Blow-ups of surfaces
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Today, we’re going to start on the classification of surfaces (as always, nonsingular over an algebraically closed field). This is a nice buzzword, but in what sense are we actually classifying anything? The program is roughly as follows.

1. Group surfaces in terms of Kodaira dimension, a rather coarse birational invariant.
2. Within each class, say something about the other birational invariants, such as the geometric genus.
3. Understand the surfaces, or at least the minimal surfaces, in each class, from a more geometric point of view.

As described, this program is incomplete: the so-called surfaces of general type, with a maximal Kodaira dimension of 2, have eluded further study. Nevertheless, we can learn a lot about the geometry of surfaces by pursuing it. I won’t get to Kodaira dimension today, but I will describe the birational methods just mentioned.

1 Algebraic recollections

We’ve defined a number of algebraic invariants of surfaces that will become useful in the sequel. First we had the (Cartier) divisor group $\text{Div} X = H^0(X, K(X)^*)/\mathcal{O}_X^*)$, whose elements we can write as $\mathbb{Z}$-linear combinations of irreducible curves on $X$. The short exact sequence

$$0 \to \mathcal{O}_X^* \to K(X)^* \to K(X)^*/\mathcal{O}_X^* \to 0$$

induces a long exact sequence

$$0 \to H^0(\mathcal{O}_X^*) \to H^0(K(X)^*) \to H^0(K(X)^*/\mathcal{O}_X^*) \to H^1(\mathcal{O}_X^*) \to H^1(K(X)^*) \to \cdots$$

where $\text{Pic} X$ is the Picard group. Thus, we identify $\text{Pic} X$ with the group of divisors mod the divisors of rational functions; two divisors are said to be linearly equivalent if they differ by the divisor of a rational function. More explicitly, the divisor $D$ is associated to the line bundle $\mathcal{O}(D)$ of rational functions with ‘poles along $D’.

I introduced a couple of relations weaker than linear equivalence. Two divisors are algebraically equivalent if they are the fibers of a relative Cartier divisor over a connected base scheme, $D \subseteq X \times S \to S$. Recall that the group $\text{Pic} X$ is the $k$-points of the group scheme $\text{Pic}_X$; geometrically, then, we see that two divisors are algebraically equivalent if they’re on the same connected component of the Picard scheme.

The Néron-Severi group is defined to be the group of divisors mod algebraic equivalence,

$$\text{NS}(X) = \text{Pic} X/\text{Pic}^0 X,$$

where $\text{Pic}^0 X$ is the ($k$-points of the) identity component of the Picard scheme. The key point about this is the following.
Theorem 1 (Néron-Severi). The Néron-Severi group is a finitely generated abelian group.

I’d wanted to prove this theorem, but the proof looks a little out of reach for us. The original proof involves embedding the variety into an abelian variety called its Albanese, and then quoting a version the Mordell-Weil theorem. Apparently you can do it with étale cohomology as well. Over ℂ, the proof is cute and easy. The exponential exact sequence

\[ 0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \]

induces a long exact sequence

\[ \cdots \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to \cdots, \]

and \( H^1(\mathcal{O}_X^*/H^1(\mathcal{O}_X) \) is identified with \( \text{Pic} X / \text{Pic}^0 X \) by the results of previous lectures. Meanwhile, \( X \) is analytically a compact complex manifold, so its singular cohomology \( H^2 \) is definitely finitely generated.

Finally, we say that two divisors are numerically equivalent if \( D \cdot E = D' \cdot E \) for all divisors \( E \). We define \( \text{Num} X \) to be the Néron-Severi group mod numerical equivalence; the intersection pairing is nondegenerate on this group. Since the intersection pairing is \( \mathbb{Z} \)-valued, all torsion classes in \( NS(X) \) has to die in \( \text{Num} X \).

Surprisingly, the converse is true.

Theorem 2. \( \text{Num} X \) is the torsion-free quotient of \( NS(X) \).

Matsusaka. By the Néron-Severi theorem, we just have to show that the kernel of \( NS(X) \to \text{Num} X \) is finite. Fix a projective embedding of \( X \), with \( H \) a hyperplane section; the degree of a divisor \( D \) is then given by \( \text{deg} D = D \cdot H \), so is invariant under numerical equivalence. Thus, it suffices to show that, up to algebraic equivalence, there are only finitely many divisors of some fixed degree.

