Classification of surfaces: ruled surfaces

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Notation 1. $K$ is a canonical divisor, $\omega_X$ is the canonical sheaf, $P_n = h^0(\omega_X^n)$ is the $n$th plurigenus. $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$ is the Euler characteristic. I’ll use ‘divisor’ and ‘line bundle’ notation interchangeably, and I’ll often write things like $h^0(D)$ where I mean $h^0(\mathcal{O}(D))$. I’ll explain everything else.

Be warned that in the literature, the letter $q$ (irregularity) is sometimes used for $h^1(\mathcal{O}_X)$ and sometimes for $\dim \text{Pic}_X$, which, as we’ve seen, can be smaller than $h^1(\mathcal{O}_X)$ in positive characteristic.

We are classifying surfaces in terms of their plurigenera. We’ve got a start via two beefy theorems of Castelnuovo: the contractibility criterion, which says that a rational curve with self-intersection $-1$ can be blown down, and the rationality criterion, which says that a surface with $h^1(\mathcal{O}_X) = P_2 = 0$ is rational. Today we’ll nearly finish proving the following robust characterization of ruled surfaces.

**Theorem 2.** The following are equivalent for a surface $X$:

1. $X$ is ruled;
2. All $P_n(X)$ are zero (for $n > 0$);
3. $P_{12}(X) = 0$;
4. (assuming $X$ minimal) $K \cdot C < 0$ for some curve $C$.

**Proof.** 2 $\Rightarrow$ 3 trivially. 1 $\Rightarrow$ 2 because the plurigenera are birational invariants, and we can calculate then for $C \times \mathbb{P}^1$. In fact, if $C$ and $D$ are any curves, and $p_1 : C \times D \to C$, $p_2 : C \times D \to D$ are the projections, then

$$\Omega_{C \times D}^1 \cong p_1^*\Omega_C^1 \oplus p_2^*\Omega_D^1 \quad \text{and thus} \quad \omega_{C \times D} \cong p_1^*\omega_C \oplus p_2^*\omega_D.$$  

So $P_n(C \times D) = P_n(C)P_n(D)$. Taking $D = \mathbb{P}^1$, whose canonical bundle is anti-ample, we see that $P_n(C \times \mathbb{P}^1) = 0$ for all $n > 0$.

That 3 $\Rightarrow$ 1, probably the most mysterious step right now, will follow by simply calculating $P_{12}$ to be nonzero for all non-rulled surfaces; in a certain sense, this is the whole remainder of the classification theorem.

1 $\Rightarrow$ 4: we’ve shown that a minimal ruled non-rational surface is geometrically ruled. Let $F$ be a fiber; the genus formula for $F$ gives

$$-2 = F^2 + F \cdot K,$$

and since distinct fibers are algebraically equivalent but disjoint, we have $F^2 = 0$. Thus, $F \cdot K = -2$. Likewise, one can show that a minimal rational surfaces is either geometrically ruled over $\mathbb{P}^1$ or is $\mathbb{P}^2$, and in the latter case, $\omega \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, which has negative self-intersection with every curve.

The remainder of the talk will deal with Mumford’s proof of 4 $\Rightarrow$ 1 in all characteristics, building on the proof in characteristic 0 by Enriques and Kodaira. Let’s first pause for a moment to run through a few new techniques.

The Albanese variety

**Definition 3.** The Albanese variety of a variety $X$ is the universal abelian variety with a map from $X$. That is, the Albanese is an abelian variety together with a map $\alpha : X \to \text{Alb}(X)$, and if $A$ is another abelian
variety over \( k \), then there is a unique arrow filling every diagram of varieties of the form

\[
\begin{array}{c}
X \\
\downarrow \alpha \\
\text{Alb}(X).
\end{array}
\]

Note that I haven’t required the diagonal map here to be a homomorphism of abelian varieties. Nevertheless, one can show (for instance, see Mumford’s *Abelian Varieties* book) that any morphism of abelian varieties is the composition of a homomorphism and a translation; contrariwise, any morphism of abelian varieties preserving the identity point is a homomorphism. Thus, if one prefers homomorphisms of abelian varieties, one can just work in a category of pointed varieties instead.

I won’t prove anything about this; given what we know, the fastest construction is to take \( \text{Alb}(X) = (\text{Pic}_X^0)_\text{red} \), the dual abelian variety to the reduction of the identity component of the Picard scheme.

**Example 4.** If \( C \) is a curve of genus \( g \), then the Albanese is just the Jacobian of \( C \), a \( g \)-dimensional abelian variety. If \( C \cong \mathbb{P}^1 \), then the Albanese is a point. Otherwise, \( C \) injects into its Jacobian.

