Curves on an algebraic surface I

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(1) We’re concerned with the geometry of some fixed surface $X$. For the purposes of the next few lectures, by “geometry of a surface” we will mean the set

$$\text{Curves}_X = \{\text{curves } C \text{ in } X\}$$

More accurately we’d like to understand the space of such curves. By that I mean we’d like to understand families of curves parameterized by some scheme $S$. In this lecture we will say what we mean by a curve, and by a family of curves, and we will discuss some functions on this space of all curves. In the next lecture, we’ll actually construct this space.

(2) Whatever a curve means, it had better include one-dimensional subschemes of projective space $\mathbb{P}^2$. Any such subscheme, $C$, is given, set-theoretically, as:

$$C = \{F(X_0, X_1, X_2) = 0\}$$

Where $F$ is a homogeneous polynomial of degree $d$. To describe its structure as a scheme, we do so locally. On an affine open, say $U_0 := \{X_0 \neq 0\} \subset \mathbb{P}^2$ we have canonical coordinate functions $\frac{X_1}{X_0}$ and $\frac{X_2}{X_0}$. The set of zeros of $F$ can then be identified with the set of zeroes of the section $f_0 \in H^0(\mathcal{O}_{U_0})$ where

$$f_0 \left( \frac{X_1}{X_0}, \frac{X_2}{X_0} \right) = F \left( 1, \frac{X_1}{X_0}, \frac{X_2}{X_0} \right)$$

This, then, has a canonical scheme structure. Similarly we can describe $C \cap U_i$ for each of the standard opens in $\mathbb{P}^2$ as the zeros of a function $f_i \in H^0(\mathcal{O}_{U_i})$. Moreover, on the overlaps, say $U_0 \cap U_1 \cong \text{Spec } k[(X_1/X_0)^{\pm 1}, X_2/X_0]$ we have

$$\frac{f_0}{f_1} = F \left( 1, \frac{X_1}{X_0}, \frac{X_2}{X_0} \right) \in H^0(\mathcal{O}_{U_0 \cap U_1})$$

We notice a few things:

1. The scheme $C$ is locally cut out by a single equations.

2. These local equations differ from each other by units. The units then give gluing data for a line bundle, in this case denoted $\mathcal{O}(d)$.

3. The local equations glue to give a section of this line bundle, and the original curve is recovered as the zero set.

\footnote{From now on, $X$ will always be a projective, nonsingular scheme of dimension 2 over an algebraically closed field.}
To formalize the notion of picking local defining equations, define $\mathcal{K}_X$ to be the sheafification of the presheaf on $X$ which assigns, to each open affine Spec($A$), the localization of $A$ at the set of non-zero-divisors. (This is a general definition- in our specific case it’s much simpler, this sheaf will always return the function field of the scheme $X$.) This is a sheaf of rings, and has an associated sheaf of units $\mathcal{K}_X^*$. So we have an exact sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{K}^* \longrightarrow \mathcal{K}^*/\mathcal{O}^* \longrightarrow 0$$

Unpacking the definition of a quotient sheaf, we find that a section of $\mathcal{K}^*/\mathcal{O}^*$ is given by specifying an open cover $U_i$ of $X$, together with sections $f_i \in H^0(\mathcal{K}_{U_i}^*)$, called local defining functions, such that $f_i/f_j \in H^0(\mathcal{O}_{U_i \cap U_j})$.

**Definition 2.1.** A Cartier divisor $D$ on $X$ is a section of $\mathcal{K}^*/\mathcal{O}^*$. A Cartier divisor is effective if we can choose local defining functions $f_i$ such that $f_i \in H^0(\mathcal{O}_{U_i})$. The set of divisors forms a group which we denote additively.

As expected from our example, we have a map $\text{Div}(X) \rightarrow \{\text{line bundles on } X\} =: \text{Pic}(X)$.