Let \( D \) be a divisor with \( \chi(\mathcal{O}(D)) > 0 \), and \( \text{deg} D > \text{deg} K \). Recall the Riemann-Roch formula:

\[ \chi(\mathcal{O}(D)) = h^0(\mathcal{O}(D)) - h^1(\mathcal{O}(D)) + h^2(\mathcal{O}(D)) = D^2 - p_a(D) + \chi(\mathcal{O}_X). \]

The condition \( \text{deg} D > \text{deg} K \) means that \( K - D \) is not effective, so

\[ h^2(\mathcal{O}(D)) = h^0(\mathcal{O}(K - D)) = 0. \]

Thus,

\[ h^0(\mathcal{O}(D)) = h^1(\mathcal{O}(D)) + \chi(\mathcal{O}(D)) > 0. \]

So \( D \) is effective. The right-hand side of the Riemann-Roch formula shows that the conditions on \( D \) are numerical invariants. Thus, for any divisor numerically equivalent to zero, we get an effective divisor numerically equivalent to \( D \), and with the same degree as \( D \).

On the other hand, by a theorem of Chow and van der Waerden, there are only finitely many algebraic equivalence classes of effective divisors of fixed degree on \( X \). (This would be clear if the submonoid of effective divisors were finitely generated, since an effective divisor has a positive degree, but I don’t think it is necessarily finitely generated.) It follows that there are only finitely many divisors in the kernel of \( NS(X) \to \text{Num} X \). \(\square\)

2 Birational geometry and blowing up

Definition 3. A rational map \( f : V \to W \), with \( V \) a variety (integral scheme of finite type over a field), is a morphism from a dense open subset of \( V \) to \( W \). A birational map is an isomorphism in the category of varieties and rational maps. We can likewise talk about birational morphisms, which are everywhere defined rational maps with partially defined inverses. The domain of definition of \( V \) is the largest open subset of \( V \) over which \( f \) extends.

Proposition 4. If \( W \) is proper, then the complement of the domain of definition of a rational map has codimension \( \geq 2 \) in \( V \).
Proof. This is just the valuative criterion of properness: the map extends uniquely over irreducible codimension 1 subschemes because their local rings are DVRs.

In particular, a rational map of surfaces \( f : S \to T \) is defined away from a finite set of points. Moreover, if \( C \subseteq S \) is an irreducible curve, then \( f(C) \) is a well-defined projective curve, namely the closure of the ‘actual’ image of \( C \).

Blow-ups are the way of resolving these indeterminacies of rational maps of surfaces. We’ll mostly be concerned with their formal properties, so I’ll just briefly discuss the construction. In the most abstract setting, we have a closed subscheme \( Z \) of a variety \( X \) cut out by a coherent sheaf of ideals \( \mathcal{I} \), and the blowup of \( X \) along \( Z \) is

\[
\pi : \tilde{X} = \text{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d \to X.
\]

When \( X \) is a (smooth) surface and \( Z = P \) a closed point, the local ring \( \mathcal{O}_{X,P} \) is regular, so there is a system of parameters \( x, y \) with \( P \) cut out by the ideal \((x, y)\). There’s a surjection

\[
\text{Proj} \mathcal{O}_X[s, t] \twoheadrightarrow \text{Proj} \bigoplus \mathcal{I}^d
\]

(with \( s, t \) in degree 1) sending \( s \mapsto x \) and \( t \mapsto y \). Thus, the blowup \( \tilde{X} \) is a closed subscheme of \( X \times \mathbb{P}^1 \), specifically that cut out by the ideal \((sy - tx)\).

Away from the point \( P \), one now checks that \( \pi : \tilde{X} \to X \) is an isomorphism. At \( P \), the coordinates \( x \) and \( y \) vanish, so \( \pi^{-1}(P) \) is a copy of \( \mathbb{P}^1 \), called the exceptional divisor \( E \). A point \([s : t]\) of \( E \) corresponds to a linear form \((sy - tx)\) in \( \mathcal{O}_{X,P} \), and thus to a tangent direction of \( X \) through \( P \).

Consider what happens to an irreducible curve, \( C \), on \( X \) passing through \( P \). Its preimage is set-theoretically a union of two components: the exceptional divisor \( E \), and the strict transform \( \tilde{C} = \pi^{-1}(C - P) \). Let’s compute the divisor \( \pi^*C \). In the complete local ring \( \mathcal{O}_{X,P} \), the equation of \( C \) is a power series

\[
f(x, y) = \sum_{i \geq 0} f_i(x, y)
\]

where \( f_i \) contains terms of degree \( i \), and where \( m \) is the multiplicity of \( C \) at \( P \) (1 if \( C \) is smooth at \( P \)). Locally on \( \tilde{X} \) away from \( t = 0 \), we can take the functions \( x \) and \( t/s \) as local coordinates. We calculate that

\[
\pi^* f = f\left(x, \frac{t}{s}\right) x = x^m \cdot \left( \sum_{i \geq m} x^{i-m} f_i(1, t) \right).
\]

The factor \( x^m \) cuts out \( mE \), while the other factor cuts out \( \tilde{C} \). Thus we obtain

\[
\pi^* C = \tilde{C} + mE.
\]

As an added bonus, the above calculation shows that the multiplicities of \( \tilde{C} \) at its intersections with \( E \) are all less than the multiplicity of \( C \) at \( P \). By enough blowups, we can resolve all the singularities at \( C \). Intuitively, we think about, say, a curve with \( P \): its different branches at \( P \) will pull back to distinct points of \( E \), at which it will be smooth.

