Quasi-coherent and coherent sheaves

Let $X \to \text{Spec } k$ be a scheme. A presheaf over $X$ is a contravariant functor from the category of open subsets of $X$ to an abelian category. For example, there are presheaf of abelian groups, presheaf of rings and presheaf of modules, etc. A sheaf $\mathcal{F}$ is a presheaf with the following gluing conditions:

1. Suppose $\{U_i\}$ is an open cover of $U$. Take $s \in \mathcal{F}(U)$ a section over $U$, and denote $s_i$ be the restriction of $s$ to $U_i$. If that $s_i = 0$ for all $i$, then $s = 0$.

2. Take a family of sections $s_i \in \mathcal{F}(U_i)$. If $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

Each presheaf can be associated with a unique sheaf. For example, let $X$ be a topological space and $G$ an abelian group with discrete topology. The constant presheaf associates each non-trivial open subset $U$ with constant functions on $U$. The constant sheaf associates $U$ with locally constant functions on $U$.

Let $O_X$ denote the structure sheaf of $X$, which is a sheaf of ring. A sheaf of $O_X$-module $\mathcal{F}$ is a sheaf of abelian groups such that $\mathcal{F}(U)$ is a $O_X(U)$ module for any open subset $U \subset X$.

Let $(X,O_X)$ be a ringed space. A quasi-coherent sheaf $\mathcal{F}$ is a sheaf of $O_X$-module such that for any point $p \in X$, there is a affine open neighborhood $p \in U = \text{Spec } A$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus_{j \in J} O_U \to \bigoplus_{i \in I} O_U.$$

Let $(X,O_X)$ be a ringed space. A sheaf $\mathcal{F}$ of $O_X$-module is coherent if the following two conditions hold:

1. $\mathcal{F}$ is of finite type,
2. for every open $U \subset X$ and every finite collection $s_i \in \mathcal{F}(U), i = 1, \ldots, n$, the kernel of the associated map

$$\bigoplus_{i=1,\ldots,n} O_u \to \mathcal{F}|_U$$

is of finite type.

Example, let $X$ be a scheme, and $Y \subset X$ is a closed subscheme. Then $Y$ can be associated with a unique sheaf of ideal $\mathcal{I}_Y$. $\mathcal{I}_Y$ is a quasi-coherent sheaf of $O_X$-module. If $X$ is Noetherian, $\mathcal{I}_Y$ is coherent.

Functorial properties: Let $X = \text{Spec } A$ be an affine scheme

1. (Adjunction) $\sim - \Gamma$. For any sheaf of $O_X$-module $\mathcal{F}$, there is an isomorphism

$$\text{Hom}_A(M, \Gamma(\mathcal{F})) \cong \text{Hom}_{O_X}(\tilde{M}, \mathcal{F}).$$

2. (Equivalence) There is an equivalence of categories

$$\{A\text{-modules}\} \leftrightarrow \{\text{Quasi-coherent sheaves of } O_X\text{-modules}\}.$$
Suppose $A$ is Noetherian (which implies $X$ is Noetherian), then there is an equivalence of categories

\[ \{ \text{Finitely generated } A\text{-modules} \} \leftrightarrow \{ \text{Coherent sheaves of } \mathcal{O}_X\text{-modules} \} \]

Let $f : X \to Y$ be a morphism of schemes,

1. If $\mathcal{G} \in \text{Qcoh}(Y)$, then $f^* \mathcal{G} \in \text{Qcoh}(X)$.
2. If $\mathcal{F} \in \text{Qcoh}(X)$, $X$ is Noetherian, then $f_* \mathcal{F} \in \text{Qcoh}(Y)$.
3. If $X, Y$ are Noetherian, $\mathcal{G} \in \text{Coh}(Y)$, then $f^* \mathcal{G} \in \text{Coh}(X)$. But if $\mathcal{F} \in \text{Coh}(X)$, $f_* \mathcal{F}$ may not be coherent.

For example, $X = \text{Spec } \mathbb{C}[x, y], Y = \text{Spec } \mathbb{C}[x]$. $f : X \to Y$ is the projection corresponding to the natural embedding $\mathbb{C}[x, y] \to \mathbb{C}[x, y]$. $\mathcal{O}_X$ is a coherent sheaf, but $\Gamma(f_* \mathcal{O}_X) = \mathbb{C}[x, y]$ is not a finitely generated $\Gamma(\mathcal{O}_Y) = \mathbb{C}[x]$-module.

