May 19: Christoph Wockel: Higher connected covers by categorified principal bundles

(joint w/ S. Porst, C. Zhu)

Higher dimensional covers

Motivation Categorification of structure groups for gauge theories.

Notion of cover X $(n-1)$-connected space

$$Y \xrightarrow{q} X \quad \text{Fibration with } \pi_k(q) \text{ iso for } k > n, \ \pi_n(Y) = 0$$

Construction $X \to K(\pi_n, n-1)$ iso. on ~~[illegible]~~ π_1

$$\rightsquigarrow Y = f^*(P(K(\pi_n, n-1))) \text{ is an } n\text{-cover of } X.$$

$\downarrow$ $\quad$ path space

X

This construction is unsatisfactory from a group perspective

Example
$n=1$ $\quad$ $Spin \to SO$
$n=2$ $\quad$ $? \to \Omega\, Spin$ (∞-dim Lie group with $\pi_2 \neq 0$)
(interesting for ∞-dim Lie theory)
$n=3$ $\quad$ $? \to Spin$

A simple but instructive example $n=1$, G connected Lie group

$\pi_1 \hookrightarrow \widetilde{G} \to G$ simply connected cover

- is a π_1-principal bundle
- central extension of G by π_1

$\Rightarrow \widetilde{G} = \pi_1 \times_{\theta_1} G$ for $\theta_1 : G \times G \to \pi_1$

$$(a,g) \times (b,h) = (ab + \theta_1(g,h), gh)$$

- associativity requires that θ_1 is a group cocycle:

$$\theta_1(g,h) + \theta_1(gh,k) - \theta_1(g,hk) - \theta_1(h,k) = 0 \quad \forall g,h,k$$

- $\theta_1(g,e) - \theta_1(e,g) = 0$

$\rightsquigarrow$ defines the group structure on $\widetilde{G}$, but how about the smooth structure?

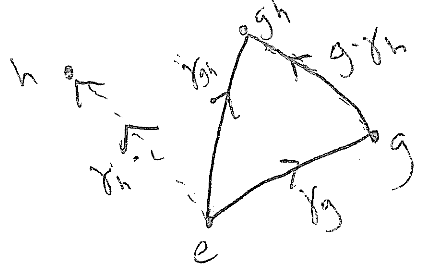
Assume that $\Theta_1|_{U\times U}$ smooth, constant map, vanishes on $U \subseteq G$ unit neighborhood.

$\Rightarrow [\tau\, \Theta_1] \in \check{H}^1(G, \pi_1)$

transgression

$\rightsquigarrow$ Endowing $\pi_1 \times_{\Theta_1} G$ with the topology making it a principal bundle with characteristic class $[\tau\, \nu_1]$ yields a Lie group structure on $\pi_1 \times_{\Theta_1} G$.

Construction ~~$\forall$ group~~ $\forall g \in G$, choose $e \longrightarrow g$ smooth path.



$= (d_{gp}\gamma)(g,h)$

$= \gamma(g) + g\gamma(h) - \gamma(gh)$ (in $C_1(G)$)

$\Theta_1 : G \times G \longrightarrow \pi_1 \longrightarrow \Theta_1(g,h) := q(d_{gp}\gamma)$

$q : Z_1(G) \to H_1(G) \cong \pi_1(G)$

Cocycle identity: $0 = d_{gp}\Theta_1 = d_{gp}(q \circ d_{gp}\gamma)$

$= q \circ d_{gp}^2 \gamma = 0$

Theorem $[\Theta_1]$ is universal for 2-cocycles that vanish on some unit neighborhood.

i.e., $f : G \times G \to A$ $\quad [f] = [\varphi_f \circ \Theta_1]$ for $\varphi_f : \pi_1(G) \to A$.

$\mathrm{Hom}(\pi_1, A) \to H^2_{gp}(G, A)$ $\quad \varphi \longmapsto [\varphi \circ \Theta_1]$ is bijective.

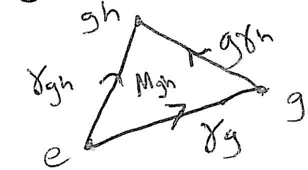
⌈Use standard covering theory, in particular, the path lifting property.⌋

(n.b. $H^n_{gp}(G,A)$ = locally smooth group cohomology)

n=2 $\quad ? \longrightarrow \Omega\,\mathrm{Spin}$ $\quad$ G simply connected

Construction of $\Theta_2 : G \times G \times G \longrightarrow \pi_2(G)$

$\forall g, h \in G$



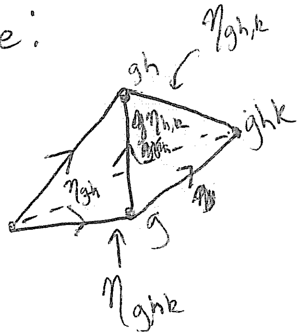
Choose M_{gh} s.t. $\partial M_{gh} = \cancel{d_{gp}}\, d_{gp}\gamma$

$\eta : G \times G \longrightarrow C^{\infty}_{*}(\Delta^2, G)$

$\uparrow$ pointed maps

and observe $(d_{gp}\eta)(g,h,k) \in Z_2(G)$

Picture:



$(d_{gp}\eta)(g,h,k) = \eta_{g,h} + \eta_{gh,k} - \eta_{g,hk} - g \cdot \eta_{h,k}$

$\Rightarrow$ define $\Theta_2 = q(d_{gp}\eta)$ $\quad q : Z_2 \longrightarrow H_2 \cong \pi_2$

<u>Theorem</u> $[\Theta_2] \in H^3_{gp}(G, \pi_2(G))$ is universal for locally constant 3-cocycles.

⌈ Use "path lifting" (i.e., parallel transport) in 2-bundles. ⌋

<u>Q</u> To what extend describes Θ_2 2-connected covering?

$\longrightarrow$ algebraically, Θ_2 gives a central extension of 2-groups.

$$\pi_2[1] \text{ “} B\pi_2 \longrightarrow \mathcal{G}_{\Theta_2} \longrightarrow \underline{G}$$

$B\pi_2 = \{ \circlearrowleft a \mid a \in \pi_2 \}$

$\mathcal{G}_{\Theta_2}$: 2-group

$\underline{G}$: only id morphisms

e.g., $\coprod U_{ij} \rightrightarrows \coprod U_i \simeq_{\text{Morita}} M \rightrightarrows M = \bigcup U_i$

$\longrightarrow$ Topologically: ???

$\rightsquigarrow [\tau\,\Theta_2] \in \check{H}^2(G, \pi_2)$

<u>Theorem</u> Principal $\mathcal{G}$-2-bundles on G are classified by $\check{H}(G, \mathcal{G})$ (up to Morita equivalence)

$\uparrow$ strict Lie 2-groups

In particular, if $\mathcal{G} = B\pi_2$, then $\check{H}(G, B\pi_2) = \check{H}^2(G, \pi_2)$

$= \text{“}\check{H}^1(G, \pi_2[1])\text{”}$

What would be nice: Lie 2-group structure on $P_{\tau\Theta_2}$.

But that's too much to ask for.

(Remedy: Invert Morita equivalence ~~⇒~~ $\rightsquigarrow$ pass to smooth stacks,
i.e., obtain $\mathcal{G}_{\Theta_2}$ as a "stacky Lie group.")

Application: ∞-dimensional Lie algebras may not come from a Lie group,
but can integrate to 2-groups.