Let’s calculate the algebraic invariants of a blowup.

**Proposition 5.** Let \( \pi : \tilde{X} \to X \) be a blowup at a point \( P \), with exceptional divisor \( E \). Then \( \text{Pic} \tilde{X} \cong \text{Pic} X \oplus \mathbb{Z} \), and \( \text{NS}(\tilde{X}) = \text{NS}(X) \oplus \mathbb{Z} \), with the new component generated by \( E \). The intersection pairing is given by

\[
(\pi^* D) \cdot (\pi^* D') = D \cdot D',
\]

\[
(\pi^* D) \cdot E = 0,
\]

\[
E \cdot E = -1.
\]

Finally, \( K_{\tilde{X}} = \pi^* K_X + E \).
Proof. First we calculate the intersection pairing. Any divisors on X can be moved to not contain P, from which the first two formulas follow. In particular, if C contains P and is smooth at P, we calculate:

$$0 = (\pi^*C) \cdot E = (\tilde{C} + E) \cdot E = \tilde{C} \cdot E + E^2.$$ 

But \(\tilde{C}\) intersects E once, so \(E^2 = -1\).

The map \(\pi^* : \text{Pic} X \to \text{Pic} \tilde{X}\) has a retraction \(\pi_*\) whose kernel is just the multiples of E. It remains to see that \(\pi^*D + nE\) is never zero for \(n \neq 0\); this follows by intersecting with E. The same goes for the Néron-Severi group.

Finally, we can choose a representative of \(K_X\) – a 2-form on X – which is regular and nonvanishing at \(P\). The pullback of \(K_X\) has the same zeros and poles away from E, so we have \(K_X = \pi^*K_X + mE\) for some \(m\). By the genus formula for \(E\),

$$-2 = E \cdot (K_X + E) = E \cdot (\pi^*K_X + (m + 1)E) = -(m + 1),$$

so that \(m = 1\).

Blowups are sufficient to resolve indeterminacy in maps of surfaces.

**Proposition 6.** Any rational map \(f : S \to X\) from a surface to a projective variety extends to a morphism \(S' \to X\), where \(S' \to S\) is a sequence of blowups.

**Proof.** Without loss of generality, \(X = \mathbb{P}^N\), and \(f\) is induced by \(N+1\) sections \(s_0, \ldots, s_N\) of a line bundle \(\mathcal{O}(D)\). \(f\) is undefined wherever the \(s_i\) all vanish – the so-called base-points of the linear system. By assumption, they do not all vanish on a common curve. Let \(P\) be a basepoint, and \(\pi^* : S' \to S\) the blowup of \(S\) at \(P\). The sections \(s_i\) induce sections of the divisor \(\pi^*D\), which vanish on a common multiple of the exceptional divisor \(E\), say \(kE\), with \(k \geq 1\); thus, they are also sections of \(D' = \pi^*D - kE\), so we get a rational map \(\tilde{S} \to \mathbb{P}^N\). Note that \((D')^2 = D^2 - k^2\). As \(D\) induces a rational map to projective space, we have \(D^2 \geq 0\). Thus, this process of blowing up and modifying the divisor must eventually terminate, giving an honest morphism from a blowup of \(S\) to \(\mathbb{P}^N\).

**Proposition 7.** Any birational morphism of surfaces \(S \to T\) factors as \(S \to T' \to T\), where \(S \to T\) is an isomorphism, and \(T' \to T\) is a sequence of blowups.

To prove this, we use the following:

**Theorem 8** (Zariski’s main theorem). Let \(f : S \to S'\) be a birational morphism of proper varieties, and let \(p \in S'\) be a normal point at which the rational map \(f^{-1}\) is undefined. Then \(f^{-1}\{p\}\) (the inverse image of \(p\) along \(f\) is connected and positive dimension.

In particular, if \(f : S \to S'\) is a birational morphism of surfaces with \(f^{-1}\) undefined at \(p\), then \(f^{-1}\{p\}\) is a connected curve.