There are two ways to see this. On the one hand, the functions $f_i/f_j \in H^0(\mathcal{O}_{U_i \cap U_j})$ satisfy the cocycle condition and so give gluing data for a line bundle, $\mathcal{O}(D)$. On the other hand, the exact sequence gives rise to a coboundary map

$$H^0(\mathcal{K}^*/\mathcal{O}^*) \rightarrow H^1(\mathcal{O}^*)$$

These are the same under the identification of $H^1(\mathcal{O}^*)$ with $\text{Pic}(X)$. From now own by curve on a surface we mean an effective Cartier divisor.

We saw that any curve in projective space is the zero set of a section of a line bundle. Similarly, given an effective Cartier divisor, $D$, we have a canonical section of $\mathcal{O}(D)$ given by the local defining functions. The dual bundle $\mathcal{O}(-D)$ is then an ideal sheaf in $\mathcal{O}_X$ whose corresponding closed subscheme is the scheme of zeros of this section. This gives

**Proposition 2.2.** There is a natural bijection

$$\{\text{effective Cartier divisors}\} \longleftrightarrow \{(L, s) \text{ where } L \text{ is a line bundle and } s \in H^0(L)\}/\sim$$

(3) At this point is clear that if we care about curves in $X$, we also care about line bundles on $X$. Let’s look at a few examples.

**Example 3.3.** $\text{Pic}(\mathbb{P}^2) = \mathbb{Z}$. 

**Example 3.4.** $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}^2$. 

**Example 3.5.** $\text{Pic}(\mathbb{P}^1 \times C) \cong \mathbb{Z} \oplus \text{Pic}(C)$, $C$ a curve.

**Example 3.6.** $\text{Pic}(C \times C) \neq \text{Pic}(C) \times \text{Pic}(C)$ if $g_C \geq 1$. So the Künneth theorem does not hold for Pic.

**Example 3.7.** $\text{Pic}(\mathbb{P}(\mathcal{E})) \cong \mathbb{Z} \oplus \text{Pic}(C)$ where $\mathcal{E}$ is a rank 2 vector bundle over a curve, $C$. So the projective bundle formula holds.

**Example 3.8.** $\text{Pic}(X) = \mathbb{Z}$ if $X$ is a double cover of $\mathbb{P}^2$ branched over a generic sextic.

(4) So far, we have only one invariant for curves, namely its associated line bundle. We can thus break up the problem of describing the space of all curves in $X$ into two steps:

1. Describe the space of curves that are mapped to the same line bundle. Such curves are called linearly equivalent, and, more generally, two divisors $D$ and $D'$ are linearly equivalent if $\mathcal{O}(D) \cong \mathcal{O}(D')$.
2. Describe the space of line bundles on $X$.

The first problem is much easier. To make sense of it, we need to define what we mean by ‘the space of curves mapped to the same line bundle.’ In particular, we will need to make sense of families of curves, line bundles, etc. parameterized by schemes $S$.

**Definition 4.9.** Let $S$ be a scheme. Then an effective divisor $D \subset X \times S$ is called a relative effective Cartier divisor if it is flat over $S$. We will also refer to this as a family of curves parameterized by $S$. We denote by $\text{Curves}_X$ the functor $\text{Sch}_{/k} \rightarrow \text{Sets}$ defined by $S \mapsto \{\text{relative effective Cartier divisors over } S\}$

**Definition 4.10.** Let $S$ be a scheme. Then a family of line bundles on $X$ is an element of $\text{Pic}(X \times S)/\text{Pic}(S)$. We denote by $\text{Pic}_X$ the functor $S \mapsto \text{Pic}(X \times S)/\text{Pic}(S)$

**Definition 4.11.** Let $\mathcal{L}$ be a line bundle on $X$, and let $S$ be a scheme. A linear system of curves on $X$ (parameterized by $S$) is a relative effective Cartier divisor $D \subset X \times S$ such that

$$O_{X \times S}(D) \cong p_1^* \mathcal{L} \otimes p_2^* \mathcal{M}$$

where $\mathcal{M}$ is a line bundle on $S$. We denote by $\text{LinSys}_{X,\mathcal{L}}$ the functor $\text{Sch}_{/k} \rightarrow \text{Sets}$ defined by $S \mapsto \{\text{linear systems of curves on } X\}$

**Lemma 4.12.** The functor $\text{LinSys}_{X,\mathcal{L}}$ is the fiber over $\mathcal{L}$ of the map $\text{Curves}_X \rightarrow \text{Pic}_X$.

