May 26: Anton Kapustin: Three-dimensional topological field theories with boundaries

2d TFT: Mirror symmetry	3d TFT:	4d TFT: S-duality
A model B model	3d mirror symmetry	Gauge theory G Gauge theory $G^\vee$

$d \doteq$ dimension

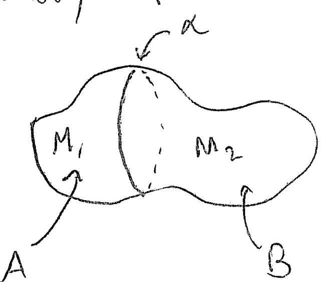
$$\underset{\text{space}}{M_{d-1}} \times \underset{\text{time}}{\mathbb{R}} \longmapsto \underset{\text{Hilbert space of theory}}{V}$$

$\partial M_{d-1} \neq \emptyset \implies$ need boundary conditions

$(M_{d-1} \times \mathbb{R}, \vec{\alpha}) \rightsquigarrow V_{\vec{\alpha} = (\alpha_1, \dots)}$

one condition for each boundary component

Domain walls divide manifold into parts

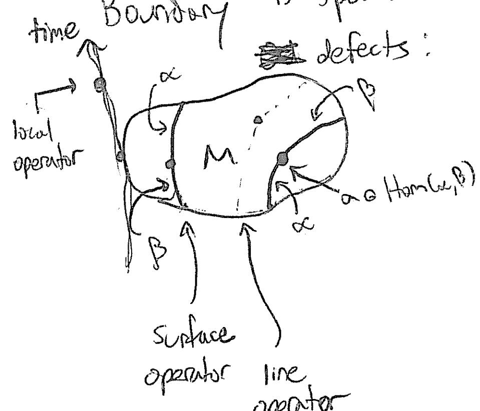


$\rightsquigarrow V_{A,B,\alpha}$

e.g. "phase boundaries"

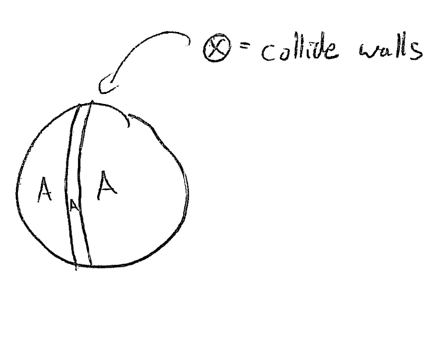
Boundary is special case (it is a wall between our theory and the "trivial" field theory.)

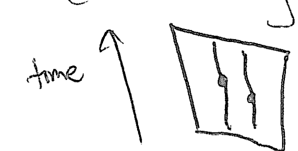
~~defects~~ defects:



tricategory:
- object: phases
- 1-mor: domain walls
- 2-mor: line operators
- 3-mor: local operators

For any particular 3d TFT, we have
(1) monoidal 2-category of domain walls
(2) 2-category of boundary conditions

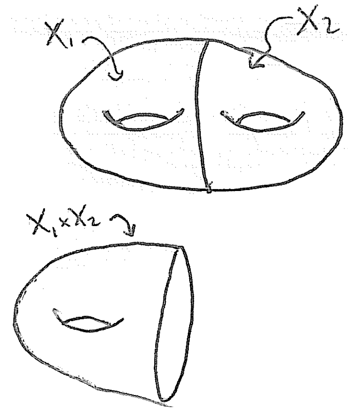




2-category of B-models: objects: Calabi-Yau manifolds $X_1, X_2, \ldots$

morphisms: $\mathrm{Hom}(X_1, X_2) = D^b(\mathrm{Coh}(X_1 \times X_2))$

(composition as kernels of functors)



We will need a generalization of this: 2-category of curved B-models:

objects: $(X, W \in \Omega^{0,\mathrm{even}}, \bar\partial W = 0)$

X: CY manifold; W: "curving"

morphisms: $\mathrm{Hom}((X_1, W_1), (X_2, W_2))$ = category of curved ~~B-branes~~ B-branes on $(X_1 \times X_2, W_1 - W_2)$