**Proof of Proposition 7** First note that Zariski’s main theorem generalizes to birational maps (not just morphisms): given a birational map \(f : S \to S'\), the graph of \(f\), a closed subvariety of \(S \times S'\), maps birationally to \(S\) and to \(S'\) via the two projections, and we can apply the theorem to get that \(f^{-1}\{p\}\) is a curve whenever \(f^{-1}\) is undefined at \(p\).

In particular, given a birational morphism of surfaces \(f : S \to T\) and a point \(p \in T\) at which \(f^{-1}\) is undefined (and suppose, for now, that this is the only one), we get a diagram

\[
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow{\pi} & & \\
T' & \xrightarrow{\pi} & T \\
\end{array}
\]

where \(\pi\) is the blowup at \(p\). (Confusingly, the diagonal map is dashed not because it’s one we’re trying to construct, but because it’s a birational map that is known to exist.) If \(g\) is undefined at \(q \in S\), then by Zariski’s main theorem, letting \(h = g^{-1}, h^{-1}\{q\} = \pi^{-1}\{f(q)\}\) is a curve \(C\) in the blowup, meaning that we
must have \( f(q) = p, C = E \). That is, \( g \) is defined everywhere except possibly at points \( q \) on the preimage of \( p \).

Now, since \( p \) pulls back to a curve along \( f \), the map \( f^* : \mathcal{O}_{T,p} \to \mathcal{O}_{S,q} \) cannot be an isomorphism – thus, there is a local coordinate \( y \in m_{T,p} \) (i.e. that isn’t in \( m_{T,p}^2 \)), with \( f^*(y) \in m_{S,q}^2 \). If \( h \) is defined at \( e \in E \) (recall that \( h(E) = \{ q \} \)), then \( h^*f^*(y) = \pi^*(y) \) is in \( m_{T,e}^2 \). But by the construction of the blowup, \( \pi^*y \) is also a local coordinate at every point of \( E \) except for one of them, the one corresponding to the tangent direction \( y \). Thus, we have a contradiction, so \( g \) must also be defined at \( q \).

What we have shown is that any birational morphism \( f : S \to T \), where \( f^{-1} \) is defined at \( p \), factors through the blowup of \( T \) at \( p \). We continue to blow up points as long as the inverse map is not everywhere defined, and we just have to show that the process ends sometime. In fact, \( f : S \to T \) only contracts finitely many irreducible curves, and we’ve shown that each blowup decreases the number of curves contracted by at least 1 (some contracted curve is mapped to an exceptional divisor), which proves the claim.

As we can blow up, so can we also blow down. Call a curve \( E \subseteq S \) exceptional if it’s the exceptional divisor of a blowup \( S \to T \). A surface is minimal if it has no exceptional curves.

**Theorem 9** (Castelnuovo’s contractibility criterion). A curve is exceptional iff it’s a rational curve \( E \) (that is, isomorphic to \( \mathbb{P}^1 \)) with \( E^2 = -1 \). (Equivalently: if \( E^2 = E \cdot K = -1 \).)

**Proof.** The ‘equivalently’ follows from the genus formula, and the fact that arithmetic genus 0 curves are automatically smooth, and thus rational. The direction (\( \Rightarrow \)) is clear.

For the direction (\( \Leftarrow \)), we need to construct a blowdown of \( E \). Let \( H \) be a very ample divisor on the surface \( S \). By Serre’s twisting theorem, we can choose \( H \) so that \( \mathcal{H}^1(\mathcal{O}(H)) = 0 \). Let \( k = H \cdot E \) and \( H' = H + kE \), so that \( H' \cdot E = 0 \). Since \( E \) is rational, there’s just one line bundle on \( E \) of each degree. In particular, the exact sequence

\[
0 \to \mathcal{O}_S(H - E) \to \mathcal{O}_S(H) \to \mathcal{O}_E(H) \to 0
\]

has cokernel \( \mathcal{O}_E(H) \cong \mathcal{O}_E(k) \). So we get exact sequences

\[
0 \to \mathcal{O}_S(H + (i - 1)E) \to \mathcal{O}_S(H + iE) \to \mathcal{O}_E(k - i) \to 0.
\]

These induce long exact sequences

\[
0 \to \mathcal{H}^0(\mathcal{O}_S(H + (i - 1)E)) \to \mathcal{H}^0(\mathcal{O}_S(H + iE)) \to \mathcal{H}^0(\mathcal{O}_E(k - i)) \to \mathcal{H}^1(\mathcal{O}_S(H + (i - 1)E)) \to \mathcal{H}^1(\mathcal{O}_S(H + iE)) \to \mathcal{H}^1(\mathcal{O}_E(k - i)).
\]

