Curves on an algebraic surface II

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(1) Last time, Paul constructed Curves$_X$ and Pic$_X$ for us. In this lecture, we’d like to compute the dimension of the Picard group for an arbitrary surface, $X$, in terms or linear data. The first step is to identify the tangent space of Pic$_X$.

**Definition 1.1.** Let $X$ be a scheme over a field $k$ and $p \in X$ be a closed point. The (Zariski) tangent space of $X$ at $p$, $T_pX$, is the set of extensions

$$
\begin{array}{c}
\text{Spec}(k) \\
\downarrow \\
\text{Spec}(k) \\
\downarrow
\end{array}
\xrightarrow{p} X
$$

This has a natural structure of a $k$-vector space.

Thus, in the case of the Picard group, Pic$_X$, we need to compute the fiber over the identity of

$$
\begin{array}{c}
\text{Pic}_X(k[\epsilon]) \\
\downarrow \\
\text{Pic}_X(k)
\end{array}
$$

that is, we need to compute the kernel of the map:

$$
H^1(\mathcal{O}^*_X) \longrightarrow H^1(\mathcal{O}^*_\bar{X})
$$

where $\bar{X} := X \times_k \text{Spec}(k[\epsilon])$. This map is split by the inclusion $k \to k[\epsilon]$, so it is actually a surjection. We may identity its kernel with $\mathcal{O}_X$ via the injective map $\mathcal{O}_X \to \mathcal{O}_\bar{X}^*$ given on functions by

$$f \mapsto 1 + \epsilon f$$

Thus, we have a split exact sequence

$$
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\bar{X}^* \longrightarrow \mathcal{O}_X^* \longrightarrow 0
$$

which gives an exact sequence

$$
0 \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_\bar{X}^*) \longrightarrow H^1(\mathcal{O}_X^*) \longrightarrow 0
$$

One has to check that the vector space structure on $H^1(\mathcal{O}_X)$ agrees with the one on the Zariski tangent space, and then we conclude our result:

$$T_\epsilon(\text{Pic}(X)) \cong H^1(\mathcal{O}_X)$$

\footnote{The ‘truncated exponential’.}
(2) Recall that the dimension of a scheme $X$ at a point $p \in X$ is the Krull dimension of the complete local ring $\mathcal{O}_{X,p}$. In the case where $X$ is finite type over a field, if

$$\dim_p(X) = r$$

then one can find $r$ independent formal power series $x_1, ..., x_r$ which describe formal curves passing through the point $p$. Their derivatives then give $r$ linearly independent tangent vectors to $X$ at $p$. This observation combined with Hensel’s lemma is essentially the reason why we have

$$\dim_p(X) \leq \dim T_p X$$

Equality then says that, for any tangent vector at $p$, we can find a formal curve with the correct derivative at $p$. That is, we can always solve the extension problem

$$\text{Spec}(k[[t]]) \rightarrow \text{Spec}(k[t]/t^2) \rightarrow X$$

When $X$ is a group scheme over a field of characteristic 0, we can use the exponential map to describe a lift. This is the essence of the proof of the main result in characteristic zero:

**Theorem 2.1** (Grothendieck-Poincare). *If $X$ is a surface over field of characteristic 0, then*

$$\dim(H^1(\Omega_X)) = \dim(\text{Pic}_X)$$

(3) In characteristic $p$, there are obstructions to solving the extension problem

$$\text{Spec}(k[[t]]) \rightarrow \text{Spec}(k[t]/t^2) \rightarrow \text{Pic}_X$$

for a tangent vector at the identity. We can break up this extension problem into a sequence of, perhaps, easier extension problems:

**Proposition 3.1.** *Let $t \in T_e \text{Pic}_X$ be a tangent vector to $\text{Pic}_X$ at the identity. Then $t \in T_e (\text{Pic}_X)_{\text{red}} \subset T_e \text{Pic}_X$ if and only if, for all $n > 0$, we can solve the extension problem*

$$\text{Spec}(k[t]/t^n) \rightarrow \text{Spec}(k[t]/t^2) \rightarrow \text{Pic}_X$$

**Proof.** Since $(\text{Pic}_X)_{\text{red}}$ is an integral scheme of finite type over a field, there is a dense open subset that is non-singular. We also know $\text{Pic}_X$ is a group scheme, so $(\text{Pic}_X)_{\text{red}}$ is homogeneous and hence non-singular everywhere. Thus, if $t \in T_e (\text{Pic}_X)_{\text{red}}$, all the extension problems can be solved.

For the other direction, note that the extension problem is equivalent to the lifting problem:

$$k[t]/t^n \rightarrow \mathcal{O}_{\text{Pic}_X, e} \rightarrow k[t]/t^2$$
Suppose that such $t_n$’s exist for all $n \geq 2$. Let $x \in \mathcal{O}_{\text{Pic}_X,e}$ be such that $x^m = 0$. Write $t(x) = \alpha \epsilon$ for some $\alpha \in k$. Then

$$0 = t_{m+1}(x^m) = t_{m+1}(x)^m = (\alpha \epsilon + O(\epsilon^2))^m = \alpha^m \epsilon^m$$

Thus $\alpha^m = 0$, whence $\alpha = 0$. In other words, $t$ annihilates all the nilpotents in $\mathcal{O}_{\text{Pic}_X,e}$ and so factors through the reduction.

