A Tale of Etale Cohomology

Elden Elmanto

March 2, 2014

1 Basic Definitions

We have defined what it means for a map to be etale. We have defined what the etale site is and what it means to be an etale sheaf. In particular if X is a fixed scheme. We define the site: X_{et}. The underlying category is the category Et/X where the objects are etale morphisms $U \to X$ and the covering are surjective families of etale morphisms $(U_i \to U)$.

Note that there is a variant of this called the big etale site where the underlying category is Sch/X, schemes over X and the coverings are surjective families of etale X-morphisms.

The plan for this talk is as follows:

1. I will define sheaf cohomology in general and say how to compute them and produce some examples of sheaves whose cohomology we will be taking throughout this talk.

2. I will then compute the Etale cohomology of a point and say how this relates to Galois theory. We will do this in detail.

3. I will indicate an intermediary computation of H^1 using Cech cohomology and explain the geometric meaning of what this group captures.

4. I will then bump up this computation to the cohomology of curves. We will do this in some detail.

5. I will then talk about cultural/historical/awesome aspects of etale cohomology — its properties, the Weil conjectures, the Bloch-Kato conjecture. I will do this in hardly any detail.

2 Basics of Cohomology

Let A, B be abelian categories with A having enough injectives. Recall that an object I is injective if $Hom(-, I)$ is right exact. In this case if we are given a left exact functor $F : A \to B$, then we can take the right derived functor of F which we denote by $R^iF : A \to B$. How do we compute this?

On an object X we may take an injective resolution: $X \to I^0 \to I^1 ...$ — this is an exact complex. Hit this with the functor F and we are looking at the complex (after removing X):

$$0 \to F(I^1) \to F(I^2) \to ...F(I^i) \to F(I^{i+1}) \to ...$$

. Then the value is given by $R^iF(X) := h^i(F(I^\bullet))$

There are a bunch of standard caveats/comments that comes with this manipulation

1. First and foremost we are really interested in $R^0F(X) = F(X)$. Examples of functors are global section $\Gamma(X, -)$ of a sheaf on X, the (continuous) fixed point of a G-module, $Hom(-, X)$ where X is an object in abelian category (like a sheaf of modules or just modules).

2. Any two injective resolutions are homotopy equivalent, this makes the above construction well-defined.

3. Having defined what the derived functors are, we can use something that is often easier to compute it: we may replace an injective resolution by an acyclic resolution, $J^\bullet$ — one whose components satsify $R^k(J^i)$ for all i and all $k > 0$.

4. One might have encountered the notion of δ-functors and universal δ-functors that comes with this package. I think an important point to note that is that if we have two functors $G, F : A \to B$ such that $G \cong F$ naturally, then their derived functors are naturally isomorphic. This is a consequence of the unversality of right derived functors. This idea will be used again and again later.

5. Here's something extremely useful. If one is given an exact sequence in A: $0 \to X' \to X \to X'' \to 0$, then one gets a long exact sequence in cohomology: $... \to R^iF(X') \to R^iF(X) \to R^iF(X'') \to R^{i+1}F(X) \to ...$

And now, I can define for you etale cohomology since we have everything already defined.

Definition 2.1. Let X be a scheme and let $F \in Shv_{et}(X)$ be an etale sheaf on X and consider the left exact functor $\Gamma(X, -) : Shv_{et}(X) \to Ab$. Then the etale cohomology of X with values in F is defined to be $H^i_{et}(X; F) := R^iF(X)$.

That isn't so bad! But in practice finding injective resolutions is really hard. This is a good definition and good for proving abstract theorems but hardly of any use for real computation. In my experience what happens is that one uses some form of Cech cohomology to get a sample of bedrock computations then piece together long exact sequences or prove duality theorems.

Let $\mathcal{U} = \{U_i \to X\}$ be an etale covering of X and let P be a presheaf of abelian groups. We define the Cech complex associated to this covering by $C^r(\mathcal{U}, P) := \prod_{i_0,\ldots,i_r} P(U_{i_0,\ldots,i_r})$ with differentials: if $s = (s_{i_0 \ldots u_r})$ we define $(d^r s)_{i_0 \ldots i_r} = \Sigma_{j=0}(-1)^j (s_{i_0 \ldots \hat{i_j} \ldots i_{r+1}}) \mid_{U_{i_0 \cdots i_{r+1}}}$.

We can verify easily that this is a complex and define $\check{H}^i(\mathcal{U}, P)$ to be the cohomology of this complex. We observe that $\check{H}^0(\mathcal{U}, P)$ is defined to be the kernel of the difference of the two arrows: $\prod P(U_i) \to \prod P(U_{ij})$ which is also known as the value of the sheaf on $P(X)$, i.e. global sections if P was a sheaf.

Explictly, we are looking at $(s_i)_{i \in I} \mapsto s_i \mid_{U_{ij}} - s_j \mid_{U_{ij}}$

There is an obvious notion of a refinement of a cover and we may define the Cech cohomology of X with coefficients in P as $\check{H}^i(X, P) := \varinjlim \check{H}^i(\mathcal{U}, P)$

The main theorem is that:

Theorem 2.2. *Assume that every finite subset of X is contained in an open affine and X is quasi-compact. Then $\check{H}^r(X, F) \simeq H^r(X, F)$.*

We will come back to this perspective later. Anyway, before we can do more, we need examples of functors that we can derive. Before we proceed let me just say that the derived functors here correspond literally to derived functors in model categories which we have been talking about all along. Injective resolutions are just some kind of cofibrant replacement in an appropriate model structure on chain complexes of an abelian category.

