

# May 26: Dennis Gaitsgory: Global ~~Langlands~~ Geometric Langlands

- I. Introduce categories
- II. KL & localization functor
- III. Whittaker category & quantum geometric Langlands conjecture
- IV. The classical case

$$\begin{aligned} \mathcal{L}: \mathfrak{g} \otimes \mathfrak{g} &\rightarrow \mathbb{C} && \text{symmetric invariant form} \\ \mathcal{L} + \frac{\hbar}{2k} \text{tr}(\rho^2): \mathfrak{h} \otimes \mathfrak{h} &\rightarrow \mathbb{C} && \\ & && \downarrow \\ & && -\mathcal{L}^{-1} \mathfrak{g}^{\vee} \otimes \mathfrak{g}^{\vee} \rightarrow \mathbb{C} \end{aligned}$$

Bun<sub>G</sub> stack of ~~infinite~~ infinite type  $\longrightarrow$  write Bun<sub>G</sub> as union of stacks of finite type  
 $D(\text{Bun}_G)_x$  twisted differential operators on Bun<sub>G</sub>

$D\text{-mod}(\text{Bun}_G)_x!$  ~~Orinfeldt: This category~~

Lemma This category is compactly generated.

Def  $D\text{-mod}(\text{Bun}_G)_{x,*} := (D\text{-mod}(\text{Bun}_G)_{-x,!})^{\vee}$

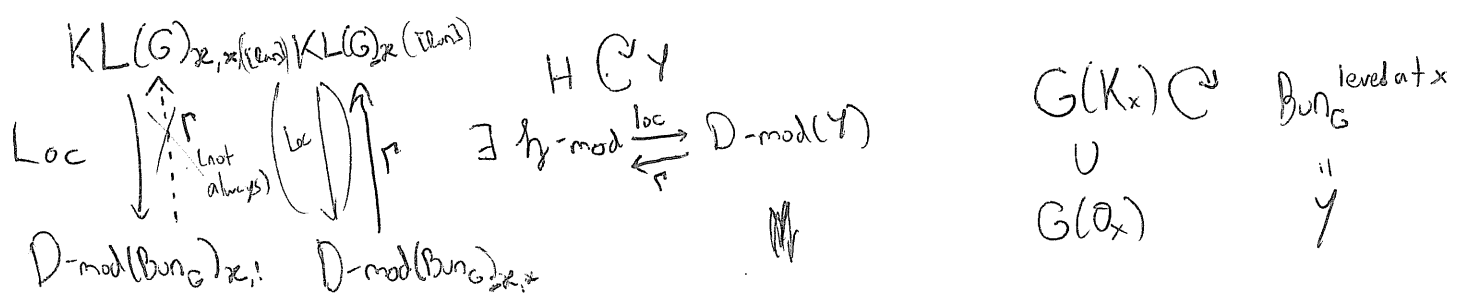
$\downarrow$   $D\text{-mod}(\text{Bun}_G)_{x,!}$  not an equivalence in general, but is when  $x$  is irrational

$$KL(G)_x = (\hat{\mathfrak{g}}\text{-mod}_x)^{G[[\hbar]]} \quad (\text{more canonically, } (\hat{\mathfrak{g}}\text{-mod}_x)^{\mathbb{F}(\mathbb{O}_x)})$$

"D-module analogue of an  $E_2$  category"



$KL(G)_x$  [Ran]



Def  $\mathcal{X}$  is positive/negative/? /irrational if

$$\mathcal{X} + \frac{1}{2} \mathcal{X}_{\text{Killing}} = C \cdot \mathcal{X}_{\text{Killing}}$$

$$C \in \mathbb{Q}^{>0}, C \in \mathbb{Q}^{<0}, C=0, C \in \mathbb{C} \setminus \mathbb{Q}$$

Lemma Let  $\mathcal{X}$  be positive or irrational. Then

(a)  $\text{Loc}$  sends compact objects to compact objects.

(b)  $\Gamma$  is fully faithful.