Let’s translate this into a condition about cohomology groups. We already know that the tangent space to Pic$_X$ is isomorphic to $H^1(\mathcal{O}_X)$. More generally, we have that

$$H^1(\mathcal{O}^*_{X \otimes k[t]/t^n}) = H^1(\mathcal{O}^*_X) \oplus H^1(1 + t\mathcal{O}_X[t]/t^n)$$

so that there is a natural isomorphism of groups:

$$H^1(1 + t\mathcal{O}_X[t]/t^n) \cong \{\text{Spec}(k[t]/t^n)-\text{valued points of Pic}_X\text{ at 0}\}$$

In other words:

**Corollary 3.2.** The tangent space of $(\text{Pic}_X)_{\text{red}}$ at the identity is the subspace of $H^1(\mathcal{O}_X)$ that lifts to $H^1(1 + t\mathcal{O}_X[t]/t^n)$ for each $n \geq 2$.

A priori, this seems like $n$ obstructions that we need to analyze. In fact, we will show that if char($k$) = $p$ then there are only $k$ obstructions to lifting a vector to $H^1(1 + t\mathcal{O}_X[t]/t^n)$ where $p^k \leq n < p^{k+1}$. In order to do this, we need to talk a little bit about Witt vectors.

(4) To any ring $R$ we may associate another ring, $R^\infty$, the product of infinitely many copies of $R$. This assignment is represented by the affine scheme $A^\infty = \text{Spec}(\mathbb{Z}[W_1, W_2, ...])$. There is also a group scheme Power$^*$ which has $R$-points given by formal power series in $R$ with leading term 1. Over $\mathbb{Q}$, we have an isomorphism of group schemes (using the additive structure on $A^\infty$)

$$A^\infty_{\mathbb{Q}} \xrightarrow{\phi} \text{Power}^*_{\mathbb{Q}}$$

given by

$$(w_1, w_2, ...) \mapsto \exp \left( -\sum \frac{w_m}{m} t^m \right)$$

Now, defining

$$W_1 = X_1$$
$$W_2 = X_1^2 + 2X_2$$
$$W_3 = X_1^3 + 3X_3$$
$$W_4 = X_1^4 + 2X_2^2 + 4X_4$$
$$W_n = \sum_{d|n} d X_d^{n/d}$$

we get a map

$$G : \mathbb{W} := \text{Spec}(\mathbb{Z}[X_1, X_2, ...]) \to A^\infty$$

The scheme $\mathbb{W}$ is called the universal Witt scheme, points are called Witt vectors, and the components of the image of these points under $G$ are called the ‘ghost components.’
Theorem 4.3. Let \( \mathcal{W} \) and \( G \) be as above.

1. \( G \otimes \mathbb{Q} \) is an isomorphism.

2. \( \mathcal{W} \) admits the unique structure of a ring scheme such that \( G \) is a ring homomorphism.

3. The composite
\[
\mathcal{W}_Q \longrightarrow \mathbb{A}_Q^\infty \longrightarrow \text{Power}_Q^*
\]
extends to an isomorphism \( \mathcal{W} \cong \text{Power}^* \) which identifies the additive group of \( \mathcal{W} \) with the group scheme \( \text{Power}^* \).

Now, the group scheme of power series \( \text{Power}^* \) fits into a tower
\[
\text{Power}^* \longrightarrow \cdots \longrightarrow \text{Power}_{\leq n+1}^* \longrightarrow \text{Power}_{\leq n}^* \longrightarrow \cdots
\]
where \( \text{Power}_{\leq n}^* \) has \( R \)-points given by the group \( 1 + tR[[t]]/t^{n+1} \subset R[[t]]/t^{n+1} \). This gives a similar tower for the Witt scheme (which is really just the same tower if we’re ignoring the extra ring structure):
\[
\mathcal{W} \longrightarrow \cdots \longrightarrow \mathcal{W}_{\leq n+1} \longrightarrow \mathcal{W}_{\leq n} \longrightarrow \cdots
\]
The reason why we introduced the Witt scheme is because this tower naturally fits into a larger diagram. The key point is this: the \( n \)th term in a product of power series is determined by the terms of degree less than or equal to \( n \). However, the \( n \)th term in a product of Witt vectors is determined by the terms \( X_d \) where \( d \mid n \).

In particular, given any set \( S \) of integers which is divisible, i.e. if \( n \in S \) and \( d \mid n \), then \( d \in S \), we can form a ring scheme
\[
\mathcal{W}_S = \text{Spec}(\mathbb{Z}[X_s]|s \in S])
\]
When \( S = \{1, p, p^2, \ldots\} \) we call this the \( p \)-typical Witt scheme, and denote it by \( \mathcal{W}_p \). We denote the truncations \( \mathcal{W}_{(1, p, \ldots, p^k)} \) by \( \mathcal{W}_{p, k} \). Given a divisible set \( S \), we denote by \( S/n \) the set of integers \( d \) where \( nd \in S \).