3 Examples of Etale Sheaves

Here's the punchline I want to make: Etale sheaves are easily checkable. First let me highlight something that always gets me confused: a Zariski cover is in particular an Etale cover because Zariski opens are etale maps. Therefore if we want to check if something is an etale sheaf, it is in general *not enough to be a Zariski sheaf.* So we ask when can we promote a Zariski sheaf to an etale sheaf.

In particular the structure sheaf is a Zariski sheaf but not necessarily an etale sheaf — but it turns out that it is an etale sheaf (phew). Anyway here's a criterion for something to be an etale sheaf. Faithfully flat

Let me say something about going the other way. There is a topology which turns out to be extremely useful called the fpqc topology. Here, the maps in the covers are required to be flat and locally quasicompact. The fpqc site is defined by demanding that the covers be surjective families as per usual. Recall from the defintion of an etale map that an etale cover is in particular an fpqc cover — the only thing that's missing is maybe unramified (locally quasicompact is part of being a finite type morphism). Anyway the maps at least have "continuously varying fibers." In this case, we note that if a sheaf is already a sheaf in the fpqc topology then it is a sheaf in the etale topology. Actually let me phrase the requirements in terms of this fpqc topology.

Theorem 3.1 (fpqc Sheaf Criterion)**.** *In order to verify that a presheaf F on X_{fpqc} is a sheaf it suffices to check that F is a Zariski sheaf and for a fpqc covers consisting of a single map, i.e. on a faithfully flat map $Spec\, B \to Spec\, A$*

The proof is more or less diagram chasing and the trick is to assemble a cover $\{U_i \to U\}$ into a single one $\{\coprod U_i \to U\}$

Let me remind you then that an fpqc sheaf is therefore, in particular, an etale sheaf!

Corollary 3.2. *The structure sheaf on $O_{X_{fpqc}}$ defined by $O_{X_{fpqc}}(U) = \Gamma(U, O_U)$ is indeed a sheaf.*

Proof. I think this is a sweet argument and it will transition nicely to the next section so let me say how it is proven. The structure sheaf is a Zariski sheaf. So it suffices to prove the theorem for an arbitrary faithfully flat map $Spec\, B \to Spec\, A$. Writing out the sheaf condiiton this is exactly the same as checking if the the sequence: $0 \to A \to B \to B \otimes_A B$ is exact. Now one can check if we have a section $s : B \to A$ then the statement is true. Now if we have a faithfully flat map: say $A \to C$ then if the sequence is exact after tensoring with C then it was exact to begin with by definition of faithfully flat or by what faithfully flat maps are designed to do! Well, we do have a faithfully flat map: $A \to B$ and the map $B \to B \otimes_A B$ has a section! □

Corollary 3.3. *Representable presheaves are sheaves*

Corollary 3.4. *Coherent sheaves in the Zariski topology are sheaves in the fpqc topology. Namely, if F is a coherent (Zariski) sheaf on X, then we can define an fpqc presheaf on X by $F(U) = f^*F(U)$. This presheaf is in fact an fpqc sheaf.*

This is proven by homming into the exact sequence above. We are now in business: here's a good list of fpqc sheaves and therefore etale sheaves!

1. The sheaf of n-th roots of unity is defined by the representable of $Spec\, \mathbb{Z}[T]/(T^n - 1) \times_{\mathbb{Z}} X$. The value of this sheaf on U is $\Gamma(U, O_U)$.

2. The sheaf of units is defined by the representable $Spec\, \mathbb{Z}[T, T^{-1}] \times_{\mathbb{Z}} X$. The value of this sheaf on U is $\Gamma(U, O_U)^\times$

3. More generally, the sheaf GL_n is defined by $Spec\,\mathbb{Z}[T, T_{11}, T_{12}, ..., T_{nn}]/(Tdet(T_{ij}) = 1) \times_{\mathbb{Z}} X$. The value of this sheaf on U is $\{s \in \Gamma(U, O_U) : s^n = 1\}$

By the way, now that we know what etale sheaves are maybe we should say a word about stalks of etale sheaves. They contain a richer algebraic structure that your typical Zariski sheaf.

So as per the usual case there two kinds of points: weird ones and the usual ones.

Definition 3.5. Let X be a scheme, a geometric point of X is defined to be a morphism $Spec(\Omega) \to X$ where Ω is a separably closed field. This is the same data as a point $x \in X$ as well as an injection of $k(x)$ into a separably closed field. If x is a point on X then we denote by $\bar{x}$ the corresponding map $Spec\,\Omega \to X$ which comes with the data of the extension $k(x) \to \Omega$.

Useful facts:

Theorem 3.6. *A morphism of abelian sheaves on X_{et} is an isomorphism, mono, epi if and only if the morphism on stalks $F_{\bar{x}}$ have the corresponding properties.*

Here $\bar{x}$ corresponds to the geometric point associated to taking the separable closure of $k(x)$ whenever $x \in X$ is a point. This is good. Now, a strictly Henselian ring is one who is Henselian (this means that in order to reduce polynomial equations over the given local ring, we can reduce it to the residue field and reduce it there) and strictly local (which means that its residue field is separably closed). One can then show that:

Theorem 3.7. *The stalk $O_{X_{et},\bar{x}}$ is the strict henselization of $O_{X_{zar},\bar{x}}$*

The tagline is that both sides are computed by the same colimit.

