The space of curves on a surface
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1 Spaces of divisors

Recall that we’re trying to describe the spaces of curves and of line bundles on a surface $X$. To be specific, we have functors

$$\text{Curves}_X(S) = \{\text{relative effective Cartier divisors on } X \times S\}$$

and

$$\text{Pic}_X(S) = \{\text{line bundles on } X \times S, \text{ mod line bundles pulled back from } S\},$$

and a natural transformation

$$\Phi : \text{Curves}_X(S) \to \text{Pic}_X(S)$$

sending a divisor $D$ to the bundle $\mathcal{O}(D)$ of rational functions with poles along $D$. In the last meeting, Dylan described the fiber of this map over a line bundle $L$: it’s the space of linear systems,

$$\text{LinSys}_{X,L} \cong \mathbb{P}(H^0(L)^\vee).$$

The goal today is to show that the functors $\text{Curves}_X$ and $\text{Pic}_X$ are, in fact, projective schemes, and hopefully to calculate their dimensions.

It’s helpful to introduce some terminology for relationships between divisors.

**Definition 1.** Two effective divisors $D, D'$ on a surface $X$ are:

- **linearly equivalent** if $D - D'$ is the divisor of a rational function; equivalently, if $\mathcal{O}(D) \cong \mathcal{O}(D')$; equivalently, if $D$ and $D'$ are fibers of a relative divisor on $X \times \mathbb{P}^1$;

- **algebraically equivalent** if $D$ and $D'$ are fibers of a relative divisor on $X \times S$, for some connected scheme $S$;

- **numerically equivalent** if $D \cdot E = D' \cdot E$ for all divisors $E$.

These definitions extend to all divisors by linearity.

The first two relations extend to more general schemes; the third does too once we develop a more general intersection theory (divisors are generally codimension 1 things, and we need to pair them with dimension 1 things). We have implications

$$\text{linear equivalence} \Rightarrow \text{algebraic equivalence} \Rightarrow \text{numerical equivalence}.$$ 

The first of these is obvious. For the second, recall that a relative effective divisor $D \subseteq X \times S$, for $S$ connected, is flat over $S$, and thus the fibers $D_s$ have constant Hilbert polynomial. By yesterday’s results, the intersection number $D_s \cdot E$ only depends on the Euler characteristics of $D_s$, $E$, $\mathcal{O}_X$, and $D_s + E$, so it too is constant for a fixed $E$.

The point of this is that $\text{Curves}_X$ and $\text{Pic}_X$ break up into subfunctors $\text{Curves}_X^\xi$ and $\text{Pic}_X^\xi$, where $\xi$ is a numerical equivalence class. If $S$ is a connected scheme, then $\text{Curves}_X^\xi(S)$ is the subset of divisors in $\text{Curves}_X(S)$ of numerical equivalence class $\xi$, and for disconnected schemes, we can take the product over their connected components. Obviously, these $\text{Curves}_X^\xi$ are going to turn out to be open and closed subschemes of the scheme $\text{Curves}_X$, and it suffices to represent each one of them.
For the Curves functor, we’ll do something even coarser, breaking it up into subfunctors $\text{Curves}_X^P$, where $P$ is the Hilbert polynomial of the sheaf $\mathcal{O}_D$. As I said above, the numerical equivalence class of $D$ only depends on $P_{\mathcal{O}_D}$, so $\text{Curves}_X^P$ is a union of functors of the form $\text{Curves}_X^S$.

The Picard functor has the added bonus of being a group; thus, all $\text{Pic}_X^\xi$ are isomorphic, and it suffices to represent just one of them, for a ‘sufficiently nice $\xi$’. For instance, we can choose to only deal with very ample line bundles with no higher cohomology.