That is, if we let $p : S \rightarrow \text{Spec}(k)$ denote the structure morphism, $\text{LinSys}_{X,\mathcal{L}}$ is isomorphic to the functor $S \mapsto \{\text{families of curves } D \subset X \times S \text{ such that } O(D) \cong \mathcal{L} \bmod \text{Pic}(S)\}$

**Proposition 4.13.** There is a natural isomorphism $\text{LinSys}_{X,\mathcal{L}} \cong \mathbb{P}(H^0(\mathcal{L})^\vee)$ (where the right hand side denotes the functor represented by the scheme.)

**Proof.** Let $V = H^0(\mathcal{L})^\vee$

$$\alpha : \text{LinSys}_{X,\mathcal{L}} \rightarrow \mathbb{P}(V)$$

$$\beta : \mathbb{P}(V) \rightarrow \text{LinSys}_{X,\mathcal{L}}$$

that are inverse. The second map is just an element of $\text{LinSys}_{X,\mathcal{L}}(\mathbb{P}(V))$, i.e. a linear system parameterized by $\mathbb{P}(V)$. The one we choose will be the ‘tautological one’, that is, start with the line bundle $p_1^* \mathcal{L} \otimes p_2^* O(1)$. Sections, by the Künneth formula and the definition of $O(1)$, correspond to elements of $H^0(\mathcal{L}) \otimes V = H^0(\mathcal{L}) \otimes H^0(\mathcal{L})^\vee \cong k$

so we choose the section corresponding to $1 \in k$. This defines $\beta$. To define $\alpha$, recall that $\mathbb{P}(V)$ represents the functor which assigns to a scheme $S$ the set of line bundles $\mathcal{M}$ on $S$ together with a surjective map $V \rightarrow H^0(\mathcal{M})$. So let $\alpha$ take a linear system parameterized by $S$, $D$, to the line bundle $\mathcal{M}$ on $S$ such that

$$O_{X \times S}(D) \cong p_1^* \mathcal{L} \otimes p_2^* \mathcal{M}$$

together with the map $V \rightarrow H^0(\mathcal{M})$ corresponding to the given section of $H^0(O(D))$ (again using the Künneth formula.) One checks immediately that $\alpha$ and $\beta$ are inverse to each other. □
For this section see Hartshorne V.1 for any omitted proofs. We have almost completed the first step in our analysis of Curves $X$. Namely, we have shown that the fibers over a line bundle in $\text{Pic}_X$ look like projective spaces. However, we would like to be able to compute which projective spaces appear. In other words, we need to know the dimension of $H^0(L)$. This is known as the Riemann-Roch problem. In order to solve this problem, we will need a new way of extracting numbers from line bundles and divisors.

The following result is a standard consequence of the ‘moving lemma’:

**Proposition 5.14.** There is a unique symmetric, bilinear pairing (the intersection product) $\text{Div}(X) \times \text{Div}(X) \to \mathbb{Z}$ such that

1. If $C$ and $D$ are nonsingular curves intersecting transversally, then $C.D = \#(C \cap D)$.
2. The pairing is invariant under linear equivalence, so that it descends to a pairing $\text{Pic}(X) \times \text{Pic}(X) \to \mathbb{Z}$

There are various ways of computing the intersection product:

**Lemma 5.15.** Let $C$ be an irreducible nonsingular curve on $X$, and let $D$ be any curve meeting $C$ transversally. Then

$$C.D = \deg_C(\mathcal{O}(D) \otimes \mathcal{O}_C)$$

**Lemma 5.16.** If $C$ and $D$ are curves on $X$ having no common irreducible component, then

$$C.D = \sum_{P \in C \cap D} \dim_k(\mathcal{O}_{X,P}/(f_P, g_P))$$

where $f_P$ and $g_P$ denote local equations for $C$ and $D$ and $P$.