4 Etale Cohomology of a Point

4.1 Descent implies Hilbert Theorem 90

The above introduced us to the idea of descent.

Let k be a field and K a finite galois extension of k. We use faithfully flat descent to prove the following result:

Theorem 4.1 (Hilbert Theorem 90). $H^1(G, K^\times) = 1$.

First, some recollection. If $f : X \to Y$ is a map of locally ringed spaces, and F is a sheaf on X, then we may define the *direct image* sheaf on Y by the formula; $f_*(F)(V) := F(f^{-1}(V))$ for V an open subset of Y. This has a left adjoint called the inverse image sheaf — so if G is a sheaf on Y, then f^*G is a sheaf on X.

Specializing in the case of schemes, If $f : X \to Y$ is a morphism of schemes and G is an O_Y-module, then we may define $f^*G := f^{-1}G \otimes_{f^{-1}O_Y} O_X$ so that it is a sheaf of O_X-modules.

Specializing even further, if we have a map $Spec\,B \to Spec\,A$ (and hence a map $\phi : A \to B$) and a quasi-coherent sheaf of A-modules, i.e. $\tilde{N}$, then we may explicitly write down $f^*\tilde{N}$ as $N \tilde{\otimes}_A B$.

Useful things:

1. We have an adjunction $Hom_{Shv_X}(f^*F, G) \simeq Hom_{Shv_Y}(F, f_*G)$

2. Right adjoints are always left exact and left adjoints are always right exact.

3. If a right adjoint has an exact left adjoint then it preserves injective objects. This is one of the standard ways by which we can show that f_* preserves injectives and hence suitable for computing resolutions: In this case, we will have that $H^i(Y, f_*F) \simeq H^*(X, F)$. This is NOT always true! We need that f_* be exact and that it preserves injectives. The latter is kind of tricky as the f^* here is the usual f^{-1} (as per Hartshorne's notation). The difference is that f^* in the usual notation is the left adjoint in O_X-modules but we are just thinking about abelian sheaves here.

Now I want to generalize the above argument:

4.1.1 Descent Datum

Let's spell this out for the affine case.

Definition 4.2. Let $A \to B$ be a ring map and N a B-module. A descent datum for N with respect to $A \to B$ is an isomorphism $\phi : N \otimes_A B \to B \otimes_A N$ of $B \otimes_A B$-modules such that the diagram of $B \otimes_A B \otimes_A B$-module:

$$\begin{array}{ccc} N \otimes_A B \otimes_A B & \longrightarrow & B \otimes_A N \otimes_A B \\ \downarrow & \swarrow & \\ B \otimes_A B \otimes_A N & & \end{array}$$

commute

If $N' = B \otimes_A M$ for some A-module M then it has a canonical descent datum given by the map $\phi : B \otimes_A M \otimes_A B \to B \otimes_A B \otimes_A M$ given by $b_0 \otimes m \otimes b_1 \mapsto b_0 \otimes b_1 \otimes m$

So the question is really: whether or not a particular B-module can be written as a lift from an A-module structure. The point of the descent datum, at least in the affine case is that it presents us with the A-modue rather explicitly! It is the kernel of the map $N \to B \otimes_A N$ given by $n \mapsto 1 \otimes n - \phi(n \otimes 1)$.

Let us examine this idea explicitly in the case of $\mathbb{R} \to \mathbb{C}$

1. if V was a real vector space we can tensor up: $V \otimes_{\mathbb{R}} \mathbb{C}$ to get a $\mathbb{C}$ vector space.

2. However this is not just some random complex vector space, we can write down an mod 2 action on this vector space: $v \otimes z \mapsto v \otimes \bar{z}$.

3. We can recover back the original V from this complex vector space by just looking at the fixed points of this action!

4. Another idea is the following: in the most basic case, we are trying to recover $\mathbb{R}$ from $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$. So the key idea observation is that we are tensored over $\mathbb{R}$ so an element in $\mathbb{C}$ will only be a real number if and only if we can balance the tensors across: $z \otimes 1 = 1 \otimes z$!

5. These two ideas coincide in the setting of Galois theory. The following is nonobvious, but conceptually it makes sense.

Lemma 4.3. *Let K be a Galois extension of k. There is bijection between descent datum on K over k and semi-linear $Gal(K/k) = G$-structure on K-vector spaces.*

A semilinear G-action is just an action of G on V such that $g(\lambda v) = \lambda^g g(v)$ where λ^g denotes the action of the Galois group

Proof. Most conceptually, the above lets us write $Spec\, K \times_k Spec\, K$ as $Spec\, K \times G$, from which semi-linearity is equivalent to the cocycle condition. See math stack-exchange post or Fields Notes by Pianzola. In any case the starting point is always the following identification:

Lemma 4.4. *Let k be a field and K a finite Galois extension. The map $K \otimes_k K \to \prod_{\sigma \in Gal(K/k)} K$ given by $a \otimes b \mapsto (a \times (\sigma(b))_{\sigma \in G}$ is an isomorphism.*

□

One can easily formulate the non-affine case and we have the following generalization of the above theorem:

Theorem 4.5 (SGA 4.5 Theorem 1.4.5)**.** *If $p : X \to Y$ be an fpqc morphism, then there is an equivalence of categories between Y-modules and quasi-coherent X-modules with descent datum.*

Now, this tells us that if had started with V' a K-vector space, then we can set $V := V'^G$ and V' is a semi linear G-vector space we have that: the natural isomorphism $V \otimes_k K \to V'$ is an isomorphism.

