James Construction

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(1) Loop spaces have extra algebraic structure coming from concatenation of loops. In particular, they are act like (associative, unital) monoids up to homotopy. On the one hand, any pointed map $X \to \Omega Z$ factors through $X \to \Omega\Sigma X$, so $\Omega\Sigma X$ is the free loop-space on X. On the other hand, there is an evident topology on the free monoid on the pointed set X, called $J(X)$. As a space this is given by

$$J(X) := \left(\bigvee_{n \geq 0} X^{\times n} \right) / \sim$$

where the relation says that $(x_1, \cdots x_k, * \wedge x_{k+1}, \cdots) \sim (x_1, \cdots x_k, x_{k+1}, \cdots)$.

This is often called the James construction, and it has an evident filtration by the images of $\bigvee_{0 \leq n \leq r} X^{\times n}$, which we denote $J_r(X) \subset J(X)$. The purpose of this talk is to prove the following two theorems about $\Omega\Sigma X$:

Theorem 1.1 (James)**.** *The spaces $J(X)$ and $\Omega\Sigma X$ are naturally weakly equivalent when X is connected.*

Theorem 1.2 (James, Milnor)**.** *There is a splitting of $\Sigma J(X)$ and of $\Omega\Sigma X$ after a single suspension:*

$$\Sigma J(X) = \Sigma\Omega\Sigma X \cong \Sigma X \vee \Sigma X^{\wedge 2} \vee \Sigma X^{\wedge 3} \vee \cdots$$

Remark 1.3. The connectedness assumption is related to the fact that $\pi_0 \Omega\Sigma X$ is a group in general while $\pi_0 J(X)$ is only a monoid. On the other hand, the monoid with one element happens to be a group.

(2) The plan for the first theorem is to build a space that interpolates between $J(X)$ and $\Omega\Sigma X$ which we'll call $\mathrm{Free}^*_{\mathcal{A}_\infty}(X)$, for reasons which will become clear in later lectures. We will then build natural maps

$$JX \leftarrow \mathrm{Free}^*_{\mathcal{A}_\infty}(X) \to \Omega\Sigma X$$

and show that each one is a weak equivalence when X is connected.

The idea is that the only thing in the way of $\Omega\Sigma X$ being an honest monoid is that we must choose a way of reparameterizing paths so that the domain of our paths remains the *unit* interval.[1]

In other words, if we tried to construct a map directly $JX \to \Omega\Sigma X$ we'd have to send the word $x_1 x_2 \cdots x_k$ to the concatenation of each of the loops around each x_i, reparameterized in some way over $[0, 1]$. Since there's no canonical choice we will just make *all* the choices.

Definition 2.4. Let $\mathbb{D}^1$ denote the unit interval and let $\mathrm{Rect}(\mathbb{D}^1, \mathbb{D}^1)$ denote the subspace in $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ of injective affine linear transformations $\mathbb{R} \to \mathbb{R}$ which send $\mathbb{D}^1$ to a subset of $\mathbb{D}^1$. We call such a map a rectilinear embedding. Let $\mathcal{A}_\infty(j)$ denote the subspace of $\prod_{1 \leq i \leq j} \mathrm{Rect}(\mathbb{D}^1, \mathbb{D}^1)$ where the images of the j copies of $\mathbb{D}^1$ are pairwise disjoint *and in order.*[2]

There is now an evident pointed map:

$$\bigvee \mathcal{A}_\infty(j) \times X^{\times j} \to \Omega\Sigma X$$

[1] In this case there is a trick called the 'Moore loop space' but we will ignore it so that the treatment more closely parallels the generalizations in later talks.

[2] This slightly conflicts with notation we'll use later, but it shouldn't be so bad.

where we use the left hand coordinate to choose a way to concatenate loops. However, the left hand side is too big because it does not encode how the basepoint behaves like a unit properly. It's easy to see how to fix it though: if one of our paths is the constant loop, then we can just ignore it in our reparameterization. Explicitly, there are maps

$$\sigma_i : \mathcal{A}_\infty(j) \to \mathcal{A}_\infty(j-1)$$

which forget the ith copy of the disk in the rectilinear embedding. So we impose the relation

$$(\gamma, x_1, \cdots, x_{i-1}, *, x_{i+1}, \cdots, x_j \sim \sigma_i\gamma, x_1 \cdots, x_{i-1}, x_{i+1}, \cdots x_j.$$

The resulting space is defined to be

$$\mathrm{Free}^*_{\mathcal{A}_\infty}(X) := \left(\bigvee \mathcal{A}_\infty(j) \times X^{\times j}\right) / \sim .$$

We now have the diagram as promised

$$JX \leftarrow \mathrm{Free}^*_{\mathcal{A}_\infty}(X) \to \Omega\Sigma X$$

where the left-hand arrow comes from collapsing each $\mathcal{A}_\infty(j)$ to a point.

Lemma 2.5. *The map* $\mathrm{Free}^*_{\mathcal{A}_\infty}(X) \to JX$ *is a weak equivalence.*

Proof. It's enough to show that each space $\mathcal{A}_\infty(j)$ is contractible. Given a rectilinear embedding in $\mathcal{A}_\infty(j)$, we can record the image of the origin in each interval and we get an ordered collection of points $p_1 < \cdots < p_j$ inside the unit interval. But of course $\{p \in [0,1]^j : p_1 < \cdots < p_j\}$ is the open j-simplex, which is contractible. It is not difficult to see that the map from $\mathcal{A}_\infty(j)$ to this simplex is a deformation retract, so we're done. □

So we just need to show that the right hand map is a weak equivalence when X is connected.