**Lemma 5.17.** Let $L$ and $M$ denote line bundles on $X$. Then

$$L.M = \chi(\mathcal{O}_X) - \chi(L^{-1}) - \chi(M^{-1}) + \chi(L^{-1} \otimes M^{-1})$$

The result we’re interested in, which helps compute $H^0(L)$, is:

**Theorem 5.18** (Riemann-Roch for surfaces). Let $D$ be a divisor on $X$ and denote by $K$ the canonical divisor on $X$. Then

$$\chi(\mathcal{O}(D)) = \frac{1}{2} D.(D - K) + \chi(\mathcal{O}_X)$$

**Proof.** We may write $D = C - E$ where $C$ and $E$ are nonsingular curves (you can find the argument in Hartshorne V.1.1). So we are reduced to proving

$$\chi(\mathcal{O}(C - E)) = \frac{1}{2}(C - E)(C - E - K) + \chi(\mathcal{O}_X)$$

We’d like to compute the left hand side by using the Riemann-Roch theorem for curves. Using the exact sequences

$$0 \to \mathcal{O}(C - E) \to \mathcal{O}(C) \to \mathcal{O}(C) \to 0$$

and

$$0 \to \mathcal{O}_X \to \mathcal{O}(C) \to \mathcal{O}(C) \otimes \mathcal{O}_C \to 0$$

we get

$$\chi(\mathcal{O}(C - E)) = \chi(\mathcal{O}(C)) - \chi(\mathcal{O}(C) \otimes D)$$

$$= \chi(\mathcal{O}(X)) + \chi(\mathcal{O}(C) \otimes C) - \chi(\mathcal{O}(C) \otimes D)$$
This we can compute using Riemann-Roch for curves and our lemma for computing intersection numbers:

\[
\chi(\mathcal{O}(C) \otimes \mathcal{O}_C) = C^2 + 1 + g_C \\
\chi(\mathcal{O}(C) \otimes \mathcal{O}_E) = C.E + 1 + g_E
\]

Putting all of this together, we see that we are reduced to proving the following lemma.

**Lemma 5.19** (Adjunction formula). Let \( C \subset X \) be a nonsingular curve. Denote by \( K \) the canonical divisor of \( X \). Then one can compute the genus as

\[
g = \frac{1}{2} (C.(C + K)) + 1
\]

**Proof.** Let’s rewrite this as

\[
\deg_C(\omega_C) = 2g - 2 = \deg_C(\mathcal{O}(C) \otimes \omega_X \otimes \mathcal{O}_C)
\]

Thus, the theorem follows from an identification:

\[
\omega_C \cong \mathcal{O}(C) \otimes \omega_X \otimes \mathcal{O}_C
\]

Consider the exact sequence

\[
0 \rightarrow \mathcal{O}(-C)/\mathcal{O}(-C)^2 \rightarrow \Omega_X \rightarrow \Omega_C \rightarrow 0
\]

Using the fact that \( \mathcal{O}(-C)/\mathcal{O}(-C)^2 \cong \mathcal{O}(-C) \otimes \mathcal{O}_C \), the result follows by taking top exterior powers of the exact sequence. \( \square \)

**Remark 5.20.** In the complex analytic setting, there is a topological justification for the adjunction formula

\[
2g - 2 = C.(C + K) = C^2 + C.K
\]

Pick a generic section \( \eta \in H^0(X, \Omega_X^2) \) and a generic normal vector field \( V \) on \( C \). The 2-form \( \eta \) defines a pairing

\[
TC \otimes N_{C/X} \rightarrow \mathbb{C}
\]

between the tangent bundle and the normal bundle of \( C \), which gives a map \( H^0(N_{C/X}) \rightarrow H^0(\Omega_C) \). Thus a generic choice of \( \eta \) and \( V \) give a generic choice \( \eta(V) \in H^0(\Omega_C) \). The Euler characteristic \( \chi(C) = 2 - 2g \) is the number of zeros of a generic vector field on \( C \), so \( -\chi(C) \) is the number of zeros of \( \eta(V) \). On the other hand, this vanishes on \( C \) precisely when, either, \( V \) vanishes on \( C \) or \( \eta \) vanishes on \( C \). These are the intersection numbers on the right hand side.