**Invertible sheaves and divisors**

A Hom sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf $U \mapsto \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$. A tensor sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is a sheaf associates to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Let $(X, \mathcal{O}_X)$ be a ringed space. Assume all stalks $\mathcal{O}_{X,x}$ are local rings. An **invertible sheaf** $\mathcal{L}$ on $X$ is a sheaf of $\mathcal{O}_X$-modules such that for each point $p \in X$, there exists an open neighborhood $U \subset X$ and an isomorphism $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. $\mathcal{L}$ is trivial if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$. Under the tensor operation, the set of invertible sheaves acquires a group structure. The inverse is given by $\mathcal{L}^{-1} = \mathcal{L}^* = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$, because

$$\mathcal{L}^* \otimes \mathcal{L} = \mathcal{H}om(\mathcal{L}, \mathcal{L}) = \mathcal{O}_X.$$

The **Picard group** $\text{Pic}(X)$ is group of isomorphism classes of invertible sheaves on $X$. The addition corresponds to tensor product.

Examples: 1. Let $X = \mathbb{P}^r_{\mathbb{C}} = \text{Proj } S, S = \mathbb{C}[z_0, \ldots, z_n]$ be the projective $n$-space. Then $\text{Pic}(X) \cong \mathbb{Z}$.

2. Let $E$ be an elliptic curve over $\mathbb{C}$. Then the Picard group is $\text{Pic}(E) \cong \mathbb{Z} \oplus \text{Pic}^0(E)$.

$\mathbb{Z}$ is the degree of the invertible sheaf. $\text{Pic}^0(E)$ consists of the degree 0 elements of $\text{Pic}(E)$.

Let $X$ be a Noetherian, integral scheme. A prime divisor is a closed subscheme $Z \subset X$, which is integral with generic point $\xi \in Z$ such that $\mathcal{O}_{X,\xi}$ has dimension 1. A **Weil divisor** on $X$ is a formal linear combination $\sum n_i[Z_i]$ of prime divisors $Z_i$ with integer coefficient. If $X$ is a curve $(\dim X = 1)$, then divisors on $X$ are points, which are zeros and poles.

A Weil divisor is effective if all non-zero $n_i$ are positive. Each prime divisor $Y$ determines a valuation ring $\mathcal{V}_Y$. Let $f \in \mathcal{K}^*$ be an invertible rational function, its associated divisor is

$$(f) = \sum \mathcal{V}_Y(f)Y.$$ 

Any divisor of such form is called a principal divisor. The **divisor class group** of $X$, denoted by $\text{Cl}(X)$, is the group of Weil divisors modulo the principal ones. I.e,

$$\text{Cl}(X) := \text{Div}(X)/\sim, \quad D \sim D' \text{ if } D - D' \text{ is principal}.$$ 

Example: 1. $X = \text{Spec } A$ is affine, then $\text{Cl}(X) = 0$. ($\mathbb{C}[x_1, \cdots, x_n]$ is UFD.) More generally, if $A$ is a UFD, $X = \text{Spec } A$, then $\text{Cl}X = 0$. 
2. $X = \mathbb{P}^r$, take $Y = \mathbb{P}^{r-1} \to X$ in the first $r$ homogenous coordinates. Then
\[ Cl(X) = \mathbb{Z}, \]
where $\mathbb{Z}$ is generated by $Y$. In fact, take $U = X - Y$, there is an exact sequence,
\[ \mathbb{Z} \to Cl(X) \to Cl(U) \to 0. \]
Since $U = A^*$, which is affine, $\mathbb{Z} \to Cl(X)$ surjectively. By construction, elements in $K^*$ are rational functions of degree 0, $nY = 0$ implies $n = 0$. Thus $Cl(X) = \mathbb{Z}$.

3. Let $X = \text{Spec } \mathbb{C}[x,y,z]/(xy - z^2)$. (This is a cone. $x, y$-axes are rulings.) Take $Y$ be the $x$-axis, on which $y = z = 0$. Thus $Y$ corresponds to the idea $(y,z)$. If fact, if $y = 0$, then $xy - z^2 = 0 \Rightarrow z^2 = 0 \Rightarrow z = 0$. The $Y$ can be cut out by the function $y$. Then $X - Y = \text{Spec } A_y$,
\[ A_y = \mathbb{C}[x,y,z^-1]/(xy - z^2) = \mathbb{C}[y,z^-1,z], \]
which is a UFD. Thus $Cl(X - Y) = 0$. By the short exact sequence,
\[ \mathbb{Z} \to Cl(X) \to Cl(X - Y) \to 0. \]

The principal divisor of $y$ is $2Y$. Because $y = 0 \Rightarrow z^2 = 0$. $z$ generates the maximal idea of the local ring at the generic point of $Y$.