## 2 Some useful theorems

There are two sets of theorems that will come in handy. One of them falls under the rubric of ‘theorems on cohomology and base change.’ Consider a pullback square

$$
\begin{array}{ccc}
X \times_S Z & \xrightarrow{h} & X \\
\downarrow q & & \downarrow p \\
Z & \xrightarrow{g} & Y.
\end{array}
$$

If $\mathcal{F}$ is a sheaf on $X$, there is always a map $g^*R^i p_*(\mathcal{F}) \rightarrow R^i q_*(h^*\mathcal{F})$.

**Theorem 2.** Suppose that $\mathcal{F}$ is flat over $S$, $s \in S$, and that one of the following two hypotheses holds.

- $R^i p_*(\mathcal{F}) \otimes k(s) \rightarrow H^i(X_s, \mathcal{F}_s)$ is surjective.
- $H^{i+1}(X_s, \mathcal{F}_s) = 0$.

Then the map $g^*R^i p_*(\mathcal{F}) \rightarrow R^i q_*(h^*\mathcal{F})$ is an isomorphism after pulling back to some open neighborhood $U$ of $s$. In particular, $R^i p_*(\mathcal{F}) \otimes k(t) \rightarrow H^i(X_t, \mathcal{F}_t)$ for all $t \in U$.

For example, if $H^1(X_s, \mathcal{F}_s) = 0$, then $\mathcal{F}$ is free of rank $h^0(\mathcal{F}_s)$ in a neighborhood of $s$.

The second set of theorems has to do with twisting sheaves on projective schemes by $\mathcal{O}(1)$. The idea is that these sheaves become cohomologically ‘nice’ after sufficient twisting. The simplest example is the following theorem of Serre.

**Theorem 3.** Suppose that $\mathcal{F}$ is a coherent sheaf on a projective scheme $X$ with very ample line bundle $\mathcal{O}(1)$. Then there is an integer $n_0$, depending on $\mathcal{F}$, such that for $n \geq n_0$ and $i > 0$, $H^i(X, \mathcal{F}(n)) = 0$, and $\mathcal{F}(n)$ is generated by global sections.

We’d like this $n_0$ to only depend on the Hilbert polynomial; we’d also like tighter vanishing conditions. Mumford defines:

**Definition 4.** A coherent sheaf $\mathcal{F}$ on a projective scheme $\mathcal{O}_X$ is $m$-regular if $H^i(\mathbb{P}^n, \mathcal{F}(m - i)) = 0$ for all $i > 0$.

**Proposition 5** (Castelnuovo). If $\mathcal{F}$ is $m$-regular, then

- $H^i(\mathbb{P}^n, \mathcal{F}(k)) = 0$ for $i > 0$ and $k + i \geq m$, and
- $H^0(\mathbb{P}^n, \mathcal{F}(k))$ is generated by $H^0(\mathbb{P}^n, \mathcal{F}(k-1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1))$ for $k > m$ (and thus by $H^0(\mathbb{P}^n, \mathcal{F}(m)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k - m))$).

If $\mathcal{F}$ is $m$-regular, then we can twist $\mathcal{F}$ up enough so that Serre’s theorem applies, and then use the second part of the proposition to show that $\mathcal{F}(k)$ is generated by global sections for $k \geq m$.

**Theorem 6.** There is a polynomial $F(x_0, \ldots, x_n)$ depending on $X$, such that if $\mathcal{F}$ is a coherent sheaf of ideals on $X$ with Hilbert polynomial

$$
P_\mathcal{F}(m) = \chi(\mathcal{F}(m)) = \sum_{i=0}^n a_i \binom{m}{i},
$$

then $\mathcal{F}$ is $F(a_0, \ldots, a_n)$-regular.
We now specialize to the case of surfaces.

**Theorem 7.** Suppose $X$ is a projective surface and that $\mathcal{L}$ satisfies one of the following properties:

- $\mathcal{L}$ is very ample,
- $\mathcal{L}$ is 0-regular,
- $\mathcal{L}$ is generated by global sections,
- $\mathcal{L} \cong \mathcal{O}(D)$ where $D$ has no multiple components.