**Example 5.** If \( X \) is a surface, the map \( \alpha : X \to \text{Alb}(X) \) need not be an injection. Often, the image of \( \alpha \) will be a curve, and then an easy category-theoretic argument shows that \( \text{Alb}(X) \) must be the Jacobian of that curve.

**Étale cohomology**

One of the reasons that we like working over \( \mathbb{C} \) is because we can treat our varieties as complex analytic spaces instead, and access all the nice tools of algebraic topology. In particular, we learn a lot about a variety from the singular cohomology groups of the associated complex analytic space. For surfaces, the key tool here is the **Noether formula**

\[
\chi(\mathcal{O}_X) = \frac{1}{12} (K^2 + \chi_{\text{top}}(X)) = \frac{1}{12} (K^2 + 2 - 2b_1 + b_2).
\]

Here \( b_i = h^i_{\text{sing}}(X) \) is the \( i \)th **Betti number**, and \( \chi_{\text{top}}(X) = b_0 - b_1 + b_2 - b_3 + b_4 \) is the **topological Euler characteristic**. By Poincaré duality, \( b_i = b_{4-i} \), giving the formula above. By Hodge theory, \( b_1 = 2h^1(\mathcal{O}_X) \), so we could further write

\[
\chi(\mathcal{O}_X) = \frac{1}{12} (K^2 + 2 - 4h^1(\mathcal{O}_X) + b_2).
\]

Étale cohomology is the positive-characteristic response to this. I’ll say nothing about what it means, but it’s written \( H^i_{\text{ét}}(X, \mathbb{Z}_\ell) \), where \( \ell \) is a suitable prime not equal to \( \text{char} k \). The following properties of singular cohomology carry over:

- If \( X \) is projective, \( H^i_{\text{ét}}(X, \mathbb{Z}_\ell) \) is a finitely generated \( \mathbb{Z}_\ell \)-module for each \( i \), concentrated in degrees \( 0 \leq i \leq 2 \dim X \).
- If \( X \) is smooth, \( 2 \dim \text{Pic}_X = h^1_{\text{ét}}(X, \mathbb{Z}_\ell) \).
- If \( X \) is a surface, there’s a Noether formula

  \[
  \chi(\mathcal{O}_X) = \frac{1}{12} (K^2 + 2 - 2b_1 + b_2),
  \]

  where \( b_i \) is the rank of \( H^i_{\text{ét}}(X, \mathbb{Z}_\ell) \).
- There’s also an exponential map \( H^1_{\text{ét}}(X, \mathbb{G}_m)_{\ell} \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell) \), whose image one can show to be \( \text{NS}(X)_{\ell} \).

  In particular, this gives a quick proof that the Néron-Severi group is finitely generated.
The rest of the proof

Proof continued. Suppose that $X$ is minimal and $K \cdot C < 0$. We’ve already seen that $K \cdot H < 0$ for some ample $H$. As effective divisors have positive degree in any projective embedding, this means that $K$ cannot be linearly equivalent to an effective divisor, so that $h^0(K) = 0$. The same argument shows that $P_n = 0$ for all $n > 0$.

By Serre duality, $h^2(\mathcal{O}_X) = h^0(K) = 0$. Thus, we already get that the Picard scheme is smooth, and of dimension $h^1(\mathcal{O}_X)$.

Case 1. $K^2 > 0$. In this case, we use the Noether formula for étale cohomology:

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K^2 + 2 - 4h^1(\mathcal{O}_X) + b_2).$$

(Since the Picard scheme is smooth, $b_1 = 2h^1(\mathcal{O}_X)$.) Also, $\chi(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X)$ since $h^2$ is zero, so we get

$$12 - 12h^1 = K^2 + 2 - 4h^1 + b_2$$

and thus

$$10 = 8h^1 + K^2 + b_2.$$

There’s an injection $NS(X) \to H^2_{\text{ét}}(X, \mathbb{Z}_t)$, so $b_2 \geq \text{rank } NS(X) \geq 1$. Thus, $h^1 = 0$ or 1. If $h^1 = 0$, then the Castelnuovo rationality criterion implies that $X$ is rational.

Consider, then, the case $h^1 = 1$. The Albanese variety is a curve, and by Zariski’s main theorem, the fibers of $X \to \text{Alb}(X)$ are also curves. Let $F$ be a fiber; then $F^2 = 0$ as before and $F \cdot H > 0$ since $F$ is effective. Thus, $F$ and $H$ are linearly independent in $NS(X) \otimes \mathbb{Q}$, so the Néron-Severi group is rank at least 2, and thus $b_2 \geq 2$. But $K^2 > 0$ by hypothesis, and these numbers can’t fit into the Noether formula. This concludes the proof in this case.