Again by rationality of \( E \), \( \mathcal{H}^1(\mathcal{O}_E(k - i)) = 0 \) for \( i \leq k \). As \( \mathcal{H}^1(\mathcal{O}_S(H)) = 0 \) by assumption, we get that \( \mathcal{H}^1(\mathcal{O}_S(H + iE)) = 0 \) by induction for \( 1 \leq i \leq k \). Thus, \( \mathcal{H}^0(S, \mathcal{O}_S(H + iE)) \to \mathcal{H}^0(E, \mathcal{O}_E(k - i)) \) is surjective for \( 1 \leq i \leq k \).

The linear system \( \mathcal{H}^0(\mathcal{O}_S(H')) \) is now filtered by the images of \( \mathcal{H}^0(\mathcal{O}_S(H + iE)) \), \( 0 \leq i \leq k \). It induces a rational map \( S \to \mathbb{P}^n \). We can think of the coordinates of \( \mathbb{P}^n \) as

\[
[a_{0,0} : \cdots : a_{0,m} : a_{1,0} : \cdots : a_{1,k - 1} : a_{2,0} : \cdots : a_{k,0}],
\]

where the \( a_{0,j} \) come from \( \mathcal{H}^0(\mathcal{O}_S(H)) \) and the \( a_{i,j} \) from \( \mathcal{H}^0(\mathcal{O}_E(k - i)) \) for \( 1 \leq i \leq k \). Since \( H \) was already very ample, and the embedding it determines is given by the first \( m \) coordinates, this map is already an embedding away from \( E \). Meanwhile, the coordinates \( a_{i,j} \) have poles of order \( k - i \) at \( E \), so clearing denominators, we see that \( E \) is mapped to the point \( P = [0 : \cdots : 0 : 1] \).

So we have constructed the alleged blowdown – let’s call it \( T \) – and we just have to prove that it’s smooth. Obviously, we just have to prove that it’s smooth at \( P \), i.e. that \( \mathcal{O}_{T,p} \cong k[[x, y]] \). We’ll use another cohomology-and-base-change style fact to prove this.

**Theorem 10** (Theorem on formal functions). Suppose that \( \pi : S \to T \) is a proper morphism of varieties (or even of locally noetherian schemes), \( \mathcal{F} \) a coherent sheaf on \( S \), \( P \in T \). Let \( S_n \) be the preimage of \( S \) the \( n \)th order formal neighborhood of \( P \), and let \( \mathcal{F}_n = \mathcal{F}|_{S_n} \). Then there is an isomorphism:

\[
(R^q\pi_*\mathcal{F})_P \cong \lim \mathcal{H}^q(S_n, \mathcal{F}_n).
\]
In our case, $T$ is normal and $\pi$ is birational, so we have that $\mathcal{O}_T = \pi_* \mathcal{O}_S$ (a variant of Zariski’s main theorem). The theorem gives

$$\mathcal{O}_{T,P} = \lim H^0(S_n, \mathcal{O}_{S_n}) = \lim H^0(E, \mathcal{O}_{S_n}).$$

Moreover, by construction, there are local coordinates around $P$ whose pullbacks generate $H^0(E, \mathcal{O}_E(1))$ – namely, $a_{k-1,0}$ and $a_{k-1,1}$. Thus, the inverse system of $H^0(E, \mathcal{O}_{S_n})$ of ‘fiber thickenings’ of $E$ is cofinal with the inverse system of $H^0(E, \mathcal{O}_E(n))$ of ‘normal thickenings.’ Finally, $E \cong \mathbb{P}^1$, so we know already that

$$H^0(E, \mathcal{O}_E(n)) \cong \text{Sym}^n H^0(E, \mathcal{O}_E(1)) \cong k[x, y]/(x, y)^n.$$

Taking inverse limits proves the claim.

Thus, $T$ is smooth, and $\pi : S \to T$ is birational and everywhere defined, and $\pi^{-1}$ is defined everywhere except $P$. By what we’ve seen already, $\pi$ is the blowup of $T$ at $P$.

**Proposition 11.** Every surface is birational to a minimal surface.

**Proof.** Blowing down decreases the rank of the Néron-Severi group by 1, and this group is finitely generated.

This all adds up to an approach to classification problems. We first construct coarse, birational invariants – in particular, the Kodaira dimension. For each value of these invariants, we find the minimal surfaces in that class. These are just surfaces without exceptional divisors, which we can identify numerically by Castelnuovo’s criterion. Any surface in the class can be constructed by one of these minimal surfaces by a sequence of blow-ups.