May 21: John Francis: E_3-geometry and 3-dimensional TFTs

Baez-Dolan:

$Z: n\mathrm{Bord} \to \mathcal{C}$ is determined by $Z(*)$

$\mathcal{C} = \mathrm{Cat}_{(\infty, n-1)}$ enriched in k-modules

This condition looks <u>hard</u> to verify.

We might need a hint as to what will produce such $Z(*)$

<u>Starting point:</u> Atiyah-style TQFT

$F(S^1)$ was ~~TQFT~~ a Frobenius algebra

Look at the values $F(S^1)$ and see what structure results.

$k \leq n \quad k\mathrm{Bord} \longrightarrow n\mathrm{Bord}$

$$\mathrm{Hom}_{k\mathrm{Bord}}(S^{k-1}, S^{k-1}) = \coprod_{\substack{[W^k] \\ \partial W = S^{k-1} \amalg S^{k-1}}} B\mathrm{Diff}(W, \partial W)$$

$B(I) :=$

$$\mathrm{Hom}\Big(\coprod_{\alpha \in I} S^{k-1}_\alpha, S^{k-1}\Big) = \coprod_{\substack{[W] \\ \partial W = \coprod_I S^{k-1} \amalg S^{k-1}}} B\mathrm{Diff}(W, \partial W)$$

$J \xrightarrow{\pi} I$

$$B(I) \times \prod_{i \in I} B(J_i) \longrightarrow B(J)$$

$i \in I$

$J_i = \pi^{-1}\{i\}$

$\{B(I)\}$ has the structure of an <u>operad</u>.

$k\mathrm{Bord} \xrightarrow{Z} \mathcal{C}$.

$Z(S^{k-1})$ becomes a B-algebra.

$$B(I) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\Big(\overset{Z}{\overset{\shortparallel}{F}}\big(\textstyle\coprod_I S^{k-1}\big), \overset{Z}{\overset{\shortparallel}{F}}(S^{k-1})\Big)$$

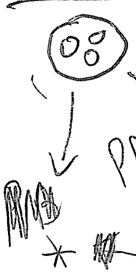
<u>Forget information</u>

<u>1st step</u>: too many manifolds

$K=3$, $\exists W^3$ s.t. $\partial W = S^2 \amalg \cdots \amalg S^2$

Restrict to $W = D^k - \coprod_{\alpha\in I} D^k_\alpha$

<u>2nd step</u>: $\mathrm{Diff}(D^k - \coprod_I D^k, \partial)$ complicated



Consider $B\mathrm{Diff}(D^k - \coprod_{\alpha\in I} D^k_\alpha, \partial)$

PB Bundle ~~Bundle~~

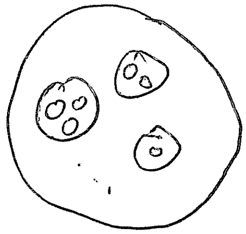
$$* \longrightarrow \mathrm{Conf}(\textstyle\coprod_I D^k, D^k) \longrightarrow B\mathrm{Diff}(D^k - \coprod_{\alpha\in I} D^k_\alpha, \partial)$$

This Fibre bundle classified by this map

$\shortparallel$

configuration space of disks in big disk

<u>Def</u> The <u>little k-disks operad</u> $\mathcal{E}_k(I)$ = config. space of I embedded k-disks in D^k.



$\Longrightarrow$ given $Z : k\mathrm{Bord} \to \mathcal{C}$

$Z(S^{k-1})$ is an $\mathcal{E}_k$-algebra.

$Z : n\mathrm{Bord} \to \mathcal{C}$, each $Z(S^{k-1})$ becomes an $\mathcal{E}_k$-algebra (in $\mathrm{Cat}_{(\infty, n-k)}$)

for $k = 0, 1, \dots, n$.

<u>Example</u> $k=0$

$V = Z(*)$ is a dualizable object

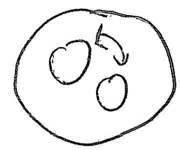
$$Z(S^0) = Z(*^+ \amalg *^-) = V \otimes V^\vee = \underline{\mathrm{End}(V)}$$

associative $= \mathcal{E}_1$-algebra

<u>$n=3$</u>

~~$Z(S^k$~~ $Z(S^1)$ is an $\mathcal{E}_2$-alg$(\mathrm{Cat}_{(\infty,1)})$

a <u>braided monoidal category</u>



<u>Ex</u> $\mathrm{Rep}(U_q(G))$

<u>Question</u> Where should such a sequence of $\mathcal{E}_k$-monoidal higher categories come from? In particular, where do $\mathcal{E}_k$-monoidal $(\infty,1)$-categories come from?

(most) Sym. monoidal categories come from algebraic geometry.

