Two Important Theorems

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(1) Today we take a short interlude to prove two important, general results that will be used in the proof of the classification theorem. The first is a criterion for a divisor to be ample. This is a generalization of the fact that, on a curve, a divisor is ample if and only if it has positive degree (as a consequence of the Riemann-Roch theorem). The analog of degree on a surface is the self-intersection number. We also require that the divisor restrict to an ample bundle on every curve (or give an equivalent intersection theoretic requirement.) The result is the following:

**Proposition 1.1 (Nakai-Moishezon).** A divisor $D$ is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for every irreducible curve $C$ in $X$.

**Proof.** First note that we can assume $D$ is effective after replacing it with $nD$ for $n$ large. Then the proof proceeds in several steps: (i) show that $O(D) \otimes O_D$ is ample, (ii) show that $H^0(O(D)^{\otimes n}) \to H^0(O(D)^{\otimes n} \otimes O_D)$ is surjective for $n$ large, (iii) the map $X \to \mathbb{P}^N$ given by lifting generating sections and taking the section of $O(D)$ vanishing only on $D$ has finite fibers, and (iv) show that if $f : Y \to Z$ is a projective map with finite fibers and $L$ is a line bundle on $Z$ then $Z$ is ample if and only if $f^* L$ is ample.

For the first step, note that, given any line bundle $L$ on $D$, it is ample if and only if it is on $D_{\text{red}}$ (use the cohomological criterion). A line bundle on $D_{\text{red}}$ is ample if and only if it is so on each component. Finally, I can always check ampleness after pulling back under a finite map (use cohomological criterion, the Leray spectral sequence, and the fact that finite maps are affine.) Thus, it’s enough to check that the line bundle is ample on a normalization $\tilde{C}$ of each component $C$ of $D_{\text{red}}$. But this is just a question of degrees, which can be computed on $C$. Finally, we recall that for $L = O(D)$,

$$\deg_C(O(D) \otimes O_C) = D \cdot C$$

By assumption, this is positive, so the first claim follows. For the second step, consider the exact sequence

$$0 \longrightarrow O((n-1)D) \longrightarrow O(nD) \longrightarrow O_D \otimes O(nD) \longrightarrow 0$$

Since $H^2(O_D \otimes O(nD)) = 0$ (we’re on a curve), the map

$$H^1(O((n-1)D) \to H^1(O(nD))$$

is surjective for all $n \geq 1$. Thus

$$\cdots \leq h^1(O((n+1)D) \leq h^1(O(nD)) \leq h^1(O((n-1)D)) \leq \cdots$$

Since these are finite dimensional vector spaces, we must have equality for large $n$, and hence, for large $n$, the map

$$H^1(O((n-1)D) \to H^1(O(nD))$$

is an isomorphism, whence $H^0(O(nD)) \to H^0(O(D) \otimes O_D)$ is a surjection. Lift generating global sections from the right hand side and we get sections of $O(nD)$ that generate at every point of $D$. Together with the
with finite fibers is actually a finite map. Then we do the same trick as in step (i) to deduce that 

\[ O \]

\[ \rightarrow \]

But, by construction, the map gives a linear equivalence 

\[ nD \sim E \]

Thus, using Riemann-Roch on 

\[ C \]

we conclude by the vanishing of 

\[ h^1(K) = 0 \]

Thus we are left with the case that 

\[ K \cdot A + B = 0 \]

First suppose there is a divisor of the form 

\[ A + B \in |−K| \]

Since \((A + B) \cdot K = −K^2 < 0\) we may assume \(A \cdot K < 0\). Then \(|K + A| = |−B| = 2\). Then Riemann-Roch applied to \(K + A\) yields:

\[ 0 \geq 1 + \frac{1}{2}(A^2 + A \cdot K) = g(A) \]

so \(A\) is rational. Moreover \(A^2 \geq −1\) by the genus formula, but we cannot have \(A^2 = −1\) since \(X\) is minimal, thus \(A^2 \geq 0\) and the proposition is proved. Thus we are left with the case that \(K^2 > 0\) and every element in

(2) The second theorem is a criterion for a surface to be birationally equivalent to \(\mathbb{P}^2\). In the case of curves, it’s enough to check that the genus is zero. The proof is stolen from Beauville’s book, almost word for word.

**Example 2.2.** Let \(C\) be an elliptic curve over \(\mathbb{C}\). Then \(C \times \mathbb{P}^1\) has geometric genus 0. Indeed, the canonical bundle is the exterior product of the canonical bundles on each factor, so the computation follows from the Kunneth formula. On the other hand, 

\[ h^1(\mathcal{O}_{C \times \mathbb{P}^1}) = 1 \]

as required.