Proof of Hilbert 90. Take a 1-cocyle, $c : G \to K^\times$. Think of K as a 1-dimensional K-vector space and choose a distinguished vector v. We define a semilinear G-action on K by sending v to $c(g)v$ and extend semilinearly. This is indeed a semilinear G-action due to the cocycle condition. Conversely if one has a semilinear G-action, then one obtains a cocycle by fixing any v

The calculation is: $\phi_g(\phi_h(v)) = \phi_g(c(h)v) = g(c(h))\phi_g(v) = g(c(h))c(g)v$. Hence we have that

$$c(gh) = c(g)g(c(h))$$

which translate to the cocycle condition:

On the other hand, we have that for any v there exists a nonzero G-invariant vector w such that $\lambda w = v$. Which means that

$$c(g)\lambda w = c(g)v = \phi_g(v) = \phi_g(\lambda w) = g(\lambda)\phi_g(w) = g\lambda w$$

, the last staement arising out of G-invariance. Hence $c(g) = \lambda g(\lambda^{-1})$ which means that it is a coboundary. □

4.2 Hilbert Theorem 90 is Cohomology of a Point

We want to show that the above computation actually has a "geometric" interpretation.

Let k be a field and k^{sep} be its separable closure. The Galois group G is a profinite group and we may consider the canonical topology on G-modules (abelian groups with G-action that is continuos), T_G. We first show that:

Proposition 4.6. *Let X be a k-scheme and denote by $X(k')$ the k' points of X where k'/k is a finite extension. Then we have an isomorphism of topologies $Spec\, k_{et} \to T_{Gal(k'/k)}$ given by $X' \to X'(k^{sep})$.*

Corollary 4.7. *We have an equivalence of abelian categories between $Shv_{Spec\, k_{et}} \to T_{Gal}$ sending a sheaf F to $\lim_{k'/k} F(Spec\, k')$*

Now, one can check that $\Gamma(Spec\, k, F) \simeq \Gamma_e(\lim_{k'/k} F(Spec\, k')) \simeq (\lim_{k'/k} F(Spec\, k'))^G$ by using what it means to have a continuous action of a profinite group G on a module.

This tells us that

$$H^q(spec(k), F) = H^q(G, \lim F((spec\, k')))$$

so that Galois cohomology is exactly the etale cohomology of a point. One thing that we have to show is that we have a natural equivalence of functors: $Shv_{X_{et}} \to Ab$ in which one way is $\Gamma_{Spec\, k}$ and $(-)^G \circ \lim\, F(-)$ where $(-)^G$ is taking fixed points which is exactly what group cohomology is about.

Corollary 4.8. *Let k be an algebraically closed field, then $H^q(spec\, k, F) = 0$ for $q > 0$.*

Corollary 4.9. *Let k be a field, then $H^q(spec\, k, G_m) \simeq H^q(Gal(\bar{k}/k), k^\times)$. In particular Hilbert 90 tells us that $H^1(spec\, k, G_m) = 0$.*

However if k is not an algebraically closed field we may have funky answers! For example $H^r(spec\, \mathbb{R}, \mathbb{Z}/2) = \mathbb{Z}/2$ for all $r \geq 0$. This is highly nontrivial! By the way the fact that this is the answer for 1 tells us that there's an interesting $\mathbb{Z}/2$-torsor: namely, $\mathbb{C}$. So let's talk about torsors.

5 H^1 and Torsors

Here H^1 has a geometric interpretation. Recall that a 1-cocyle is for a cover $\mathcal{U}$ of X on the site X_{et} with values in a sheaf of groups G is defined to be a family (g_{ij}) that satisfies compatibility on triple intersections.

Definition 5.1. Let X be a scheme. A G-torsor on X is the data of an etale sheaf of abelian groups G on X and a sheaf of F of sets on X such that (1) there exists a cover of X, $\{U_i \to X\}$ for which $F(U_i) \neq \emptyset$ for all i, (2) whenever $U \to X$ is an etale map, and $s \in \Gamma(U, S)$ we have an isomorphism $G\mid_U \to F\mid_U$ defined by sending $g \mapsto sg$.

Let's given the geometric interpretation immediately:

Proposition 5.2. *We have a bijection from the set of isomorphism classes of G-torsors and $H^1_{et}(X, G)$.*

Proof. Of course, the above assumes that H^1 can be computed using Cech cohomology and independent of cover. So suppose that U_i is a cover for which $F(U_i) \neq \emptyset$ for all i. Pick $s_i \in F(U_i)$ for each i and notice that on interesections, $s_i g_{ij} = s_j$. Now, on triple intersections we have that $s_i g_{ij} g_{jk} = g_{ik} s_k$ and thus the g_{ij} satisfy the cocycle conditions and thus form Cech cocycles. We have to check that isomorphism clases of G-torsors define the same cocycle and then injection and surjection. □

Now the conditions above is defined so that G-torsors are etale-locally trivial. But it turns out that under some conditions on F we have that etale triviality coincides with zariski triviality.

By $L_n(X_\tau)$ we denote isomorphism classes of locallly free sheaves of rank n on X with the topology τ.