Lemma Let  $\mathcal{X}$  be negative or irrational. Then  $\Gamma$  is a fully faithful embedding & admits a left adjoint.

$$x \in X$$

$$Gr_{G,x} = G(K_x) / G(\mathcal{O}_x)$$

$$\text{Whit}(G)_{\mathcal{X}} \subseteq D\text{-mod}(Gr_{G,x})_{\mathcal{X}}$$

the category of objects equivariant with respect to  $N(K_x)$  against a non-degenerate character  $N(K_x) \rightarrow \mathbb{G}_a$ .

$$Gr_{G,x} \quad \pi_x: D\text{-mod}(Gr_{G,x})_{\mathcal{X}} \rightarrow D\text{-mod}(Bun_G)_{\mathcal{X},*}$$

$$\text{Poinc}: \text{Whit}(G)_{\mathcal{X},x} \rightarrow D\text{-mod}(Bun_G)_{\mathcal{X},*}$$

$$\pi_x^!: D\text{-mod}(Bun_G)_{\mathcal{X},!} \xrightarrow{\text{coeff}} \text{Whit}(G)_{\mathcal{X},x}$$

$$\text{Whit}(G)_{\mathcal{X}}[\text{Ran}] \xrightarrow{\text{Poinc}} D\text{-mod}(Bun_G)_{\mathcal{X},x}$$

$$D\text{-mod}(Bun_G)_{\mathcal{X},!} \xrightarrow{\text{coeff}} \text{Whit}(G)_{\mathcal{X}}[\text{Ran}]$$

$$\text{Def } D\text{-mod}(Bun_G)_{\mathcal{X},!}^{\text{non-degen}} = D\text{-mod}(Bun_G)_{\mathcal{X},!} / \ker(\text{coeff})$$

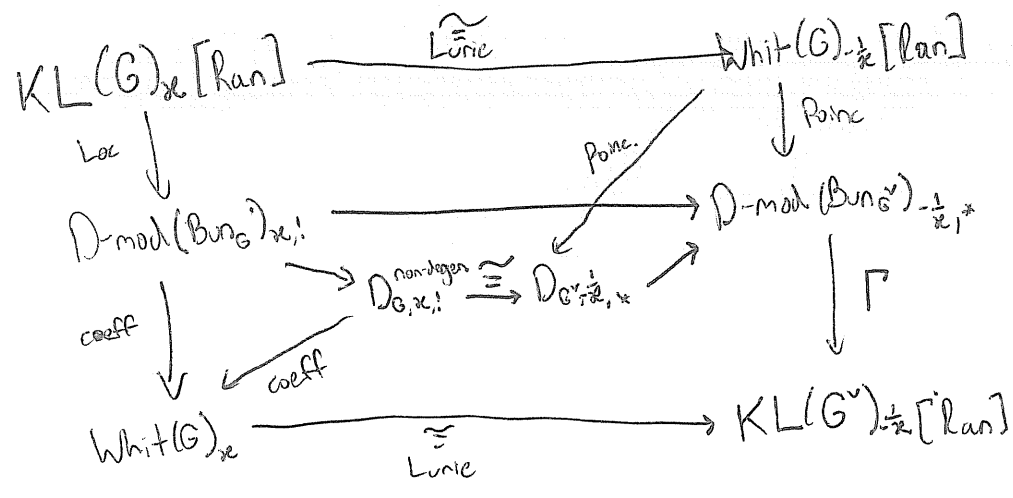
Conjecture The resulting functor

$$D\text{-mod}(Bun_G)_{\mathcal{X},!}^{\text{non-degen}} \xrightarrow{\text{coeff}} \text{Whit}(G)_{\mathcal{X}}[\text{Ran}]$$

is fully faithful.

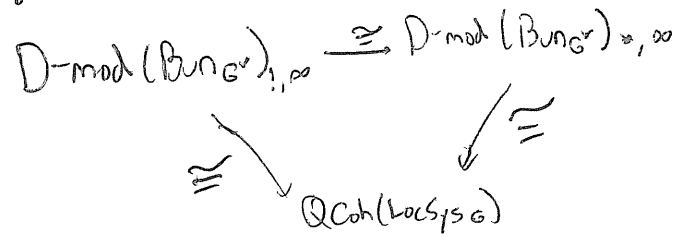
Def  $D\text{-mod}(Bun_G)_{\mathbb{R},*}^{\text{non-degen}} = \langle \text{Poinc} \rangle$      $\text{Poinc}: \text{Whit}(G)_{\mathbb{R}}[\text{Ran}] \rightarrow D\text{-mod}(Bun_G)_{\mathbb{R},*}^{\text{non-degen}}$