Case 2. $K^2 \leq 0$.

Claim. For all $n > 0$, there is an effective divisor $D$ with $h^0(D + K) = 0$ and $h^0(D) \geq n$.

Proof of claim. Fix $n$. Since $K \cdot H < 0$, for sufficiently large $m$ we have $(nH + mK) \cdot H < 0$, and thus that $h^0(nH + mK) = 0$. Let $m$ be the largest nonnegative integer with $h^0(nH + mK) > 0$; that is, there is an effective divisor $D'$ linearly equivalent to $nH + mK$. Write $D' = D + D''$, where $D$ is the sum of components $C$ of $D'$ with $C \cdot K < 0$, and $D''$ the sum of those components $C''$ with $C'' \cdot K \geq 0$. By construction, $h^0(D' + K) = 0$, and since global sections only go up as we add in effective divisors, $h^0(D + K) = 0$ as well. Furthermore, $h^2(K) = h^0(K - D) = 0$ as well; if not, then $K - D$ is linearly equivalent to an effective divisor, so $K$ is linearly equivalent to an effective divisor, which contradicts $K \cdot H < 0$.

Next we calculate $D^2$. By the genus formula, $K \cdot D + D^2 \geq -2$, and $K \cdot D < 0$, so if $D^2 < 0$ as well, we must have $K \cdot D = D^2 = -1$ and $g(D) = 0$, contradicting minimality. (Even if $D$ is reducible or has multiple components, the genus formula successfully computes its arithmetic genus, and one can show the only curves with arithmetic genus zero are in fact rational.) So $D^2 \geq 0$.

Finally, we use Riemann-Roch to find $h^0(D)$. We have

$$\chi(D) = h^0(D) - h^1(D) + h^2(D) = \frac{D \cdot (D - K)}{2} + \chi(\mathcal{O}_X).$$

Since $h^2(D) = 0$, we can simplify this to an inequality:

$$h^0(D) \geq \frac{D^2}{2} - \frac{D \cdot K}{2} + \chi(\mathcal{O}_X) \geq -\frac{D \cdot K}{2} + \chi(\mathcal{O}_X) \geq -\frac{D' \cdot K}{2} + \chi(\mathcal{O}_X) \geq \frac{n}{2} + \chi(\mathcal{O}_X).$$

By increasing $n$, we can make $h^0(D)$ arbitrarily large, proving the claim. \(\square\)

The reason we introduced $D$ is to work with its Picard scheme. Note that $h^2(D) = h^0(K + D) = 0$, so that the exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$
induces a surjection $H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_D)$. We already know that $\text{Pic}^0_X$ is smooth, and $H^1(\mathcal{O}_X)$ is its tangent space at the identity; $\text{Pic}^0_X$ is smooth since $D$ is a curve; so the natural map $\text{Pic}^0_X \rightarrow \text{Pic}^0_D$, being a surjection on tangent spaces, is a surjection of group schemes.

One may ask how $\text{Pic}^0_D$ behaves when $D$ has singularities. In general, singularities introduce non-properness into the Jacobian of a curve. However, $\text{Pic}^0_X$ is proper since $X$ is smooth, and so $\text{Pic}^0_D$, being a quotient of $X$, is proper as well.

Now write $D = \sum n_i E_i$ with the $E_i$ integral curves. Then $\text{Pic}^0_D$ surjects onto $\text{Pic}^0_{E_i}$, so by the same token, $\text{Pic}^0_{E_i}$ is proper, and so $E_i$ is smooth. The $E_i$ together form a graph in $X$, and the same argument shows that this graph is a tree – if it were not, we’d get a copy of $\mathbb{G}_m$ in $\text{Pic}^0_D$ generated by a line bundle that twists as you go around the loop, contradicting properness of $\text{Pic}^0_D$.

Assume that for some $i$, $n_i \geq 2$. By the same argument, $\text{Pic}^0_{E_i} \rightarrow \text{Pic}^0_{2E_i}$, and $\text{Pic}^0_{2E_i} \cong \text{Pic}^0_{E_i}$. On tangent spaces, then, $H^1(2E_i) \cong H^1(E_i)$. Looking at the exact sequence

$$0 \rightarrow \mathcal{O}_{E_i}(-E_i^2) \rightarrow \mathcal{O}_{2E_i} \rightarrow \mathcal{O}_{E_i} \rightarrow 0,$$

we get $H^1(\mathcal{O}_{E_i}(-E_i^2)) = 0$. If $E_i^2 \geq 0$, then $\mathcal{O}_{E_i}(-E_i^2)$ is a nonpositive-degree sheaf on a smooth curve, and its $h^1$ can only be zero in a few restricted cases. We conclude that the curves $E_i$ that appear multiply in $D$ satisfy one of the following:

(a) $E_i^2 < 0$,

(b) $E_i$ is rational and $E_i^2 = 0$ or 1,

(c) or $E_i$ is elliptic and $E_i^2 = 0$, but $\mathcal{O}_{E_i}(-E_i^2) \not\cong \mathcal{O}_{E_i}$.