So that didn’t work. The next guess is that we must require 

\[ h^1(\mathcal{O}_X) = h^0(\omega_X) = 0 \]

This also turns out to be insufficient— we will see later that Enriques surfaces are irrational but satisfy this condition. The correct statement is as follows:

**Theorem 2.3** (Castelnuovo’s Rationality Criterion). A minimal surface \(X\) is rational if and only if 

\[ h^1(\mathcal{O}_X) = h^0(\omega_X) = 0 \]

\[ \Rightarrow \]

**Proof.** Suppose first that we can find a smooth rational curve \(C \subset X\) such that \(C^2 \geq 0\). Then, by the exact sequence 

\[ 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_C \otimes \mathcal{O}_X(C) \rightarrow 0 \]

we conclude by the vanishing of 

\[ H^1(\mathcal{O}_X) \]

that we have an exact sequence 

\[ 0 \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_X(C)) \rightarrow H^0(C, \mathcal{O}_C(C)) \rightarrow 0 \]

Thus, using Riemann-Roch on \(C \cong \mathbb{P}^1\), we conclude 

\[ h^0(\mathcal{O}_X(C)) = 1 + C^2 + 1 \geq 2 \]

In particular, the complete linear system \(|C|\) contains a line. Blowing up base points we get a nonconstant map \(\tilde{X} \rightarrow \mathbb{P}^1\) with one fiber isomorphic to \(C\). By the Noether-Enriques theorem we conclude that \(X\) is rational. Thus the theorem will follow if we can prove the below proposition.

**Proposition 2.4.** Suppose that \(X\) is a minimal surface with 

\[ h^1(\mathcal{O}_X) = P_2 = 0 \]

Then there exists a smooth, rational curve \(C\) on \(X\) such that \(C^2 \geq 0\).

**Proof sketch.** The proof goes by cases: 

\[ K^2 > 0, \ K^2 = 0, \text{ and } K^2 < 0 \]

The first case is the hardest so we’ll do that one and leave the rest to the reader. Notice that, by Riemann-Roch,

\[ h^0(−K) \geq K^2 + \chi(\mathcal{O}_X) = K^2 + 1 \geq 2 \]

First suppose there is a divisor of the form \(A + B \in |−K|\). Since \((A + B) \cdot K = −K^2 < 0\) we may assume \(A \cdot K < 0\). Then 

\[ |K + A| = |−B| = 2 \]

Then Riemann-Roch applied to \(K + A\) yields:

\[ 0 \geq 1 + \frac{1}{2}(A^2 + A \cdot K) = g(A) \]

so \(A\) is rational. Moreover \(A^2 \geq −1\) by the genus formula, but we cannot have \(A^2 = −1\) since \(X\) is minimal, thus \(A^2 \geq 0\) and the proposition is proved. Thus we are left with the case that \(K^2 > 0\) and every element in
the system $| - K|$ is indecomposable. From here we have two subcases: Either $\text{Pic}(X) = \mathbb{Z}(K)$ or else there is some effective divisor $D$ and $n > 0$ such that $H + nK$ is not linearly equivalent to zero and $|D + nK| \neq \emptyset$ while $|D + (n+1)K| = \emptyset$. In the latter case, take $E \in |H + nK|$ and note that $K \cdot C \leq 0$ for some component of $E$ (since we know $-K \cdot E \geq 0$ by computing in some projective embedding). But then $|K + C| = \emptyset$, by assumption. The same calculation as above shows $C$ is rational, and $C^2 = -2 - K \cdot C$ by the genus formula. If $K \cdot C \leq -2$ the proposition is proved, if $K \cdot C = -1$ this contradicts minimality, and if $K \cdot C = 0$ then a Riemann-Roch calculation shows $h^0(-K - C) \geq 1$ so there is an effective divisor $A$ with $A + C \in |-K|$, contradicting the assumption that every element was indecomposable.

So we are left with the case that $\text{Pic}(X) = \mathbb{Z}(K)$. In fact, this cannot happen. In characteristic 0, the exponential exact sequence implies that $H^2(X, \mathbb{Z}) \cong \mathbb{Z}(K)$. By unimodularity of the intersection form on integral cohomology (a result of Poincaré duality), we have $K^2 = 1$. But then Noether’s formula gives:

$$12 = 1 + c_2(X) = 1 + 2 - 2b_1 + b_2 = 1 + 2 - 2b_1 + 1$$

so that $b_1 = -4$, which is absurd. In characteristic $p$ there is a more complicated argument using étale cohomology, but the result still holds. 

$\Box$