Tannakian duality:

$$\begin{tikzcd}
R \in \mathrm{CRings}_k \arrow[rr] \arrow[dr, bend right, "\mathrm{Mod}"'] & & \mathrm{Schemes/Stacks} \arrow[dl, "\mathrm{QC}"] \\
& \mathrm{Sym\ mon.\ Cat}/k \arrow[ul, bend right, "\mathrm{End}(\mathbb{1})"'] \arrow[ur, dashed, bend right] &
\end{tikzcd}$$

$(\mathrm{Mod}_R, \otimes_R)$

1st Example $\mathcal{C} = \mathrm{Rep}\, G$, G affine algebraic group

Can we recover G? Not quite. We can recover BG.

$QC_{BG} \cong \mathrm{Rep}(G)$.

Recover G from BG with the choice of a point.

$\longleftrightarrow$ Choice of forgetful functor $\mathrm{Rep}\, G \longrightarrow k\text{-mod}$.

Generality

X large class of stacks, can recover X from QC_X.

Theorem (Lurie) $X(A) = \mathrm{Fun}^{\otimes}(QC_X, \mathrm{Mod}_A)$

Pre-Theorem This works in the derived setting.

Derived algebraic geometry $\xleftrightarrow{\text{Tannakian}}$ Sym. monoidal ∞-cat/k + t-structure

$$
\begin{tikzcd}
\mathcal{E}_n\text{-rings} \arrow[r, "\mathrm{Spec}"] \arrow[dr] & \mathcal{E}_n\text{-scheme or stack} \arrow[d, "QC"] \\
& \mathcal{E}_{n-1}\text{-monoidal } \infty\text{-cat}
\end{tikzcd}
$$

$R \overset{\mathrm{Mod}}{\longmapsto} \mathrm{Mod}_R$

This is what we might want.

Reason for $n \longrightarrow n-1$: (Dunn)

$\mathcal{E}_n \simeq \mathcal{E}_1(\mathcal{E}_1(\cdots \mathcal{E}_1))\cdots)$

i.e., $\mathcal{E}_n\text{-alg} = \mathcal{E}_1\text{-alg}(\mathcal{E}_1\text{-alg}(\cdots \quad))$

$n>1$

Def An <u>$\mathcal{E}_n$-scheme</u> is a topological space X with a sheaf of $\mathcal{E}_n$-rings s.t. locally, it is of the form Spec A, for A an $\mathcal{E}_n$-algebra.

What is Spec A?

Def $A \in \mathcal{E}_n$-ring

$$\operatorname{Spec} A = (\operatorname{Spec} \pi_0 A, \mathcal{O})$$

s.t. $f \in \pi_0 A$, $\mathcal{O}(U_f) = A[f^{-1}]$.

Theorem $\mathcal{E}_n$-geometry ~~works~~ works as well as $\mathcal{E}_\infty$-geometry 90% of the time.

Given X an $\mathcal{E}_n$-scheme or stack, QC_X is an $\mathcal{E}_{n-1}$-monoidal ∞-cat.

$\{(\infty,k)\text{-cat}/X\}$ is an $\mathcal{E}_{n-k-1}$-monoidal $(\infty,k+1)$-category.

<u>Applications?</u>

(1) $\operatorname{Rep}(U_q(G))$. Where's the actual group?

Take the "Tannakian dual" of this $\mathcal{E}_2$-monoidal category. ~~in the~~

Produces $\mathcal{E}_3$-stack B, candidate for the "classifying stack of the quantum group."

(2) In char 0, n odd, $\mathcal{E}_n$-schemes come from symplectic geometry.

Theorem (Tamarkin, Kontsevich) $\mathcal{E}_n$-operad is formal.

$$\mathcal{E}_n(2) \simeq \operatorname{Conf}(2\text{pts. in } D^n) \cong S^{n-1}$$

$A \in \mathcal{E}_n\text{-alg}(\mathrm{Mod}_k)$ $\qquad C_*(\mathcal{E}_n(2)) \otimes A^{\otimes 2} \to A$

$$\rightsquigarrow H_*A \otimes H_*A \xrightarrow[\deg\ n-1]{\{,\}} H_*A$$

<u>Theorem</u> (F. Cohen) This defines a Poisson algebra, $\{\ ,\ \}$ is a Lie bracket.

<u>Theorem</u> $\mathcal{E}_n \simeq H_*(\mathcal{E}_n)$ in char. 0.

For A a Poisson algebra, $A[q] \otimes A[q] \xrightarrow{\{,\}_q} A[q]$

$|q| = n-1$ even $\quad f \otimes g \longmapsto \{f, g\}_q$

~~These~~ makes $A[q]$ into an $\mathcal{E}_n$-algebra.

This is <u>local</u> in A.

X a Poisson variety, can do the same. e.g., X = holomorphic symplectic manifold.

<u>2nd Proposal</u>

3d TQFT Z

$Z(S^1)$ is an ∞-cat $/k$, maybe ~~QC(X)~~. QC_X? No, "weird"

But we can thicken X given a symplectic structure.

People thinking about this: Ben-Zvi, Costello, Nadler