May 26: Dennis Gaitsgory: Global ~~Langlands~~ Geometric Langlands

I. Introduce categories
II. KL & localization functor
III. Whittaker category & quantum geometric Langlands conjecture
IV. The classical case

$\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ symmetric invariant form

$\kappa + \frac{1}{2}\cancel{\mathrm{kill}}^{\kappa_{\mathrm{killing}}}\big|_{\mathfrak{h}} : \mathfrak{h} \otimes \mathfrak{h} \to \mathbb{C}$ $\qquad h^\vee \otimes h^\vee \to \mathbb{C}$

$$\Big\downarrow$$

$$-\kappa^{-1} \quad \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \to \mathbb{C}$$

Bun_G stack of ~~infinity~~ infinite type $\longrightarrow$ write Bun_G as union of stacks of finite type

$D(\mathrm{Bun}_G)_\kappa$ twisted differential operators on Bun_G

$D\text{-mod}(\mathrm{Bun}_G)_{\kappa,!}$ ~~Drinfeld: This category~~

<u>Lemma</u> This category is compactly generated.

<u>Def</u> $D\text{-mod}(\mathrm{Bun}_G)_{\kappa,*} := \left(D\text{-mod}(\mathrm{Bun}_G)_{-\kappa,!}\right)^\vee$

$$\downarrow$$

$D\text{-mod}(\mathrm{Bun}_G)_{\kappa,!}$ $\quad$ not an equivalence in general, but is when κ is irrational

$KL(G)_\kappa = (\hat{\mathfrak{g}}\text{-mod}_\kappa)^{G[[t]]}$ $\quad$ (more canonically, $(\hat{\mathfrak{g}}\text{-mod}_\kappa)^{G(\mathcal{O}_x)}$)

"D-module analogue of an E_2 category"

X, x

$$KL(G)_\kappa[\mathrm{Ran}]$$

$$\begin{array}{ccc} KL(G)_{\kappa,*}[\mathrm{Ran}] & & KL(G)_\kappa[\mathrm{Ran}] \\ \mathrm{Loc}\downarrow \; \uparrow \Gamma \text{ (not always)} & & \mathrm{Loc} \downarrow \; \uparrow \Gamma \\ D\text{-mod}(\mathrm{Bun}_G)_{\kappa,!} & & D\text{-mod}(\mathrm{Bun}_G)_{\kappa,*} \end{array}$$

$H \circlearrowright Y$

$\exists \; \mathfrak{h}\text{-mod} \underset{\Gamma}{\overset{\mathrm{Loc}}{\rightleftarrows}} D\text{-mod}(Y)$

$G(K_x) \circlearrowright \mathrm{Bun}_G^{\text{level at } x}$

$\cup \qquad\qquad \shortparallel$

$G(\mathcal{O}_x) \qquad\qquad Y$

<u>Def</u> $\varkappa$ is positive/negative/?/irrational if

$$\varkappa + \tfrac{1}{2}\varkappa_{Killing} = C\cdot \varkappa_{Killing}$$

$$C\in\mathbb{Q}^{>0},\quad C\in\mathbb{Q}^{<0},\quad C=0,\quad C\in\mathbb{C}\setminus\mathbb{Q}.$$

<u>Lemma</u> Let $\varkappa$ be positive or irrational. Then

(a) Loc sends compact objects to compact objects.

(b) Γ is fully faithful.

<u>Lemma</u> Let $\varkappa$ be negative or irrational. Then Γ is a fully faithful embedding & admits a left adjoint.

$x\in X$

$$Gr_{G,x} = G(K_x)/G(\mathcal{O}_x)$$

$$\mathrm{Whit}(G)_\varkappa \subseteq \text{D-mod}(Gr_{G,x})_\varkappa$$

the category of objects equivariant with respect to $N(K_x)$ against a non-degenerate character $N(K_x)\longrightarrow \mathbb{G}_a$.

$$\begin{tikzcd} Gr_{G,x} \arrow[d, "\pi", "\text{(forget triv.)}"'] \\ Bun_G \end{tikzcd}$$

$$\pi_*: \text{D-mod}(Gr_G)_{\varkappa,x} \longrightarrow \text{D-mod}(Bun_G)_{\varkappa,*}$$

$$\mathrm{Poinc}: \mathrm{Whit}(G)_{\varkappa,x} \longrightarrow \text{D-mod}(Bun_G)_{\varkappa,*}$$

$$\pi^!: \text{D-mod}(Bun_G)_{\varkappa,!} \xrightarrow{\text{coeff}} \mathrm{Whit}(G)_{\varkappa}$$

$$\mathrm{Whit}(G)_\varkappa[\mathrm{Ran}] \xrightarrow{\mathrm{Poinc}} \text{D-mod}(Bun_G)_{\varkappa,*}$$

$$\text{D-mod}(Bun_G)_{\varkappa,!} \xrightarrow{\text{coeff}} \mathrm{Whit}(G)_\varkappa[\mathrm{Ran}]$$

<u>Def</u> $\text{D-mod}(Bun_G)^{\text{non-degen}}_{\varkappa,!} = \text{D-mod}(Bun_G)_{\varkappa,!}/\ker(\text{coeff})$

<u>Conjecture</u> The resulting functor

$$\text{D-mod}(Bun_G)^{\text{non-degen}}_{\varkappa,!} \xrightarrow{\text{coeff}} \mathrm{Whit}(G)_\varkappa[\mathrm{Ran}]$$

is fully faithful.