$Y$ is not principal. $A$ is integral closed, it is equivalent to show $(y,z)$ is not principal. Let $m = (x,y,z)$, then $m/m^2$ is a three dimensional $\mathbb{C}$-vector space generated by $(\bar{x}, \bar{y}, \bar{z})$. The image of $(y,z)$ in $m/m^2$ contains at least $(\bar{y}, \bar{z})$, which cannot be principal.

Summarize the arguments, $Cl(X) = \mathbb{Z}/2\mathbb{Z}$.

A Cartier divisor is a global section of $K^*/\mathcal{O}^*$. In other words, it is a consist choice of \{U_i, f_i\}, $f_i \in \Gamma(U_i, K^*)$ such that \{U_i\} is a cover for $X$ and $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ for all $i, j$.

If $X$ is integral separated Noetherian scheme, all of whose local rings are UFDs (locally factorial), then
\[ \text{Div}(X) \cong \Gamma(X, K^*/\mathcal{O}^*). \]
The correspondence is given by valuation.

A Cartier divisor is principal if it is in the image of $\Gamma(X, K^*) \to \Gamma(X, K^*/\mathcal{O}^*)$. Similarly, there is a divisor class group defined via Cartier divisors:
\[ \text{CaCl}(X) := \Gamma(X, K^*/\mathcal{O}^*)/\{\text{principal divisors}\}. \]

For any scheme $X$, there is a map $\text{CaCl}(X) \to \text{Pic}(X)$. A Cartier divisor $D$ is a collection \{U_i, f_i\}, $f_i \in \Gamma(U_i, K^*)$, which can be viewed as a trivialization of the line bundle,
\[ \mathcal{O}_{U_i} \to \mathcal{L}(D)|_{U_i}, \]
\[ 1 \mapsto f_i^{-1}. \]
The transition map from $U_i$ to $U_j$, is thusly given by $f_i/f_j \in \mathcal{O}^*$. The trivialization and the transition maps define an invertible sheaf $\mathcal{L}(D)$. Moreover, principal divisors defines a trivial invertible sheaf, via $1 \mapsto f^{-1}$.

If $X$ is integral, then $\text{CaCl}(X) \to \text{Pic}(X)$ is an isomorphism.

Given an invertible sheaf $\mathcal{L}$, we can associate a Weil divisor by the intersection of any meromorphic section $s$,
\[ [D] := \sum v_Y(s)[Y]. \]
Example: Elliptic curve $E$. Suppose $D$ is a divisor of degree $n$,

$$\text{Pic}(E) \cong \mathbb{Z} \oplus \text{Pic}^0(E),$$

$$[D] \mapsto (n, [D] - n[O]).$$

**Sheaf of differentials**

Let $f : X \rightarrow Y$ be a morphism of schemes. The **sheaf of relative differentials** of $X$ over $Y$, $\Omega_{X/Y}$ is defined in the following equivalent ways,

1. $\Omega_{X/Y}$ is the sheaf of differentials of $f$ viewed as a morphism of ringed spaces equipped with it universal $Y$-derivation
   $$d_{X/Y} : \mathcal{O}_X \rightarrow \Omega_{X/Y}.$$

2. Let $\mathcal{F}$ be any sheaf of $\mathcal{O}_X$-module, then
   $$\text{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{F}) \cong \text{Der}_Y(\mathcal{O}_X, \mathcal{F}).$$

3. Then $\Delta : X \rightarrow X \times_Y X$ is a locally closed embedding. I.e. There is a open subscheme $W \subset X \times_Y X$ such that $\Delta : X \rightarrow W$ is a closed embedding. Let $\mathcal{I}$ be the sheaf of ideal of $\Delta(X)$ in $W$. Then define $\Omega_{X/Y} = \Delta^* (\mathcal{I}/\mathcal{I}^2)$ over $X$.

