May 18: David Ben-Zvi: Topological Field Theory, Loop Spaces, and Representation Theory: Lecture I

Gauge theory $\longleftrightarrow$ Representation theory

- 2d gauge theory $\leftarrow$ reps. of finite groups "toy model"
- 3d gauge theory $\longleftrightarrow$ real & complex Lie groups
- 4d gauge theory $\longleftrightarrow$ geometric Langlands

What is gauge theory?

Physical theory (quantum field theory) in which fundamental fields are principal G-bundles w/ connection on spacetime M

<u>Classical</u> look for connections satisfying equations (flatness, Yang-Mills)

<u>Quantum</u> look at all connections by attaching $\mathbb{C}$-linear data (w/ a weighting by Yang-Mills action)

<u>Topological</u> study "course" features depending only on topology of M

Dijkgraaf-Witten theory

G finite group $\rightsquigarrow$ ~~M~~ Z_G 2d field theory

M manifold of dim'n 2, 1, 0 (oriented)

$\rightsquigarrow$ space of gauge fields $\mathcal{M}_G(M) = \{\text{G-bundles on } M\}$

$= \{\text{G-Galois covers of } M\}$

$= \{\pi_1(M) \to G\}/\text{conj.}$

Think of as a finite orbifold: keep track of automorphisms.

$\mathcal{M}_G(\cdot) = \dot{\circ}_G = BG$ pt. orbifold with symmetry G ✓

$\mathcal{M}_G(S^1) = \frac{G}{G}$ ← G acts on G by conjugation

$\frac{G}{G} = \coprod_{[g]} \dot{\circ}_{Z_G(g)}$

$$\mathcal{M}_G(\Sigma_g) = \left\{ \begin{matrix} A_1, \ldots, A_g \\ B_1, \ldots, B_g \end{matrix} \in G, \quad \prod [A_i, B_i] = 1 \right\} / G$$

Partition function:

$M = \Sigma$ 2-manifold

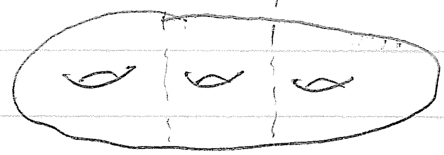
$$Z_G(M) = \text{“}\int_{\substack{\text{fields}\\ \text{on } M}} e^{-S(\varphi)}\, d\varphi\text{”}$$

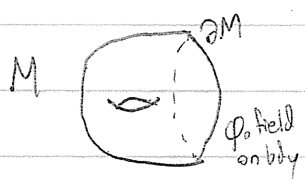
In our case, $Z_G(\Sigma) = \# \mathcal{M}_G(\Sigma)$

$= $ weighted # of G-bundles on Σ

$$= \sum_{P \in \mathcal{M}_G(\Sigma)} \frac{1}{|\mathrm{Aut}\, P|}$$

<u>Locality</u> "calculate by cut-and-paste"



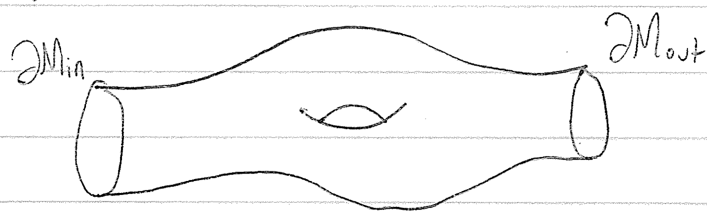


$$Z(M)(\varphi_0) = \text{“}\int_{\varphi|_{\partial M} = \varphi_0} e^{-S(\varphi)}\, d\varphi\text{”}$$

$Z(M) \in \mathrm{Fun}(\mathrm{Fields}(\partial M)) =: Z(\partial M)$

2-manifold $\rightsquigarrow \mathbb{C}$

1-manifold $\rightsquigarrow$ Vect



generalization:

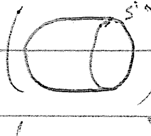
$$\begin{array}{ccc} & \mathrm{Fields}(M) & \\ \swarrow & & \searrow \\ \mathrm{Fields}(\partial M_{in}) & & \mathrm{Fields}(\partial M_{out}) \end{array}$$

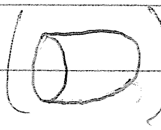
$$Z(\partial_{in}) \xrightarrow{Z(M)} Z(\partial_{out})$$