Def $\text{D-mod}(\mathrm{Bun}_G)^{\text{non-degen}}_{\kappa,*} = \langle \mathrm{Poinc} \rangle$ $\qquad \mathrm{Poinc}: \mathrm{Whit}(G)_\kappa[\mathrm{Ran}] \longrightarrow \text{D-mod}(\mathrm{Bun}_G)^{\text{nondegen}}_{\kappa,*}$

Theorem $\exists$ a functor $\Psi^{\kappa, -\frac{1}{\kappa}}_{G,G^\vee}$:

$$\begin{tikzcd}
\mathrm{KL}(G)_\kappa[\mathrm{Ran}] \arrow[rrr, "\overset{\sim}{\text{Lurie}}"] \arrow[d, "\mathrm{Loc}"'] & & & \mathrm{Whit}(G)_{-\frac{1}{\kappa}}[\mathrm{Ran}] \arrow[d, "\mathrm{Poinc}"] \arrow[ddl, "\mathrm{Poinc.}"'] \\
\text{D-mod}(\mathrm{Bun}_G)_{\kappa,!} \arrow[rrr] \arrow[dr] \arrow[dd, "\mathrm{coeff}"'] & & & \text{D-mod}(\mathrm{Bun}_{G^\vee})_{-\frac{1}{\kappa},*} \arrow[dd, "\Gamma"] \\
& D^{\text{non-degen}}_{G,\kappa,!} \arrow[r, "\cong"] \arrow[dl, "\mathrm{coeff}"] & D_{G^\vee,-\frac{1}{\kappa},*} \arrow[ur] & \\
\mathrm{Whit}(G)_\kappa \arrow[rrr, "\cong", "\text{Lurie}"'] & & & \mathrm{KL}(G^\vee)_{-\frac{1}{\kappa}}[\mathrm{Ran}]
\end{tikzcd}$$

$\kappa = -\frac{1}{2}\kappa_{\mathrm{Kill}}$

$-\frac{1}{\kappa} = \infty$

$$\begin{tikzcd}
\text{D-mod}(\mathrm{Bun}_{G^\vee})_{!,\infty} \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & \text{D-mod}(\mathrm{Bun}_{G^\vee})_{*,\infty} \arrow[dl, "\cong"] \\
& \mathrm{QCoh}(\mathrm{LocSys}_G) &
\end{tikzcd}$$

$\mathrm{KL}(G^\vee)_\infty = \mathrm{Rep}(G^\vee)$

opers

$\mathrm{Whit}(G^\vee)_{\infty,x} = \mathrm{QCoh}(\mathrm{Op}_{G^\vee,x})$

$\mathrm{Loc}: \mathrm{Rep}(G^\vee)_x \longrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee})$

$\overset{\shortparallel}{p^*}$

$p: \mathrm{LocSys}_{G^\vee} \longrightarrow BG^\vee$

$$\begin{tikzcd}
& \mathrm{Op}_{G^\vee}(X,x) \arrow[dl] \arrow[dr] & \\
\mathrm{LocSys}_{G^\vee} & & \mathrm{Op}_{G^\vee,x}
\end{tikzcd}$$

Conjecture The functor $\mathrm{coeff}: \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}) \longrightarrow \mathrm{QCoh}(\mathrm{Op}_{G^\vee})[\mathrm{Ran}]$ is fully faithful

$\Updownarrow$

$\mathrm{Poinc}: \mathrm{QCoh}(\mathrm{Op}_{G^\vee})[\mathrm{Ran}] \longrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee})$ is quotienting.

$$
\begin{tikzcd}
KL(G)_{crit} \arrow[rr, "\cong"] \arrow[d, "Loc"'] & & QCoh(Op_{G^\vee})[Ran] \arrow[d, "Poinc"] \\
D\text{-mod}(Bun_G) \arrow[r, two heads] \arrow[d, "\Gamma"'] & D\text{-mod}(Bun_G)_!^{non\text{-}degen} \arrow[r, "\cong"] & QCoh(LocSys_{G^\vee}) \arrow[d, "\Gamma"] \\
Whit(G) \arrow[rr, "\cong"'] & & Rep(G^\vee)[Ran]
\end{tikzcd}
$$

$$
\begin{tikzcd}
KL(G)_{crit} & & QCoh(Op_{G^\vee})[Ran] \arrow[ll, "\cong"'] \\
D\text{-mod}(Bun_{G,x}) \arrow[u, "\Gamma"] & D\text{-mod}(Bun_G)_*^{non\text{-}degen} \arrow[l, hook'] & QCoh(LocSys_{G^\vee}) \arrow[l, "\cong"'] \arrow[u, "coeff"'] \\
& Whit(G)[Ran] \arrow[u, "Poinc"'] & Rep(G^\vee)[Ran] \arrow[u, "Loc"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& QCoh(Y)^! = Ind(Coh(Y)) \arrow[dl] \\
QCoh(Y) &
\end{tikzcd}
$$