Suppose $\Omega_{X/Y}$ is locally free of rank $n$. Define the **canonical sheaf** to be the top exterior power of the sheaf of differentials,

$$\omega_{X/Y} := \wedge^n \Omega_{X/Y}.$$

Examples:

1. Let $X = \mathbb{P}^r$, then $\omega_{X/k} = \mathcal{O}(-r - 1)$.

2. Let $X = \mathbb{P}^r$ and $Y$ is a divisor defined by a degree $d$ polynomial. Then
   $$\omega_Y = \mathcal{O}_X \otimes \mathcal{L}(Y) \otimes \mathcal{O}_Y = \mathcal{O}_Y(d - r - 1).$$

**$K_3$ surfaces:** Suppose $r = 3$ and $Y$ is defined by $z_0^d + z_1^d + z_2^d + z_3^d$. For $Y$ to be a $K_3$ surface, its canonical sheaf has to be trivial. Then we have to take $d = 4$.

**Elliptic curves:** Any elliptic curve $E$ can be embedded into $\mathbb{P}^2$. For $r = 2, d = 3$, we can calculate the canonical sheaf of $E$ is $\omega_E = \mathcal{O}_E(0)$. Hence the Canonical divisor is $[O]$.

**Sheaf cohomology**

Taking global sections is a left exact covariant function. Suppose

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

is an exact sequence of sheaves over $X$, then

$$0 \rightarrow \Gamma(X, \mathcal{F}') \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}'') \rightarrow 0.$$  

Let $\mathcal{F}$ be a sheaf over $X$. Then the sheaf cohomology is defined as the right derived functors of $\Gamma$.

$$H^i(X, \mathcal{F}) := R^i \Gamma(X, \mathcal{F}).$$

Let $(X, \mathcal{O}_X)$ be a ringed space, then the cohomology of sheaf of modules coincides with the cohomology of sheaf of abelian groups.

Let $Y \subset X$ be a closed subspace of $X$. $\mathcal{F}$ is a sheaf of abelian groups on $Y$, $j : Y \rightarrow X$ is the inclusion. $j_* \mathcal{F}$ coincides with the extension by zero. Then $H^i(Y, \mathcal{F}) = H^i(X, j_* \mathcal{F})$.

Examples:
1. Over affine schemes. Let $X = \text{Spec } A$ be an affine scheme of a Noetherian ring $A$. $\mathcal{F}$ is an quasi-coherent sheaf over $X$. Then

$$H^i(X, \mathcal{F}) = 0, \quad \text{for all } i > 0.$$ 

2. $X = \mathbb{P}^r = \text{Proj } S$, $S = \mathbb{C}[z_0, \cdots, z_n]$. Then

(a) $S \cong \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$.

(b) $H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and $n \in \mathbb{Z}$.

(c) $H^r(X, \mathcal{O}_X(-n-1)) = \mathbb{C}$.

(d) There is a perfect paring of finitely generated free $\mathbb{C}$-modules,

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to \mathbb{C}, \quad \text{for all } n \in \mathbb{Z}.$$

Smooth morphisms

A morphism $f : X \to Y$ is a flat morphism if the induced map on every stalk is a flat map of rings, i.e.,

$$f_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

is a flat map for all $p \in X$.

Let $X \to Y$ be a morphism of schemes. The the higher direct image functor $R^i f_* \mathcal{F}$ is defined to be the right derived functors of $f_*$. Let $\mathcal{F}$ be a sheaf over $X$, $R^i f_*(\mathcal{F})$, the higher direct images sheaves, is the sheaf associated with the presheaf on $Y$

$$V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) \quad \text{for all } i \geq 0.$$ 

The global sections of this sheaf are the regular sheaf cohomologies.

Flat morphisms commute with higher direct cohomologies. Let $f : X \to Y$ be a separated morphism of finite type of Noetherian schemes. $u : Y' \to Y$ is a flat base change of Noetherian schemes. $X' = X \times_Y Y'$.

$$\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
g \downarrow & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}$$

$\mathcal{F} \in \text{Qcoh}(X)$ is a quasi-coherent sheaf over $X$. Then for all $i \geq 0$,

$$u^* R^i f_*(\mathcal{F}) \cong R^i g_*(v^* \mathcal{F}).$$ 

Let $f : X \to S$ be a morphism of schemes. $f$ is smooth at $p \in X$ if there exists a affine open neighborhood Spec $A = U \subset X$ and affine open Spec $R = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is smooth. $f$ is smooth if it is smooth at every point of $X$.

Let $f : X \to S$ be a morphism of schemes. $f$ is étale at $p \in X$ if there exists a affine open neighborhood Spec $A = U \subset X$ and affine open Spec $R = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is étale. $f$ is smooth if it is étale at every point of $X$.

References