$\alpha \in \mathrm{Ob}(\mathrm{Hom}(\ldots))$

$\alpha = (E, D : \Gamma(E \otimes \Omega^{0,\cdot}) \to \Gamma(E \otimes \Omega^{0,\cdot}))$

$\deg_{\mathbb{Z}_2} D = 1, \quad D^2 = 1 \cdot (W_1 - W_2)$

E: $\mathbb{Z}_2$-graded vector bundle on $X_1 \times X_2$

L. Rozansky, N. Saulina, A. Kapustin:

(1) Rozansky-Witten model (review)

$(X, \Omega) \rightsquigarrow$ action

$(X, \Omega) \rightsquigarrow$ 2-category of boundary conditions, monoidal 2-category of domain walls, braided monoidal category of codimension-2 defects = End($\mathbb{1}$) in

(X, Ω): complex symplectic manifold

$\varphi : M \to X$

$\rho \in \Gamma(\varphi^* TX \otimes \Omega^1_M)$

~~RW, Roberts~~

Rozansky-Witten, Roberts-Willerton:

Category of Wilson lines $\cong D^b_{\mathbb{Z}_2}(\mathrm{Coh}(X))$

We study the 2-category of boundary conditions.

__Remark__ RW theory on a 3-manifold of the form $S^1 \times \Sigma$ is equivalent to the B-model with target space X. But B-model does not show the braided monoidal structure.

Simplest objects of 2-category of boundary conditions:

Complex Lagrangian submanifolds $Y \hookrightarrow X$.

A slight generalization: (Y, W) $\qquad$ $\mathrm{Hom}((Y_1, W_1), (Y_2, W_2)) = ?$

W = curving

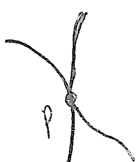
Remark (1) RW is perturbative
(2) $\mathbb{Z}_2$-graded

Extreme cases $(\hbar = 0)$

(1) $Y_1 = Y_2$. $\quad \mathrm{Hom}((Y, W_1), (Y, W_2)) = D^{*}_{\mathbb{Z}_2}(\mathrm{Coh}(Y), W_1 - W_2)$

(2) $W_1 = W_2$. $\quad \mathrm{Hom}(Y_1, Y_2) = D_{\mathbb{Z}_2}(\mathrm{Coh}(Y))$

(3) Y_1 & Y_2 intersect at isolated points. (Let $W_1 = W_2 = 0$ for simplicity.)

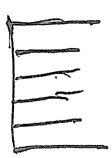


$$\mathrm{Hom}_p(Y_1, Y_2) = D_{\mathbb{Z}_2}(\underset{\mathbb{C}^n}{\overset{\cap}{U}}, F_1 - F_2)$$

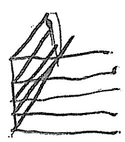
Choose Darboux coordinates near p s.t. $\Omega = dx_i \wedge dy^i$

$$Y_1 \cap V = \left\{ y_i = \frac{\partial F_1}{\partial x^i} \right\}$$
$$Y_2 \cap V = \left\{ y_i = \frac{\partial F_2}{\partial x^i} \right\}$$

More general boundary conditions



1d TFT = vector space (i.e., vector bundle)



2d TFT

One can associate a bdy condition to a fibration over $Y \hookrightarrow X$ whose fibre is a CY, Z.

$$\begin{array}{c} Z \longrightarrow \mathcal{Y} \\ \qquad\downarrow \\ \qquad Y \end{array}$$

$$\mathrm{Hom}\left(\begin{array}{c}\mathcal{Y}_1 \\ \downarrow \\ Y\end{array}, \begin{array}{c}\mathcal{Y}_2 \\ \downarrow \\ Y\end{array} \right) = D_{\mathbb{Z}_2}(\mathcal{Y}_1 \times_Y \mathcal{Y}_2)$$