Theorem  $\exists$  a functor  $\Psi_{G,G^\vee}^{\mathbb{R},-\frac{1}{2}}$



$\mathbb{R} = -\frac{1}{2}$  kill

$-\frac{1}{2} = \infty$



$KL(G^\vee)_\infty = \text{Rep}(G^\vee)$

$\text{Whit}(G^\vee)_{\mathbb{R},x} = \text{Qcoh}(\text{Op}_{G^\vee,x})$      $\swarrow$  opers

$\text{Loc}: \text{Rep}(G^\vee)_x \rightarrow \text{Qcoh}(\text{LocSys}_{G^\vee})$

$\overset{p^*}{\parallel}$

$p: \text{LocSys}_{G^\vee} \rightarrow BG^\vee$

$\text{Op}_G^{\text{loc}}(X,x)$

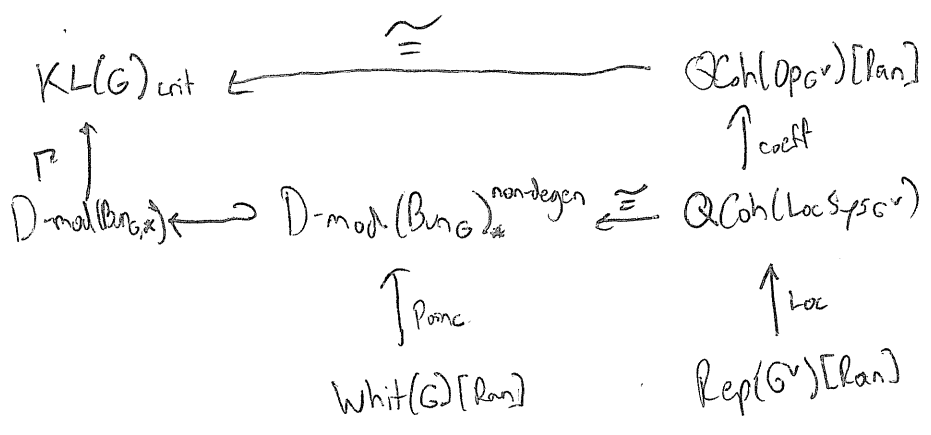
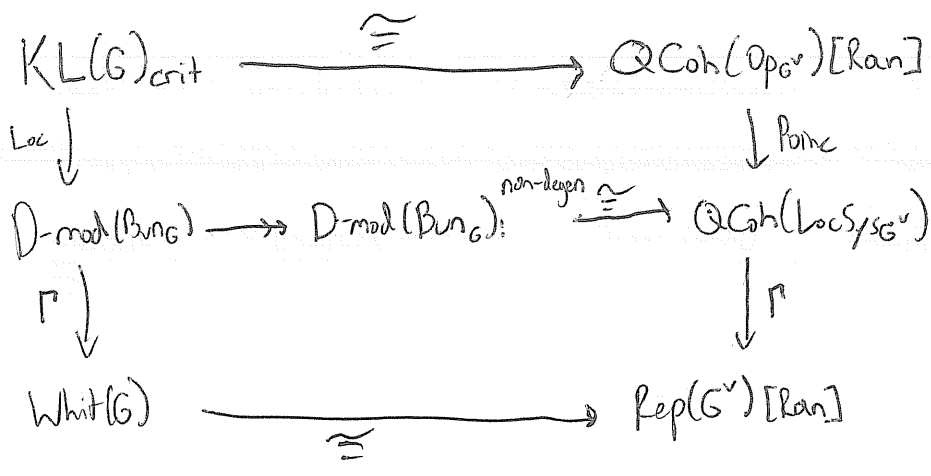
$\text{LocSys}_{G^\vee}$

$\text{Op}_{G^\vee,x}$

Conjecture The functor  $\text{coeff}: \text{Qcoh}(\text{LocSys}_{G^\vee}) \rightarrow \text{Qcoh}(\text{Op}_{G^\vee})[\text{Ran}]$  is fully faithful

$\Downarrow$

$\text{Poinc}: \text{Qcoh}(\text{Op}_{G^\vee})[\text{Ran}] \rightarrow \text{Qcoh}(\text{LocSys}_{G^\vee})$  is quotienting.



$$QCoh(Y)^! = \text{Ind.}(Coh(Y))$$