Theorem 5.3 (Hilbert 90 is a Geometric Statement)**.** *We have a canonical isomorphism between $L_n(X_{et})$ and $L_n(X_\tau)$*

Sketch Proof. Unsurprisingly, the ideas of descent comes in to play again. Given a Zariski sheaf F, we may define an etale sheaf $F^{et}(U)$ by first defining at the presheaf $U \mapsto \Gamma(U, F)$ and then taking associated sheaf. In order to show the correspondence above we need to show three things:

1. Every locally free sheaf of $O_{X_{et}}$-module is of the form M^{et} for some coherent sheaf M of $O_{X_{Zar}}$-module
2. If M is a coherent sheaf of $O_{X_{Zar}}$-module then if M^{et} is locally free then so was M
3. $M = N$ as locally free sheaves of $O_{X_{zar}}$-module if and only if $M^{et} = N^{et}$

The first one is again a descent problem.

□

This gives us yet another proof of Hilber theorem 90.

Corollary 5.4. $0 = Pic(Spec\, k) = H^1(X_{et}, G_m)$

Alternatively we may use the Leray spectral sequence which starts with $H^p(X_{Zar}, R^q\pi_* G_m)$ converging to $H^{p+q}(X_{et}, G_m)$. This spectral sequence gives an edge homomorphism $H^1(X_{zar}, G_m) \to H^1(X_{et}, G_m)$ which is an isomorphism if we have collapse. It suffices then to check that the stalks of $R^1\pi_*(G_m)$ are zero which means that we have to compute $H^1(Spec\, O_{X,x}, G_m)$ which is zero.

6 Etale Cohomology of a Curve

Outline:

Firstly, the etale cohomology of a point may be computed in the following way. If X is a smooth projective variety over an algebraically closed field k and $K := k(X)$, i.e. the rational functions on X, then we have Tsen's theorem which is a generalization of Hilbert theorem 90:

Theorem 6.1. *If $K = k(X)$ then $H^i(Gal(K^{sep}/k), K^*) = 0$ for all $i > 0$.*

Now, the above can be interpreted as a purely algebraic computation. To make it a geometric one, we need to make sense of the statement about canonical topologies on the category on $Gal(K^{sep}/k)$-modules and the etale topology over a point. That was done:

Theorem 6.2. *$H^i(X, j_*G_m) \cong H^i(Gal(K^{sep}/K), K^*)$.*

which identifies the etale cohomology of a curve with a constant coefficient system as above and Galois cohomology which is the cohomology of its generic point.

An analogous (but, really, the content is different) analysis will also give us:

Theorem 6.3. *$H^i(X_{et}, Z) = 0$ for $i > 0$*

The content of the above theorem is also a local-to-global thing. To piece all our computations together we need the Weil divisor sequence:

$0 \to G_m \to j_*G_m \to \oplus_p Z \to 0$ and look at the long exact sequence in cohomology.

Therefore our answer is:

Theorem 6.4. *$H^i(X_{et}, G_m) = k^*$ if $i = 0$, $Pic(X)$ if $i = 1$ and 0 for $i > 1$.*

Let's carry out this program.

Theorem 6.5 (Weil Divisors Exact Sequence)**.** *Let X be a connected normal variety (all its local rings are integrally closed) and $g : \eta \to X$ be the inclusion of the generic point of X into X and $i_z : z \to X$ be the inclusion of a point z into X. Then there is an exact sequence of Zariski and etale sheaves:*

$$0 \to G_m \to g_*G_m \to \oplus_{z\, closed} i_{z*}\mathbb{Z} \to 0$$

.

In general, this is derived from the exact sequence $0 \to O_X^\times \to K^\times \to Div \to 0$. One notes a divisor as an integral, closed codimension 1 subscheme is completely determined by its generic point — so that an open set U contains z if and only if $U \cap Z$ is nonempty and a codimension subscheme of U. Therefore $\Gamma(U, \oplus_{codim\,1} i_{*,z}\mathbb{Z}) = Div(U)$. By the way the global sections of these sheaves are: $0 \to \Gamma(X, O_X^\times) \to K^\times \to Div(X)$ and the last map sends a rational function $f \mapsto div(f) = \Sigma_z ord_z(f)[z]$

The Zariski case is well known and we can show that if a sequence of etale sheaves on X_{et} is exact when retricted to U_{zar} for all $U \to X$ etale, then it is also exact on X_{et}

Lemma 6.6. $H^r(X_{et}, \oplus_{z\,closed} i_{z*}\mathbb{Z}) = 0$ *for* $r > 0$

Proof. Step by step:

1. For z a closed point, i_* is exact by computing the stalks (the etale stalks are trickier: if $i : Z \to X$ is a closed immersion and F a sheaf on Z, then $i_* F_{\bar{x}} = \oplus_{y \mapsto x} F_{\bar{y}}^{d(y)}$ where $d(y)$ is the separable degree of $k(y)$ over $k(x)$ — in the Zariski case the stalks are like delta functions). Furthermore, i has an exact left adjoint in the etale topology (again, not true in general — but flatness helps us out here) which means that it preserves injectives just by nonsense.
2. This tells us that $H^r(X_{et}, i_{z*}\mathbb{Z}) = H^r(z_{et}, \mathbb{Z})$ and geometric points satisfy the dimension axiom (this requires the identification of cohomology of a point with Galois cohomology)
3. Now cohomology commutes with direct sum in our case so we have the lemma.