Then $\mathcal{L}(n)$ satisfies the same properties as $\mathcal{L}$ for $n \geq 0$, and $\mathcal{L}(n)$ satisfies all four properties for $n \gg 0$. In fact, any $\mathcal{L}$ with sufficiently large Euler characteristic and degree, depending only on $X$ and $\mathcal{O}(1)$, satisfies all four conditions.

3 Representing the Curves functor

In fact, we’re going to represent Curves$_X^P$, where $P$ is a fixed Hilbert polynomial. For $D$ an effective divisor on some $X$ with Hilbert polynomial $P$, $\mathcal{O}(-D)$ is a coherent sheaf of ideals, and so we can apply the $m$-regularity theorem to find an $m$, depending only on $P$, such that $\mathcal{O}_X(-D)$ is $m$-regular. As a result,

$$H^1(\mathcal{O}_X(-D + m)) = H^2(\mathcal{O}_X(-D + m)) = 0,$$

and $\mathcal{O}_X(-D + m)$ is generated by global sections. In fact, as $\mathcal{O}_X$ is also a coherent sheaf of ideals, we might as well take $m$ large enough that $\mathcal{O}_X$ is also $m$-regular. In this case, using the exact sequence

$$0 \to \mathcal{O}_X(-D + m) \to \mathcal{O}_X(m) \to \mathcal{O}_D(m) \to 0,$$

we find that $H^1(\mathcal{O}_D(m)) = 0$.

Now consider $D \subseteq X \times S$, a relative effective Cartier divisor with Hilbert polynomial $P$ on each fiber. We’re now in a position to use the theorem on flat base change, to the diagram

$$X \times \{s\} \longrightarrow X \times S \longrightarrow S.$$

By the above, $R^1p_{2, *} \mathcal{O}_D(m) = 0$, so $p_{2, *} \mathcal{O}_D(m)$ is locally free of rank $h^0(\mathcal{O}_D(m))$. By the exact sequence above, and the 0-regularity of $\mathcal{O}_X(-D + m)$ and $\mathcal{O}_X(m)$, this number is

$$r = h^0(\mathcal{O}_D(m)) = h^0(\mathcal{O}_X(m)) - h^0(\mathcal{O}_X(-D + m + 1)) = P_{\mathcal{O}_X}(m) - P(m).$$

Not only is this independent of the fiber, it’s also independent of $S$.

Likewise, since $R^2p_{2, *} \mathcal{O}_{X \times S}(-D + m) = 0$, we get $R^1p_{2, *} \mathcal{O}_{X \times S}(-D + m) \cong H^1(\mathcal{O}_{X \times S}(-D + m)) = 0$. Thus, we can push forward the exact sequence

$$0 \to \mathcal{O}_{X \times S}(-D + m) \to \mathcal{O}_{X \times S}(m) \to \mathcal{O}_D(m) \to 0$$

to get

$$0 \to p_{2, *} \mathcal{O}_{X \times S}(-D + m) \to p_{2, *} \mathcal{O}_{X \times S}(m) \to p_{2, *} \mathcal{O}_D(m) \to 0.$$

By the projection formula, the middle term is isomorphic to $\mathcal{O}_S \otimes_k H^0(\mathcal{O}_X(m))$, a free sheaf of rank $N + 1 = h^0(\mathcal{O}_X(m))$. Summing up, we’ve produced from $D \subseteq X \times S$ a locally free sheaf $p_{2, *} \mathcal{O}_D(m)$ on $S$ of rank $r$, with a spanning set of $N + 1$ global sections, where both $N$ and $r$ are independent of $S$. That is, we have produced a map $S \to \text{Gr}_{N,r}$, where $\text{Gr}_{N,r}$ is the Grassmannian parametrizing locally free sheaves with a spanning set of $N + 1$ global sections.
On the other hand, \( \text{Gr}_{N,r} \) naturally maps to the Hilbert scheme. A map \( S \to \text{Gr}_{N,r} \) corresponds to an exact sequence