(3) The loopspace $\Omega\Sigma X$ acts on $P\Sigma X$, the space of paths on ΣX which begin at the basepoint, by precomposition. As before, this action really only makes sense up to higher homotopies. We will need an analogous action of $\mathrm{Free}^*_{\mathcal{A}_\infty}(X)$ on some contractible space. To motivate the construction let's unwind how the action of $\Omega\Sigma X$ on $P\Sigma X$ works.

Let CX denote the reduced cone on X and view ΣX as CX/X. Then, given a loop β and a path α, we get a new path $\alpha * \beta$ which would like to be parameterized over two copies of the unit interval. So, in fact, we need a choice of rectilinear embedding $\mathbb{D}^1 \amalg \mathbb{D}^1 \to \mathbb{D}^1$ which preserves the order of the intervals.

Expanding on this theme, we are lead to consider the subspace

$$\mathrm{Free}^*_{\mathcal{A}_\infty}(X, CX) \subset \mathrm{Free}^*_{\mathcal{A}_\infty}(CX)$$

consisting of points $(e : \coprod \mathbb{D}_1 \hookrightarrow \mathbb{D}_1, y_1, ..., y_j)$ such that $y_1, ..., y_{j-1} \in X$.

This is meant to be reminiscent of the bar construction: if M is a monoid and Y is an M-set then the action is encoded by the simplicial set with j-simplices given by $M^{\times j} \times Y$.

Now, collapsing $X \subset CX$ gives a commutative diagram:

$$\begin{array}{ccccc} \mathrm{Free}^*_{\mathcal{A}_\infty}(X) & \longrightarrow & \mathrm{Free}^*_{\mathcal{A}_\infty}(X, CX) & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \| \\ \Omega\Sigma X & \longrightarrow & P\Sigma X & \longrightarrow & \Sigma X \end{array}$$

The proof of Theorem 1.1 is now reduced to two lemmas.

Lemma 3.6. *The space* $\mathrm{Free}^*_{\mathcal{A}_\infty}(X, CX)$ *is contractible. Thus, the middle vertical arrow is a homotopy equivalance.*

Proof. Filter by j and show that each step in the filtration deformation retracts onto the previous one. The idea is that the last coordinate is just a path in CX, which is contractible, so we can just reel it in to the basepoint and we're back down a step in the filtration. □

The next lemma is the crucial one, but we skip it because it's technical. The idea is to flesh out the feeling that $\mathrm{Free}^*_{\mathcal{A}_\infty}(X, CX)$ is a torsor for $\mathrm{Free}^*_{\mathcal{A}_\infty}(X)$.

Lemma 3.7. *The top row is a quasi-fibration when X is connected. (That is, the map from the fiber to the homotopy fiber is a weak equivalence.)*

(4) Finally, we indicate a couple approaches to the James-Milnor splitting. On the one hand, we could approach this via the James construction (when X is connected).

Lemma 4.8. $\Sigma(X \times Y) \cong \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$.

Proof. We have a cofiber sequence

$$\Sigma X \vee \Sigma Y \to \Sigma(X \times Y) \to \Sigma(X \wedge Y)$$

and we need to show that it splits. □

Using this lemma we can inductively split the cofiber sequences

$$J_{r-1}(X) \to J_r(X) \to X^{\wedge r}$$

after suspending.

Alternatively, we can work directly with $\Omega\Sigma X$. We will show that

$$\Sigma\Omega\Sigma X \cong \Sigma(X \wedge \Omega\Sigma X) \vee \Sigma X$$

and then iteratively expand the right hand side. I learned the following argument from Gijs Heuts.

The first observation is that the possibly funky right hand side arises for a very simple reason.

Lemma 4.9. *There is a homotopy pushout diagram of pointed spaces:*

$$\begin{array}{ccc} X \times Y & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \longrightarrow & \Sigma(X \wedge Y) \vee \Sigma Y \end{array}$$

where the left vertical map is projection.

Proof. Consider the diagram:

$$\begin{array}{ccccc} X \times Y & \longrightarrow & Y & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \Sigma(X \wedge Y) & \longrightarrow & \Sigma(X \wedge Y) \vee \Sigma Y \end{array}$$

Since X and Y are pointed, the first vertical and top horizontal maps admit sections. It follows that, for example, the middle vertical arrow is null. This shows that the right hand square is a homotopy pushout. We leave it to the reader to check that the left hand square is also a homotopy pushout. □

Now take the homotopy pushout square:

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

and pull it back along the map $* \to \Sigma X$. Homotopy pullbacks preserve homotopy pushouts in pointed spaces, so we get a homotopy pushout diagram:

$$\begin{array}{ccc} X \times \Omega\Sigma X & \longrightarrow & \Omega\Sigma X \\ \downarrow & & \downarrow \\ \Omega\Sigma X & \longrightarrow & * \end{array}$$

In particular, the cofibers of the horizontal maps are equivalent, so by the lemma:

$$\Sigma\Omega X \cong \Sigma(X \wedge \Omega\Sigma X) \vee \Sigma X.$$

Notice that this decomposition after suspension implies that

$$\widetilde{H}_*(\Omega\Sigma X) \cong \bigoplus_{j>0} \widetilde{H}_*(X)^{\otimes j}$$

for homology with field coefficients. Moreover, the tensor algebra structure agrees with the product coming from the H-space structure on $\Omega\Sigma X$. It follows formally from a diagram chase that the map $\mathrm{Free}^*_{\mathcal{A}_\infty}(X) \to \Omega\Sigma X$ induces an isomorphism on reduced homology with any field coefficients. This gives another proof of Theorem 1.1, as promised.