□

Lemma 6.7. $H^r(X_{et}, g_* G_m) = 0$ *for* $r > 0$.

Proof. Step by step again:

1. I claim that this reduces to the computation on a generic point. I would be okay if the point was closed but it is not. The Leray spectral sequence measures this failure: $H^p(X_{et}, R^q g_* G_m) \Rightarrow H^{p+q}(\eta_{et}, G_m)$. In particular we have the comparison of the two groups via edge morphisms: $H^p(X_{et}, g_* G_m) \to H^p(\eta_{et}, G_m)$
2. To collapse, we have to compute $R^r g_* G_m$. This guy is a sheaf. It is zero if it is locally zero (the stalks over all geometric points are zero). This deserves a separate discussion below, but let's assume this for now so that we just have to compute: $H^p(\eta_{et}, G_m)$
3. Since η is the generic point $\eta = Spec\, K$ where K is the field of rational functions on X
4. By identification of cohomology of a point and Galois cohomology $H^p(\eta_{et}, G_m) = H^p(Gal(K^{sep}/K), (K^{sep})^\times)$
5. For $k = 1$, this is Hilbert theorem 90.
6. For $k > 1$ this is Tsen's theorem which we will blackbox and make a comment that $k = 2$ is the Brauer group of K. This can be found in Shatz: profinite groups, arithmetic and geometry.

□

Theorem 6.8 (Cohomology of Curves with Coefficient in the Multiplicative Group)**.** *For a connected nonsingular curve X above over an algebraically closed field, $H^0(X_{et}, G_m) = \Gamma(X, O_X^\times), H^1(X_{et}, G_m) = Pic(X) = Div(X)/K^\times$ and all other groups are zero*

7 The Kummer Sequence, the Picard group of a Curve and the Final Answer

We aren't done yet. We still would like to get the result that we know and love about cohomology of surfaces over the complex numbers. Where's the genus?

This computation comes out of Kummer theory. Here's a punchline: *the Kummer exact sequence used to compute etale cohomology of curves will not usually be exact in the Zariski topology* and this arises because *the etale stalks contain more algebraic information than the Zariski stalks.*

Let X be connected and normal over an algebraically closed field and let n be an integer that is invertible in all fields in sight.

Lemma 7.1. *The following sequence of sheaves is exact:* $0 \to \mu_n \to G_m \to G_m \to 0$ *where the second to last map is defined by* $t \mapsto t^n$

Proof. We claim that the sequence is exact on stalks at geometric points: $0 \to \mu_n \to O^{\times}_{X_{et},x} \to O^{\times}_{X_{et},x} \to 0$ and recall that $O^{\times}_{X_{et},x}$ is a strictly Henselian local ring (the ring is Henselian and the residue field is separably algebraically closed). By Henselian property we see that the polynomial $T^n - a$ for a a unit is separable (the derivative being nonzero because n is invertible!) and therefore we can solve the polynomial in the ring by solving it in the residue field. □

7.1 Picard Group of a Curve

If you still think at this point that our computation is abstract nonsense, then let me convince you otherwise. The point is that we need to really understand the structure of line bundles on a curve — its Picard group. This is serious geometry. Let me add one more assumption to my curve — I want it to be proper (over the base field). Such varieties are called complete varieties. The key idea is that the map $X \to Spec\, k$ is universally closed so that any base change of this map is closed. This is the analog of a map where the inverse image of a compact set is compact. Denote by Pic^0, the quotient of $Div^0(X)$ (divisors of degree 0) modulo the principal divisors (which are degree 0) since rational functions have degree 0. The punchline here is that to understand Pic, we might as well understand Pic^0. The latter can be computed more geometrically as per the proof presented below.

Proposition 7.2. *For any integer* n *relatively prime to the characteristic of* k, $z \mapsto nz : Pic^0(X) \to Pic^0(X)$ *is surjective with kernel a free* Z/nZ *module of rank* $2g$.

Fake Proof. Suppose we are working over the complex numbers. First, choose a basis $\omega_1, ..., \omega_g$ of holomorphic differentials on X (these correspond to a basis on H^2), and a basis $\gamma_1, ..., \gamma_{2g}$ for H^1. Denote by Γ the subgroup of $\mathbb{C}^g$ generated by vectors $V_i := (\int_{\gamma_i} \omega_1, ..., \int_{\gamma_i} \omega_g)$ for $i = 1, ..., 2g$. For each points z_0, z_1 on the surface we choose a path $\gamma 01$ and let $I(z_0, z_1) := (\int_{\gamma 01} \omega_1, ...) \in \mathbb{C}^g$. Its image in $\mathbb{C}^g/\Gamma$ is independent of choice of paths.

Define a map: $Div^0(X) \to \mathbb{C}^g/\Gamma$ by $[z_1] - [z_0] \mapsto I(z_0, z_1)$ and extend linearly. A classical theorem of Abel says that D is principal if and only if $i(D) = 0$ and i is onto by a classical theorem of Jacobi. Therefore we have an isomorphism: $Pic^0(X) \to \mathbb{C}^g/\Gamma$.