(Note also that $\mathcal{O}_{E_i}(-E_i^2)$ is the normal bundle of $E_i$ in $X$.)

We now start to count these curves. By the Chow-van der Waerden theory quoted several lectures ago, in a fixed projective embedding, there are only finitely many curves of fixed degree $d$, up to algebraic equivalence. However, a curve $C$ with $C^2 < 0$ must be in an algebraic equivalence class of its own; for if $C$ is algebraically equivalent to $C'$, then $C^2 = C \cdot C' > 0$. Thus, there are only finitely many curves $E_i$ of degree $d$ of type (a). Likewise, there are only finitely many curves $E_i$ of degree $d$ of type (c). I’m a little shaky on this point, but my intuition is that an algebraic family $C_i$ of curves represents a normal deformation of $C_0$, so if $C_0^2 = 0$, then $C_0$ is algebraically equivalent to another curve only if it has trivial normal bundle.

(Mumford skips some steps here, and Masayoshi Miyanishi’s *Open Algebraic Surfaces* was helpful for the rest of the argument.) Our aim, at last, is to prove the following:

**Lemma 6.** Every point in an open subset of $X$ lies on a rational curve.

Suppose this is proven, and consider the Albanese map $\alpha : X \rightarrow \text{Alb}(X)$. If $h^1 = 0$, then since $P_2 = 0$, $X$ is rational by Castelnuovo. So we can take $h^1 > 0$. Since $\alpha$ maps rational curves to points (\text{Alb}(\mathbb{P}^1) being trivial), the generic fiber of $X \rightarrow \alpha(X)$ is a rational curve by the claim, and so $\alpha(X)$ is a curve and $X$ is ruled.

Now let’s prove the lemma. There is a Hilbert scheme $H_{X,d}$ parametrizing rational curves of fixed degree $d$ (in some fixed projective embedding), which is projective and in particular quasi-compact. If $\dim H_{X,d} = 1$, then the universal family of curves in $X$ living over $H_{X,d}$ is 2-dimensional, and so the union of these curves is a 2-dimensional subscheme of $X$, that is, an open subscheme of $X$ in which every point is on a rational curve. Thus, if the lemma is false, the Hilbert scheme is 0-dimensional and finite; so there are only countably many rational curves (in total) on $X$. In sum, there are only countably many curves of type (a)–(c) above.

Mumford, at this point, says that these curves can’t exhaust $X$. If the ground field $k$ has uncountably many elements, this is true just by cardinality, so we reduce to this case. Again assuming $h^1 > 0$, we want to prove that the map $\alpha : X \rightarrow \text{Alb}(X)$ is ruled over its image, i. e. that the function field extension $k(\alpha(X)) \rightarrow k(X)$ is purely transcendental of dimension 1. One can show, using Tsen’s theorem, that this assertion is stable under replacing $k$ with an algebraically closed field extension of $k$, so we can reduce to the case where $k$ is uncountable.

Finally, we get that the curves (a)–(c) do not exhaust $X$. We are still assuming that $q = h^1 > 0$. Let $P_1, \ldots, P_q$ be points that don’t lie on the above curves. Let $D$, as above, be a nonzero effective divisor with $h^0(K + D) = 0$, $h^0(D) \geq 3q + 1$. The condition that $P_i$ is a multiple point of $D$ is codimension 2: generically,
divisors linearly equivalent to \( D \) vanish to order 0 at \( P_i \), and we want them to vanish to order 2. So there’s an effective \( D' \sim D \) with double points at all \( P_i \). The only multiple components of \( D' \) are of type (a)–(c), so each \( P_i \) lies on only simple components; since each component is smooth, each \( P_i \) lies on exactly two components; since \( D' \) is a tree, the \( P_i \)s in total lie on \( q + 1 \) components, all of genus \( \geq 1 \) by hypothesis. The Picard scheme of each component is thus positive-dimensional. So we get

\[
q \geq \dim \text{Pic}_X^0 \geq \dim \text{Pic}_{D'}^0 \geq q + 1,
\]

a contradiction. This proves the lemma, and the theorem. \( \square \)