\[
0 \to \mathcal{K} \to \mathcal{O}_S \otimes_k H^0(\mathcal{O}_X(m)) \to \mathcal{E} \to 0,
\]

where \( \mathcal{E} \) is locally free of rank \( r \). Pulling back to \( X \times S \) gives \( p_2^* \mathcal{K} \to p_2^* p_{2,*} \mathcal{O}_{X \times S}(m) \to \mathcal{O}_{X \times S}(m) \), so that \( p_2^* \mathcal{K}(-m) \) maps to a sheaf of ideals in \( \mathcal{O}_{X \times S} \), and thus to a closed subscheme of \( X \times S \). If \( S \to \text{Gr}_{N,r} \) originally came from a curve, the closed subscheme we get is just that curve.

We can now stratify \( \text{Gr}_{N,r} \) by locally closed subschemes, according to the Hilbert polynomial of the closed subscheme of \( X \times S \) induced. Some subscheme \( Y \) corresponds to the Hilbert polynomial \( P \). The condition that \( S \to Y \) induces a relative effective Cartier divisor is shown to be open. Thus, Curves\( ^r_X \) is represented by a locally closed subscheme of \( \text{Gr}_{N,r} \). Though I won’t prove this, this scheme is in fact closed in \( \text{Gr}_{N,r} \), and thus projective.

4 Representing the Picard functor

We’ll now represent \( \text{Pic}^\xi_X \), where \( \xi \) is a numerical equivalence class. By the last of the useful theorems above, we can take \( \xi \) to be very ample and 0-regular (the condition in the theorem depends only on the Hilbert polynomial and degree of a line bundle, and in particular, on its numerical equivalence class.)

We already know that Curves\( ^\xi_X \) is a scheme, and that there’s a map \( \Phi : \text{Curves}^\xi_X \to \text{Pic}^\xi_X \). Suppose that \( \Phi \) has a section \( s \). Then \( \text{Pic}^\xi_X \) is the pullback

\[
\begin{array}{ccc}
\text{Pic}^\xi_X & \to & \text{Curves}^\xi_X \\
\downarrow & & \downarrow (1,s\Phi) \\
\text{Curves}^\xi_X & \to & \text{Curves}^\xi_X \times \text{Curves}^\xi_X,
\end{array}
\]

and in particular, a scheme (even a closed subscheme of \( \text{Curves}^\xi_X \)).

It remains to construct this section. In other words, given \( \mathcal{L} \) over \( X \times S \) of numerical equivalence class \( \xi \), we must find a relative effective Cartier divisor \( D \) over \( X \times S \) with \( \mathcal{O}_{X \times S}(D) = \mathcal{L} \otimes p_2^* \mathcal{M} \), for some \( \mathcal{M} \in \text{Pic}(S) \). Moreover, \( D \) should be natural in \( S \), and independent of twists by line bundles \( p_2^* \mathcal{N} \) pulled back from \( S \).

The basic construction is as follows. We’ve chosen \( \mathcal{L} \) to be very ample and 0-regular, so it has no higher cohomology. By the base change theorem, \( \mathcal{F} = p_{2,*} \mathcal{L} \) is locally free on \( S \) of rank \( h^0(\mathcal{L}_s) = r \), a number depending only on \( \xi \).