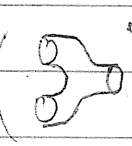
$$Z(\partial_{in}) \ni f \longmapsto \left\{ \varphi_{out} \longmapsto \int_{\varphi|_{\partial_{out}} = \varphi_{out}} f(\varphi_{in}) e^{-S(\varphi)}\, d\varphi \right\}$$

$$= \pi_{out\,*}(\pi_{in}^{*} f \cdot e^{-S})$$

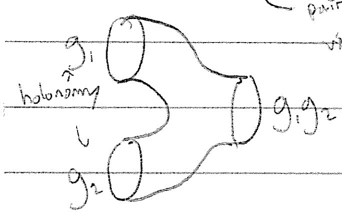
$Z_G(S^1) = \mathbb{C}\left[\frac{G}{G}\right] = \mathbb{C}[G]^G$ = class functions

$Z\Big($  $\Big) =: 1 = \delta_1 \in \mathbb{C}[G]^G$

$Z\Big($  $\Big) =: eval_1 : \mathbb{C}[G]^G \xrightarrow{tr} \mathbb{C}$

$Z\Big($  $\Big) = Z\Big($  $\Big) : \mathbb{C}[G]^G \otimes \mathbb{C}[G]^G \longrightarrow \mathbb{C}[G]^G$ Convolution

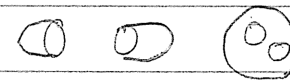
pair of pants viewed on side



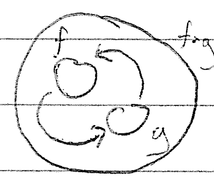
$$\left.\begin{array}{c} G\times G \longrightarrow G \\ \mathbb{C}[G]\otimes\mathbb{C}[G] \to \mathbb{C}[G] \end{array}\right\}$$

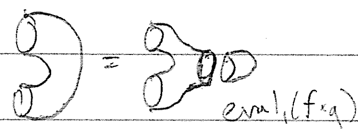
G finite: functions on G = measures on G

$\mathbb{C}G$ associative algebra, $\mathbb{C}G^G \subset \mathbb{C}G$

Given any surface, can cut up into pieces 

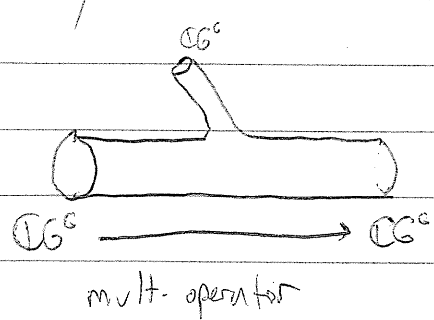
Commutative Frobenius algebra $\mathbb{C}G^G$ commutative:



~~Canomat~~ Canonical trace: 

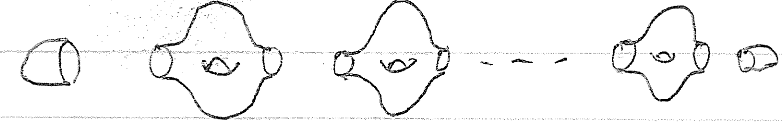
$\rightsquigarrow$ nondegenerate inner product

Claim: given $\mathbb{C}G^G$, 1, $eval_1$, can "solve this system": evaluate on any surface.



$\longleftrightarrow$ diagonalize simultaneously
joint spectrum
$= \operatorname{Spec} \mathbb{C}G^G = \{\mathbb{C}G^G \to \mathbb{C}\} = \widehat{G}$
irred. characters of G

<u>Exercise</u> Calculate what is assigned to closed surface.

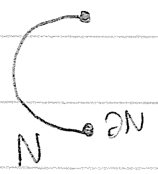


<u>Answer:</u> $$Z(\Sigma_g) = \sum_{\substack{\chi \in \text{irred} \\ \text{characters} \\ \text{of } G}} \left(\frac{|G|}{\dim \chi}\right)^{2g-2}$$

2-manifold $\rightsquigarrow$ $Z(M) = \int_\varphi e^{-S(\varphi)}\, d\varphi$

1-manifold $\rightsquigarrow$ $Z(N) \in \mathrm{Vect}$ = functions on bdy fields

1-manifold w/ boundary:

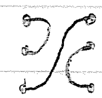


"fix boundary values" $\quad Z(N): \left\{\begin{matrix}\text{fields} \\ \text{on} \\ \partial N\end{matrix}\right\} \longrightarrow \mathrm{Vect}$

$Z(N)$ is a vector bundle, or sheaf, on the space of fields on ∂N.

