a perfect paring

$$H^1(X, T_X) \otimes H^1(X, \Omega_X^2 \otimes \Omega_X^1) \to H^2(X, \Omega_X^2)$$

and in our case $\Omega_X^2 \cong \mathcal{O}_X$, since X is a K3 surface, so we have a perfect pairing

$$H^1(X, T_X) \otimes H^1(X, \Omega_X^1) \to H^2(X, \mathcal{O}_X)$$

By the very definition of perfect pairing, the adjoint map is an isomorphism. One can unwind the construction of Serre duality to show tat this adjoint coincides with the cup product map considered above. $\square$

(2) For the rest of the talk, we'll focus on Kummer surfaces. These arise from the geometric rigidification of the procedure we outlined at the beginning of the talk. Let A be an abelian variety, and consider the involution ι given by $x \mapsto -x$. We have the diagram:

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\sigma"] \arrow[d, "\overline{p}"'] & A \arrow[d, "p"] \\ \mathrm{Km}(A) \arrow[r] & A/\iota \end{tikzcd}$$

where $\widetilde{A}$ is formed by blowing-up the sixteen 2-torsion points. We're interested in proving the following:

Theorem 2.4 (Torelli theorem for Kummer Surfaces)**.** *Let $X = \mathrm{Km}(A)$ be a projective Kummer surface and Y a projective K3 surface. Then any isometry of (polarized) Hodge structures $\phi : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ is induced by an isomorphism $f : Y \to Z$.*

We'll need to make a series of reductions to prove this theorem. First, we'd like to recognize Y as some Kummer surface, and then deduce the result by considering the associated abelian varieties. So we'd like to know how to recognize Kummer surfaces $\mathrm{Km}(A)$.

The exceptional curves in $\widetilde{A}$ project down to (-2)-curves in $\mathrm{Km}(A)$, so any Kummer surface has these 16 distinguished 'nodal' classes (the terminology arises from Kummer surfaces arising as elliptic fibrations). In fact, these classes essentially characterize Kummer surfaces amongst K3 surfaces:

Proposition 2.5. *Let X be a K3 surface with 16 distinct, disjoint (-2)-curves $C_1, ..., C_{16}$ suc that $\mathcal{O}_X\left(\sum C_i\right)$ is divisible by 2 in $\mathrm{Pic}(X)$. Then X has the structure of a Kummer surface with $\{C_1, ..., C_{16}\}$ as the distinguished set of (-2)-curves.*

Proof. Let $\mathcal{L}$ be a line bundle with

$$\mathcal{L}^{\otimes 2} = \mathcal{O}_X\left(\sum C_i\right)$$

We have a projection $p : L \to X$ from the total space of the line bundle, and we may consider $p^*\mathcal{L}$ and $p^*\mathcal{O}_X(\sum C_i)$. The first has a tautological section, call it t, and the second has a section s pulled back from the section defining the divisor. Let $Y \subset L$ be the zeros of the section $s - t^n \in H^0(p^*\mathcal{O}_X(\sum C_i))$. Then $Y \to X$ is a branched double covering. The only exceptional curves come from the pre-images of the C_i, and contracting them gives a surface Z for which one may calculate that $\omega_Z \cong \mathcal{O}_Z$ and $c_2(Z) = 0$. By the classification theorem, Z must be an abelian variety. The involution of switching sheets of Y defines an involution on Z which acts by -1 on $H^1(Z,\mathbb{Q})$ (a diagram chase), and since morphisms of abelian varieties are determined by their action on H^1, we conclude that $X \cong \mathrm{Km}(Z)$. ☐

In particular, the assumptions of the Torelli theorem, above, imply that $Y = \mathrm{Km}(B)$ and, moreover, that ϕ induces a bijection between these distinguished (-2)-curves. We'd like to lift ϕ to a Hodge isometry $H^2(A) \to H^2(B)$, and for that we need to examine more closely the relationship between the Hodge structures of A and $\mathrm{Km}(A)$.

The map $\widetilde{A} \to A \times \mathrm{Km}(A)$ coming from the definition of a Kummer surface is a correspondence between A and $\mathrm{Km}(A)$. It provides an actual map on cohomology:

$$\alpha : \; H^2(A,\mathbb{Z}) \xrightarrow{\overline{p}_! \sigma^*} H^2(\mathrm{Km}(A),\mathbb{Z})$$

Proposition 2.6. *For any $x, y \in H^2(A,\mathbb{Z})$ we have*

$$(\alpha(x), \alpha(y)) = 2(x,y)$$

In particular, α is a monomorphism. Moreover, $\alpha_{\mathbb{C}}$ induces an isomorphism on $H^{2,0}$.

Proof. Exercise in the projection formula. □

Remark 2.7. It turns out that there is also a monomorphism from $H^2(\mathrm{Km}(A))$ into $H^2(T)$ where T is a product of copies of A, but this is harder to construct (the so-called Kuga-Satake variety). In fact, such a monomorphism exists for *any* K3 surface, not just Kummer surfaces, and it is what allows us to prove many theorems about them by using theorems about abelian varieties. For example, Deligne proved the Weil conjectures first for K3 surfaces by using essentially this idea.

Sketch proof of the Torelli theorem. First one shows that the image of α is the orthogonal complement of the span of the exceptional curves. It follows that the Hodge isometry considered in the statement of the theorem induces a (polarised) Hodge isometry:

$$H^2(A, \mathbb{Z}) \to H^2(B, \mathbb{Z})$$

Unfortunately, abelian varieties are characterized by their H^1s, not H^2. So we have to do some linear algebra to get a verifiable condition for when such an isometry arises as the second exterior power of an isometry between the H^1s. It turns out that this is a condition on the map after reduction mod 2. Moreover, we may relate both cohomologies and the map to the $\mathbb{F}_2$-affine space formed by the 2-torsion points of A and B, and a little care yields the result. □

Next time Paul will set up the period map and introduce the Torelli theorem for K3 surfaces.