Pick \( r-1 \) closed points \( x_1, \ldots, x_{r-1} \) of \( X \), and let \( \mathcal{M}_i \) be the pullback of \( \mathcal{L} \) to \( \{x_i\} \times S \). There’s a map \( \mathcal{F} \to \mathcal{M}_i \): a section of \( \mathcal{F}(U) \) is a section of \( \mathcal{L}(X \times U) \), which restricts to a section of \( \mathcal{M}_i(U) \). This induces a map

\[
\mathcal{F} \to \bigoplus_{i=1}^{r-1} \mathcal{M}_i,
\]

and thus

\[
\bigwedge_{i=1}^{r-1} \mathcal{F} \to \bigotimes \mathcal{M}_i.
\]

Dualizing gives

\[
\mathcal{O}_S \to \left( \bigwedge_{i=1}^{r-1} \mathcal{F} \right)^{\vee} \otimes \bigotimes \mathcal{M}_i,
\]

and via the perfect pairing \( \mathcal{F} \otimes \bigwedge_{i=1}^{r-1} \mathcal{F} \to \bigwedge \mathcal{F} \),

\[
\mathcal{O}_S \to \mathcal{F} \otimes \left( \bigwedge_{i=1}^{r} \mathcal{F} \right)^{\vee} \otimes \bigotimes \mathcal{M}_i.
\]
Pulling back to $X \times S$ induces
\[ \mathcal{O}_{X \times S} \to p_2^*(\mathcal{F} \otimes \left( \bigwedge^r \mathcal{F} \right) \otimes \bigotimes \mathcal{M}_i) \cong \mathcal{L} \otimes p_2^*(\left( \bigwedge^r \mathcal{F} \right) \otimes \bigotimes \mathcal{M}_i), \]
the final isomorphism by the projection formula. This is a global section $\sigma$ of a locally free sheaf on $X \times S$. It is clearly natural in $S$, and by counting tensor exponents, one sees that it is invariant under twisting by sheaves $p_2^*(\mathcal{M})$. To check that $\sigma$ defines a relative effective Cartier divisor, it remains to show that $\sigma$ is not identically 0 on any fiber over $s \in S$. Since $\mathcal{L}_s$ is very ample, it defines $\phi_s(X \times \mathbb{P}^{r-1})$, and $\sigma$ is not identically 0 over $s$ iff $\phi_s(x_1), \ldots, \phi_s(x_{r-1})$ are in general position in $\mathbb{P}^{r-1}$.

In fact, Mumford doesn’t quite show this, but instead shows that you can get general position after picking a bunch more points and tensoring by a bunch more pullbacks of $\mathcal{L}$. I won’t go into this any further – morally, we’ve seen how to build the desired section of $\Phi$, and why this means that $\text{Pic}_X$ is representable.

5 Smoothness

Now that we’ve proved that $\text{Pic}_X$ and $\text{Curves}_X$ are schemes, and identified the fiber of $\Phi : \text{Curves}_X \to \text{Pic}_X$, it remains to calculate their dimensions. (Note that the dimension of $\text{Curves}_X$ will vary from component to component, depending on $h^0(\mathcal{L})$, while $\text{Pic}_X$, being a group scheme, is equidimensional.) This will take two steps: show that these schemes are smooth, and calculate their tangent spaces. I’ll do this in characteristic 0, and next meeting, Dylan will do it in general characteristic.

For smoothness, note that $\text{Pic}_X^0$ is a group scheme, and thus always smooth in characteristic 0. (This fails in characteristic $p - \mu_p$, for example, is not smooth.)

Now consider the Curves functor. We’ve seen that the fibers of $\Phi$ are projective spaces $\mathbb{P}(H^0(\mathcal{L}))$; clearly, to show that Curves is smooth, it suffices to show that it’s a projective space bundle over Pic, i.e. locally free over small open subsets of Pic. In other words, we want to show that $H^0(\mathcal{L})$ varies continuously over points of Pic. When $H^1(\mathcal{L}) = 0$, this follows from the flat base change theorem, applied to the projection map $X \times \text{Pic}_X \to \text{Pic}_X$. That is, over an open set $U \subseteq \text{Pic}_X$ in which $H^1(\mathcal{L}_z) = 0$ for all $\mathcal{L}_z \in U$ (these exist again by the flat base change theorem), the preimage of $\Phi$ is the projective space bundle $\mathbb{P}(p_2^*(\mathcal{L}))$.