So to ∂N, assign $\left\{\begin{matrix}\text{vector bundles} \\ \text{on Fields}(\partial N)\end{matrix}\right\}$ a linear category

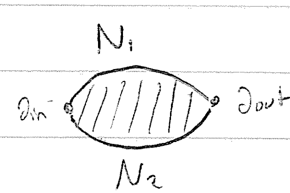
$$\begin{tikzcd} & \mathrm{Fields}(N) \arrow[dl, "\pi_{in}"'] \arrow[dr, "\pi_{out}"] & \\ \cancel{\mathrm{Fields}(\partial N)} \ \mathrm{Fields}(\partial_{in}) & & \mathrm{Fields}(\partial_{out}) \end{tikzcd}$$



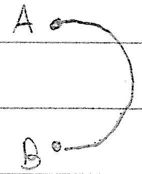
$$Z: \quad \mathrm{Vect}(\mathrm{Fields}(\partial_{in})) \xrightarrow{Z(N_1)} \mathrm{Vect}(\mathrm{Fields}(\partial_{out}))$$

$$V \longmapsto \pi_{out *}\left(\pi_{in}^* V \otimes \text{“}e^{-S}\text{”}\right)$$

$$\begin{tikzcd} & \mathrm{Fields}(N_1) \arrow[dl] \arrow[dr] \arrow[d, Rightarrow] & \\ \mathrm{Fields}(\partial_{in}) \arrow[r, leftrightarrow] & \mathrm{Fields}(N_2) \arrow[r] & \mathrm{Fields}(\partial_{out}) \end{tikzcd}$$



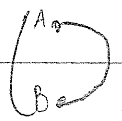
$$Z_G(\cdot) = \mathrm{Vect}(\mathrm{Fields}(\cdot)) = \mathrm{Vect}(\cdot_G) = \mathrm{Rep}_{\mathbb{C}}\, G = \mathbb{C}G\text{-mod}$$

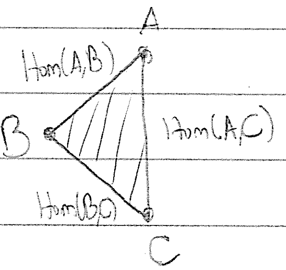


$Z(\emptyset) = \mathrm{Vect}$

$Z : \mathrm{Rep}\, G \otimes \mathrm{Rep}\, G \longrightarrow \mathrm{Vect}$

$A, B \in \mathrm{Rep}_{\mathbb{C}}\, G$

$Z\Big($  $\Big) = \mathrm{Hom}_{\mathrm{Rep}G}(A, B)$

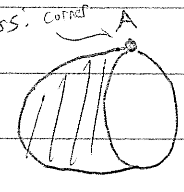


$$\mathrm{Hom}(A,B) \otimes \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$$

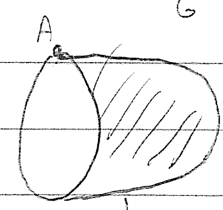
$= \mathrm{End}(A)$

glue: add disk:

2-manifold with corners: corner



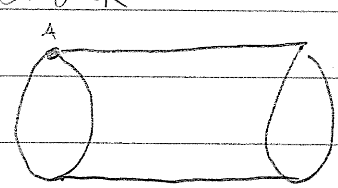
$$\mathbb{C} \longrightarrow \mathrm{End}_G(A)$$



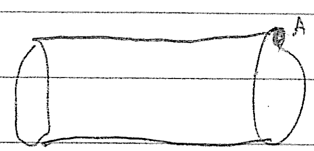
$$\mathrm{End}_G(A) \xrightarrow{tr} \mathbb{C}$$

$$\mathrm{Id}_A \longmapsto \dim A$$

Cut open disk:



$$\mathrm{End}_G A \rightleftarrows \mathbb{C}G^G$$



(in fact, $\mathbb{C}G^G \xrightarrow{\cong} \mathrm{End}\, \mathrm{Id}_{\mathrm{Rep}G}$

$Z(\mathbb{C}G) =$ Center or Hochschild cohomology

$$\mathrm{Id}_A \longmapsto \chi_A$$

Dually, $\mathrm{End}_G A \longrightarrow \mathbb{C}G^G$

– encodes fact that $\mathbb{C}G^G$ = Hochschild homology of $\mathbb{C}G$

$\mathbb{C}G \xrightarrow{tr} HH_*$ "universal trace" $tr(a*b) = tr(b*a)$

$HH^* \longrightarrow \mathbb{C}G$ $a*b = b*a$