So we are looking at a map $\mathbb{C}^g/\Gamma \to \mathbb{C}^g/\Gamma$ which is multiplication by n. This is surjective with kernel $(1/n)\Gamma/\Gamma \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ (should try to explain better) □

We have an exact sequence $0 \to Pic^0(X) \to Pic(X) \to \mathbb{Z} \to 0$. This lets us compute things like $ker(Pic(X) \to Pic(X)$ and $coker(Pic(X) \to Pic(X))$ in terms of Pic^0

7.2 Putting it all together

Theorem 7.3. *Let X be a complete, connected nonsingular curve over an algebraically closed field, then $H^0(X_{et}, \mu_n) = \mu_n(k)$, $H^1(X_{et}, \mu_n) = (Z/nZ)^{2g}$, $H^2(X_{et}, \mu_n) = Z/nZ$ and $H^r(X_{et}, \mu_n) = 0$ for $r > 0$.*

8 Abelian Varieties = Tori in complex land

Here's a nice result:

Theorem 8.1. *Let A be an abelian variety of dimension g over a separably closed field and let ℓ be invertible. The cup product pairings define isomorphisms $\Lambda H^1(A_{et}, \mathbb{Z}_\ell) \to H^r(A_{et}, \mathbb{Z}_\ell)$ for all r. Hence $H^r(A_{et}, \mathbb{Z}_\ell)$ is a free $\mathbb{Z}_{ell}$-module of rank $\binom{2g}{r}$*

The proof of this theorem isn't so bad. If A is an abelian variety, then we have an addition map $m : A \times A \to A$ which indues a map on cohomology: $m^* : H^*(A) \to H^*(A \times A) \simeq H^*(A) \otimes H^*(A)$. One checks that this defines a graded Hopf algebra structure on the cohomology ring and we can just check against the classification theorem for Hopf algebras (they tell us that H^* is canonically isomorphic to an exterior algebra on H^1).

9 The Weil Conjectures

Although I am not a number theorist but I feel like the best math around really revolves around the Weil conjectures. This is just one guy's very biased perspective — but it's such an amazing way to prove a theorem. I call this the value of wishful thinking.

Here's the point: I am interested in the number of solutions of polynomial equations over finite fields; let's fix $k = F_q$. That's a reasonable thing to study. It's much easier, however, to study the variety that this polynomial cuts out, let's call it X, over its algebraic closure. So we are looking at

$$\bar{X} = X \times_k \bar{k}$$

where $\bar{k}$ is the algebraic closure of k. For each integer r, denote by N_r the number of points of $\bar{X}$ which are rational over F_{q^r}. This is the number of points of $\bar{X}$ whose coordinates lie in k_r. The case of $r = 1$ is really asking for how many points have coordinates that lie in the field that we had started with. One strategy to tackle this problem is to look form the zeta function: $Z(t) = Z(X; t) = \exp(\Sigma N_r(t^r/r))$. This seems unweildy — it lives in $Q[[t]]$ — but at least this really encodes what we are interested in. Let's look at an example. If $X = P^1$, then P^1 has one more element than the number of elements in the field (think one-point compactification of the line) — so that $N_r = q^r + 1$. Now, $Z(P^1, t) = \exp(\Sigma(q^r + 1)t^r/r)$ and mucking around tells us that this expression is equal to $1/(1 - t)(1 - qt)$. So it is natural to ask: is $Z(t)$ a rational function? A priori this has nothing to do with cohomology, but one way to rephrase this problem is to a question about fixed points.

Okay a side comment: we rig (choice word here) Galois extensions so that the fixed field of our Galois group is exactly the field that we started with. So, a priori, we already know what the fixed point of our action is. So that's where fixed points come into the picture. We define the frobenius: $f : \bar{X} \to \bar{X}$ by sending the point P with coordinates (a_i), $a_i \in \bar{k}$ to (a_i^q). Now P is a fixed point of f if and only if its coordinates lie in k (that's what I meant by rigged, I guess). More generally, P is a fixed point of f^r if and only if it lies in the field F_{q^r}. Thus we have that N_r = number of fixed points of f^r. Who are the experts at fixed points? Topologists! We have long developed the machinery of fixed point detecting and fixed point counting by way of cohomology. So one prays for a cohomology theory that comes equipped with such a thing!

Such a thing is ℓ-adic cohomology theory of Grothendieck. We've seen coefficients in finite groups, now we can define:

$$H^i(X, \mathbb{Q}_\ell) := \varprojlim H^i(X_{et}, Z/\ell^r) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

. These guys behave a lot like the cohomology we know and love:

1. We have a cup product structure: $H^i(X, \mathbb{Q}_\ell) \otimes H^j(X, \mathbb{Q}_\ell) \to H^{i+j}(X, \mathbb{Q}_\ell)$
2. There is poincare duality: if X is smooth, complete variety of dimension n, then $H^{2n}(X, \mathbb{Q}_\ell)$ is 1-dimensional and the cup-product pairing $H^i(X, \mathbb{Q}_\ell) \otimes H^{2n-i}(X, \mathbb{Q}_\ell) \to H^{2n}(X, \mathbb{Q}_\ell)$ is a perfect pairing.

Furthermore we also have a comparison theorem in this setting:

Theorem 9.1 (Comparison). *If X is smooth and proper over the complex numbers the $H^i(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \simeq H^i(X_{complex}, \mathbb{C})$*

Of course what makes all even better is the Lefschetz fixed point formula

Theorem 9.2 (LFPT). *Let X be a smooth proper variety over k and $f : X \to X$ a morphism with isolated fixed points with "multiplicity 1", let the number of fixed points be denoted $L(f, X)$, then we have that*

$$L(f, X) = \Sigma(-1)^i Tr(f^*)$$

Anyway playing around with power series and plugging in the appropriate values gives us rationality! In any case, we know how a really great standard technique to understand something as monstrously complicated as, say, solutions to Diophantine equations!