Definition 4.4. For each \( n \) there is a map
\[
V_n : \mathcal{W}_{S/n} \to \mathcal{W}_S
\]
defined by
\[
V_n^*(X_m) = \begin{cases} 
X_{m/n} & \text{if } n \mid m \\
0 & \text{otherwise}
\end{cases}
\]
This is called the “Verschiebung”. We also have a map called the Frobenius:
\[
F_n : \mathcal{W}_S \to \mathcal{W}_{S/n}
\]
which, on ghost components, is the map
\[
(w_1, w_2, \ldots) \mapsto (w_n, w_{2n}, \ldots)
\]

Theorem 4.5. Let \( S \) be a divisible set of integers.

1. The kernel of the truncation map \( \mathcal{W}_S \to \mathcal{W}_{S-n2} \geq 0 \) is the Verschiebung.

2. \( F_n \circ V_n = [n] \), multiplication by \( n \) on the group scheme.

3. Over \( \text{Spec}(\mathbb{Z}[1/n]) \), \( [n] \) is invertible and so \( V_n \) is a split injection.
Corollary 4.6. The kernel of the truncation map $\mathbb{W}_{p,k+1} \to \mathbb{W}_{p,k}$ is the additive group $\mathbb{A}^1$.

Corollary 4.7. Let $k$ be a field of characteristic $p$. Suppose that $p^k \leq n < p^{k+1}$. Then the truncation map $\mathbb{W}_{\leq n} \to \mathbb{W}_{p,k}$ has a splitting by the Verschiebung.

(5) We can use the Witt vectors to define a sheaf of rings on any variety.

Definition 5.1. Let $X$ be a scheme. Define a sheaf $\mathcal{W}$ by

$$U \subset X \mapsto \text{Hom}(U, \mathcal{W})$$

Similarly, we get sheaves $\mathcal{W}_{\leq n}$, $\mathcal{W}_{p,k}$, etc.

Corollary 5.2. The tangent space of $(\text{Pic}_X)_{\text{red}}$ is the subspace of $H^1(\mathcal{O}_X)$ that lifts to $H^1(\mathbb{W}_{\leq n})$ for all $n$.

Over a field of characteristic $p$, we can do better: $\mathcal{W}_{p,k}$ is a factor of $\mathcal{W}_{\leq n}$ when $p^k \leq n < p^{k+1}$. So lifting to $H^1(\mathcal{W}_{p,k})$ immediately gives a lift to $H^1(\mathbb{W}_{\leq n})$.

Corollary 5.3. If $X$ is a surface over a field $k$ of characteristic $p > 0$ then the tangent space of $(\text{Pic}_X)_{\text{red}}$ is the subspace of $H^1(\mathcal{O}_X)$ that lifts to $H^1(\mathcal{W}_{p,k})$ for all $k$. Equivalently, it is the image of $H^1(\mathcal{W}) := \text{lim}_k H^1(\mathcal{W}_{p,k}) \to H^1(\mathbb{W}_{\leq n})$.

Definition 5.4. We denote the differentials $d_r : E^r_{0,1} \subset H^1(\mathcal{O}_X) \to E^r_{1,2-r} \cong H^2(\mathcal{O}_X)$ by $\beta_r$. They are called Bocksteins.

Putting all of this together we get:

Theorem 5.5. Let $X$ be a surface over a field of characteristic $p > 0$. Then dim$(\text{Pic}_X)$ is the dimension of the subspace of $H^1(\mathbb{W}_{p,k})$ given by the intersection of the kernels of the Bocksteins, $\bigcap \ker(\beta_r)$.

Corollary 5.6. If $H^2(\mathcal{O}_X) = 0$, then $\text{Pic}_X$ is smooth of dimension equal to that of $H^1(\mathbb{W}_{p,k})$.

(6) The example is motivated by the following calculation in group cohomology. Let $C_p$ denote the cyclic group of order $p$. Then

$$H^*(C_p, \mathbb{F}_p) = \Lambda(x) \otimes \mathbb{F}_p[y], \quad |x| = 1, |y| = 2$$

One can check that the (topological) Bockstein $\beta$ acts like:

$$\beta(x) = y$$

So now the game is to construct some surface that approximates $BC_p$, and then, if there’s justice in the world, the Bocksteins we constructed earlier will match up with the topological Bockstein, and we’ll get the result we want. Indeed, we can do this:
Theorem 6.7 (Serre). Let $p \geq 5$. There is a non-singular surface $Y \subset \mathbb{P}^3$ with a free action of $C_p$, and hence $X = Y/C_p$ exists and is a non-singular surface with a $p$-fold cover by $Y$.

Corollary 6.8. There exists a surface $X$ with $\dim(\text{Pic}_X) < \dim(H^1(\mathcal{O}_X))$.

Sketch. One examines the spectral sequence

$$H^*(C_p, H^*(Y, \mathcal{O}_Y)) \Rightarrow H^*(X, \mathcal{O}_X)$$

and checks via naturality that the topological Bockstein converges to the algebraic Bockstein. \qed