May 26: David Nadler: Langlands duality for character sheaves

Joint w/ D. Ben-Zvi, arXiv 0904.1247

<u>Motivation</u> Place representation theory of Lie groups in the context of 3d TFT.
(relation to 4d TFT and geometric Langlands)

<u>Application</u> duality for <u>character sheaves</u>

<u>Outline</u>

I. Review of character sheaves
II. TFT interpretation for character sheaves
III. Relation to geometric Langlands

<u>Ground rules</u> everything derived

I. Fix G reductive group$/\mathbb{C}$ $(GL_n, SL_n, T, \ldots, GL_2)$

quotient by adjoint action $\rightarrow$ $\frac{G}{G}$ adjoint quotient stack

D-modules on $\frac{G}{G}$

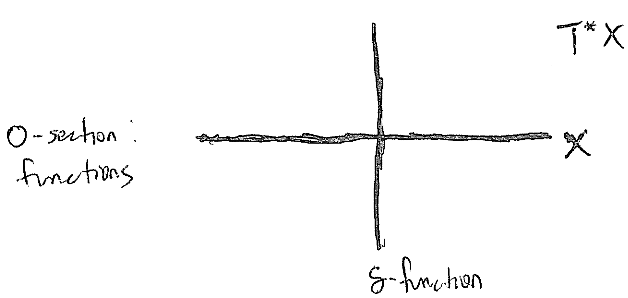
<u>D-modules</u> X smooth scheme$/\mathbb{C}$

<u>Def</u> A <u>D-module</u> on X is a quasicoherent sheaf on X with a compatible action of D_X = differential operators.

(1) M is a qcoh sheaf with "infinitesimal parallel transport"

<u>Examples</u> · vector bundles with flat connections

(2) M is a noncommutative module on T^*X.



rough picture:
coisotropic submanifold

(3) M is an A-brane in T^*X.

Def M is <u>holonomic</u> if its singular support is Lagrangian. (+ finitely generated)

Theorem

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

schemes

$$D(X) \underset{D(Z)}{\otimes} D(Y) \cong D(X \times_Z Y) \cong \mathrm{Fun}_{D(Z)}(D(X), D(Y))$$

Exercise G reductive group$/\mathbb{C}$. Calculate $D(BG)$.

Answer: $D(BG) \cong C_{-*}(G)\text{-mod}$

Back to $\frac{G}{G}$. $\quad Ch_G \underset{\text{full}}{\subseteq} D(\frac{G}{G})$

nilpotent cone

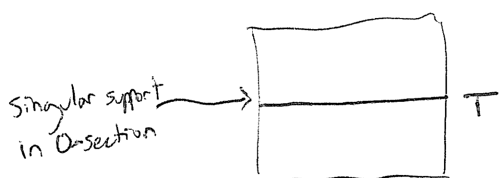
$$T^*G \cong G \times \mathfrak{g}^* \overset{\text{Killing}}{\cong} G \times \mathfrak{g} \supset G \times \mathcal{N}$$

Def $Ch_G = \{ M \in D(\frac{G}{G}) \mid \text{singular support of } M \subset \mathcal{N} \}$

Examples (1) $G = T$ $\quad T \circlearrowright T$ is trivial

$$\longrightarrow \frac{T}{T} = T \times BT$$

all elements semisimple



$Ch_T = \{\text{local system on } T\} \otimes \{C_{-*}(T)\text{-module}\}$

$\cup$

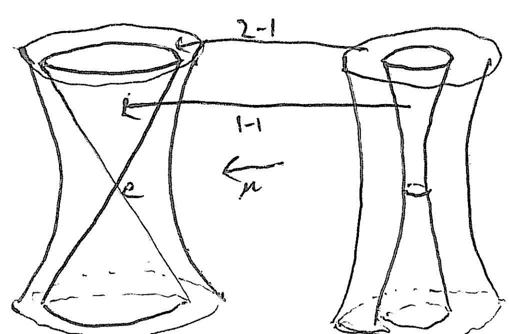
We will only consider these $\longrightarrow$ unipotent char. sheaves $Ch^u_T = C^*(T) \otimes C_{-*}(T)\text{-mod}$

(2) Construct our favorite character sheaf in Ch_G.

<u>Springer sheaf</u> $\quad \frac{\tilde{G}}{G} \xrightarrow{\mu} \frac{G}{G}$ $\quad \tilde{G}$ Grothendieck-Springer resolution

$\tilde{G} = \{ g \in B \} \subset G \times \mathcal{B}$

SL_2:



<u>Springer sheaf</u> $\doteq \mu_* \mathcal{O}_{\frac{\tilde{\mathfrak{g}}}{G}} \in Ch_G$

<u>Lusztig's construction</u> All character sheaves arise by the correspondence given below.

$$\underbrace{B\backslash G/B}_{} \xleftarrow{\delta} \frac{G}{B} \xrightarrow{p} \frac{G}{G}$$

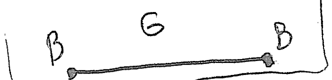
discrete set of orbits

$\rightsquigarrow$ all D-modules holonomic

$$D(B\backslash G/B) \xrightarrow{p_* \delta^*} D\left(\frac{G}{G}\right)$$

<u>Goal</u> Understand Ch_G in terms of the Hecke category $\mathcal{H}_G = D(B\backslash G/B)$.

(1) monoidal



(2) Everything is holonomic.

(3) Kazhdan-Lusztig duality: composition of Verdier duality with $g \mapsto g^{-1}$.

Role of this category $\mathcal{H}_G$ in rep. theory: natural "braid category" of symmetries/intertwiners acting on $\mathfrak{g}$-mod.

<u>Theorem</u> There is a (partial: 0,1,2) 3d TFT, (or equivalently, categorified 2d TFT) χ_G called the <u>character theory</u> s.t.

$$\chi_G(S^2) = H_T^*(pt)$$

$$\chi_G(S^1) = Ch_G$$

$$\chi_G(pt) = \text{2-cat of } \mathcal{H}_G\text{-modules}$$

<u>Examples</u> $G \circlearrowright Y$, $D(B\backslash Y)$

<u>Implication for Ch_G</u>

rich structure of TFT

$$Ch_G = \begin{Bmatrix} Z \\ \ddot{HH}^* \\ HH_* \end{Bmatrix} (\mathcal{H}_G)$$

<u>Langlands duality</u> $\quad G \longleftrightarrow G^\vee$

χ_G : character theory for G associated to $\mathcal{H}_G = D(B\backslash G/B)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \wr\!\cong$ $\quad$ (Beilinson–Ginzburg–Soergel)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathbb{Z}/2$-graded

$\chi^{mon}_{G^\vee}$ character theory ~~for~~ assoc. to $\mathcal{H}^{mon}_{G^\vee} = D(U^\vee\backslash G^\vee/U^\vee)$

$B^\vee/U^\vee \circlearrowright \quad\quad \circlearrowleft B^\vee/U^\vee$

<u>Cor</u> $\quad Ch_{G,per} \cong Ch_{G^\vee, per} \quad$ (per = periodiz)

<u>Example</u> $\quad G = T,\ G^\vee = T^*$

$$C^*(T) \otimes C_{-*}(T)\text{-mod} \underset{\mathbb{Z}/2}{\cong} C^*(T^*) \otimes C_{-*}(T^*)\text{-mod}$$