6 Calculating the dimension

It’s standard that we can identify $\text{Pic}_X(k)$ with $H^1(X, \mathcal{O}_X^\times)$: a line bundle on $X$ is a locally free sheaf of rank 1, whose transition functions are thus in $\mathcal{O}_X^\times$, and one checks that these functions form a 1-cocycle, with cohomologous cocycles corresponding to isomorphic line bundles. We need to identify the identity component of this space, which is the group $\text{Pic}_X^0$ of line bundles algebraically equivalent to 0 mod linear equivalence. Over $k = \mathbb{C}$, we can do this analytically via the exponential exact sequence
\[ 0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0. \]

The induced map $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ sends a line bundle to its first Chern class, and the kernel of this map is precisely the space of line bundles algebraically equivalent to 0. Thus we get $\text{Pic}_X^0 = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$. As $H^1(X, \mathbb{Z})$ is finitely generated, we get $\dim \text{Pic}_X = h^1(\mathcal{O}_X)$ in this case. This was proved by Poincaré in 1909. Unfortunately, the general conjecture, that $\dim \text{Pic}_X$ is always $h^1(\mathcal{O}_X)$, is false in positive characteristic, as shown by Igusa and Serre; Dylan will talk about this next time.

More generally, though still in characteristic 0, we can use smoothness and just compute the dimension of the tangent space, $\text{Pic}_X^0(\mathbb{C}[\epsilon]/\epsilon^2)$ – that is, the space of deformations of line bundles. Write $T = \text{Spec} k[\epsilon]/\epsilon^2$. We have an exact sequence of sheaves of groups on $X$
\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X^\times \to \mathcal{O}_X^\times \to 0, \]
where the first map sends $f$ to $1 + \epsilon f$. Moreover, this sequence splits, because a unit of $\mathcal{O}_X(U)$ is also a unit of $\mathcal{O}_{X \times T}(U)$. Thus, the long exact sequence of cohomology groups breaks up into split short exact
sequences. In particular, we have

\[ 0 \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_{X \times T}) \to H^1(\mathcal{O}_{X}^+) \to 0. \]

The last group is \( \text{Pic}_X(k) \), and the middle group is \( \text{Pic}_X(T) \). Thus, the kernel \( H^1(\mathcal{O}_X) \) is none other than the space of tangent vectors at the identity element, which proves the claim.

This is closely connected to another cohomology computation we can make for Curves. For a sufficiently good divisor \( D \), i.e. one with no multiple components and corresponding to a line bundle with no higher cohomology, we have an exact sequence

\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{N}_D \to 0, \]

where \( \mathcal{N}_D \) is the normal bundle \( \mathcal{O}_D(D) \). We get an exact sequence on cohomology

\[ 0 \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{N}_D) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X(D)). \]

The last group is 0 by hypothesis on the goodness of \( D \). The first group is \( k \) since \( X \) is projective. Thus we really have

\[ 0 \to H^0(\mathcal{O}_X(D))/k \to H^0(\mathcal{N}_D) \to H^1(\mathcal{O}_X) \to 0. \]

Now a global section of \( \mathcal{O}_X(D) \) defines a first-order linear deformation of \( D \), i.e. a point of \( \text{LinSys}_{X,D}(T) \) – indeed, this section \( f \) is just a rational function on \( X \) with the property that \( (f) + D \) is an effective divisor. As we’ve seen, \( H^1(\mathcal{O}_X) \) is the tangent space of \( \text{Pic}_X \). Thus, \( H^0(\mathcal{N}_D) \) is the tangent space of \( \text{Curves}_X \).

Intuitively, one should think of a normal vector field on \( D \) corresponding to a first-order deformation of it in the space of curves. Over line bundles whose \( H^1 \) is 0, \( \text{Curves}_X \) is a smooth projective variety of dimension \( H^0(\mathcal{N}_D) \).