10 The Bloch-Kato Conjectures

This is I guess the point of the talk in relation to motivic homotopy theory. Recall the Kummer sequence above: $0 \to \mu_\ell \to G_m \to G_m \to 0$ where ℓ is an invertible element in the field I am working over. Now if we are looking at this over a field $Spec\, k$, then we have that $H^1(Spec\, k, G_m) = 0$ by Hilbert 90 so our long exact sequence breaks up into $k^\times \to k^\times \to H^1(k, \mu_\ell)$. We are interested in the connecting homomorphism $\partial : k \to H^1(k, \mu_\ell)$. One checks that we can extend this homomorphism into $\partial^n : k^\times \times ... \times k^\times \to H^1(k, \mu_\ell^{\otimes n})$ and it satisfies: $\partial^n(..., a, ..., 1-a, ...) = 0$.

The Milnor K-groups are defined by $K_n^M(k) = T^n(k)/(a \otimes (1-a))$ so we have a well map for each n: $\partial_n : K_n^M(k) \to H^1(k, \mu_\ell^{\otimes n})$. We ask if this is an isomorphism. The case of $k = 2$ is due to Suslin and Merkurjev and the higher cases was proven by Voevodsky and collaborators using motivic homotopy theory. To make an even more explicit connection the following is a conjecture of Beilinson and Lichtenbaum which turns out to be equivalent:

Let X be a smooth variety over a field containing $1/\ell$, then the motivic cohomology group $H^{p \cdot q}(X, Z/\ell) \cong H^p(Spec\, k_{et}, \mu_\ell^{\otimes q})$

11 A Skinny on Derived Functors of Direct Image

This section is meant to make sense of the computation that

Lemma 11.1. $(R^r g_* G_m))\bar{x} = 0$ *for* $r > 0$

First, we note that:

Proposition 11.2. *For any morphism of schemes:* $\pi : Y \to X$ *and* F *a sheaf on* Y_{et} *then* $R^r\pi_*F$*, as a sheaf on* X_{et} *is the associated sheaf to the presheaf* $U \mapsto H^r(U \times_X Y, F)$*. Hence the stalk of* $R^r\pi_*F$ *at a geometric point* $\bar{x} \to X$ *is computed as* $\lim H^r(U \times_X Y, F)$ *where* U *are etale neighborhoods of* $\bar{x}$*.*

This can be a little tricky when you see it at first — it is telling us that the right derived functors (this is now a sheaf! We can derive functors valued in a category of sheaves) are computed as the "local" cohomology of the etale opens. A little thought buys us the desired result.

12 Spectral Sequences

Theorem 12.1 (Composition of Functors Spectral Sequence)**.** *Let* A, B, C *be abelian categories such that* A, B *have enough injectives. Let* $F : A \to B$ *and* $G : B \to C$ *be left exact functors such that* $R^rG(FI) = 0$ *for* $r > 0$ *and* I *injective (i.e.* FI *is* G*-acyclic for all* I*), then for each object* X *of* A *there exists a spectral sequence:*

$$E_2^{p,q} = (R^pG)(R^qF)(X) \Rightarrow R^{p+q}(GF)(X)$$

.

Nice cases happen when F takes injectives to injectives (happens whenever F admits an exact left adjoint!)

by general bla:

Proposition 12.2. *This is a first quadrant spectral sequence and hence we have:*

1. *edge morphism:* $E_2^{p,0} = R^p(G)(F(X)) \to R^p(GF)(X) = E_\infty^p$
2. *edge morphism:* $E_\infty^p = R^q(GF)(X) \to E_2^{0,q} = G(R^qF(X))$
3. *An exact sequence* $0 \to E_2^{1,0} \to E_\infty^1 \to E_2^{0,1} \to E_2^{2,0} \to E_\infty^2$ *where the second to last arrow is a differential.*

Theorem 12.3 (Leray Spectral Sequence)**.** *Let* $\pi : Y \to X$ *be a morphism of schemes. For any etale sheaf* F *on* Y *there is a spectral sequence* $H^p(X_{et}, R^q\pi_*F) \Rightarrow H^{p+q}(Y_{et}, F)$

Proof. We are looking at the functors π_*, which preserves injectives in the etale case and always left exact, and $\Gamma(X, -)$ which is left exact. □

Theorem 12.4 (Zariski to Etale Spectra Sequence)**.** *Let* $F \in Shv_{X_{et}}$ *and let* $\epsilon : Shv_{X_{et}} \to Shv_{X_{zar}}$ *be the forgetful functor (which is left exact). Then there is a spectral sequence* $H^p(X, R^q\epsilon(F)) \Rightarrow H^{p+q}(X_{et}, F)$*.*

Proof. We are looking at the functors ϵ, which sends injectives to acyclics and always left exact, and $\Gamma(X, -)$ which is left exact. □

13 Some References

Lei Fu has a book on etale cohomology and that is very detailed and gives a very terse but comprehensive treatment. Gunter Tamme's book and Milne's notes are excellent — gives less details and more ideas. The stacks projects has an exposition too at the same level but it does read like an encyclopeadia (duh). For expository articles look up Tom Sutherland's notes and Donu Arapura's course notes, Edgar Costa's master's thesis, Evan Jenkins' notes for background. All these were used in the production of these notes.