May 19: Bertrand Toën: Secondary K-theory: Lecture II

Reminders (+ complements)

k commutative ring — dualizable = saturated

$(H_0^{mor}(dgcat/k), \overset{L}{\otimes}_k)$ $\qquad [T,T'] = \{E \in D(T \overset{L}{\otimes}_k T'^{op}) \mid \forall x, E_x \text{ is compact}\}$

$\cap|$

$(H_0^{ct}(dgcat/k), \overset{L}{\otimes}_k)$ $\qquad$ Same objects $[T,T'] = D(T\otimes T'^{op})/\text{isomorphism}$

rigid $\otimes$-cat

Prop $T \in H_0^{mor}(dgcat/k)$ is saturated $\Leftrightarrow$ $T \sim B$, where B is a dg-algebra$/k$

s.t. $\quad$ * B is compact $\in D(k)$ $\qquad$ ("proper")

$\quad$ ** B is compact $\in D(B \otimes_k B^{op})$ $\qquad$ ("smoothness")

$\alpha \in H^2_{et}(X, \mathbb{G}_m)$ $\qquad X = \operatorname{Spec} k$

$\rightsquigarrow A_\alpha$ Azumaya k-algebra $\qquad$ smooth + proper

$\rightsquigarrow L_{pe}(A_\alpha)$ is saturated $\Longrightarrow L_{pe}(A_\alpha) \in K_0^{(2)}(k)$

$\Longrightarrow H^2(X, \mathbb{G}_m) \longrightarrow K_0^{(2)}(k)$

More generally, A is a k-algebra, proj of finite type over k

$\forall L$ field, $k \longrightarrow L$, $A \otimes L$ has finite cohomological dimension

$\Rightarrow L_{pe}(A)$ is saturated $\qquad$ (e.g., A = quiver algebra)

Basic * $k \longmapsto K_0^{(2)}(k)$ is a functor in k

$k \to k'$ $\quad - \otimes_k k'$ $\quad H_0^{mor}(dgcat/k) \longrightarrow H_0^{mor}(dgcat/k')$

- $K_0^{(2)}(k) = \pi_0(K^{(2)}(k))$ $\quad$ where $K^{(2)}(k)$ is a commutative ring spectrum

(Theorem) $k = \operatorname{colim}_{\text{filtered}} k_\alpha$ $\qquad K^{(2)}(k) = \operatorname{colim} K^{(2)}(k_\alpha)$

We want $\begin{cases} (*) \text{ extend } K^2(k) \text{ to schemes and algebraic stacks} \\ (**) \ ch: K^{(2)}(X) \longrightarrow ? \end{cases}$

need ∞-cat theory

(scheme, stack)

Uses cobordism hypothesis in dim 1.

ch = "de Rham realization of n.c. motive$/X$"

Lecture 2: Segal categories

Ideas of what higher categories are $\longrightarrow$ Jacob's talk.

"(1,∞)-category" (or (∞,1)) = ∞-category s.t. n-morphisms are invertible for $n>1$.
= ∞-category where Hom are ∞-groupoids

$$\text{∞-groupoids} \underset{\Pi_\infty}{\overset{|\cdot|}{\rightleftarrows}} \text{homotopy types} = \text{SSet}, \quad \cong$$

Naive definition A (1,∞)-category is a SSet-enriched category = $\mathcal{S}$-category.

$\mathcal{S}$-categories are badly behaved with respect to category of functors.

Def A Segal category ("∞-category") is $A: \Delta^{op} \longrightarrow \text{SSet}$

s.t. (1) A_0 is discrete (i.e., a set)

(2) $A_n \longrightarrow A_1 \times_{A_0} A_1 \times_{A_0} \cdots \times_{A_0} A_1$ is a weak equivalence.

$[1] \to [n]$, $0 \mapsto i$, $1 \mapsto i+1$

Remark If A is an $\mathcal{S}$-category,

$$A: \Delta^{op} \longrightarrow \text{SSet}$$

$$m \longmapsto \coprod_{(a_0,\dots,a_m)} A(a_0,a_1) \times A(a_1,a_2) \times \cdots \times A(a_{m-1},a_m)$$

$$\Longrightarrow \mathcal{S}\text{-Cat} \xrightarrow{\text{full}} \text{Segal Cat} \overset{\text{full}}{\subseteq} \text{Fun}(\Delta^{op}, \text{SSet})$$

A Segal cat. A comes from an $\mathcal{S}$-cat

$\Longleftrightarrow$ $A_n \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ is an isomorphism.

Def If $[A]$ is an ∞-category, $[A]$ = cat
- same objects as A
- $[A](x,y) = \pi_0(x,y)$

$[A]$ = "1-truncation" (or "homotopy category")

$f: A \to B$ a morphism between ∞-cats is

$\begin{cases} - \underline{\text{fully faithful}} \text{ if } A(x,y) \to B(fx,fy) \text{ is a W. equiv.} \\ - \underline{\text{ess. surjective}} \text{ if } [A] \to [B] \text{ is so.} \end{cases}$

<u>Main theorem</u> (Hirschowitz-Simpson/ Pellisier/ Bergner)

There is a model category structure on $\underbrace{\{F: \Delta^{op} \to SSet \mid F_0 \text{ is discrete}\}}$. $\quad$ Pr Se Cat $/\!/$

s.t. * W. equiv $\cap$ {Segal cat} = Fully faithful + ess. surjective,

* cofibration = monomorphism
* fibrant objects = Segal cat + Reedy fibrancy condition
* $\boxed{\text{monoidal model category for the direct product}}$ (. "allows us to define functor categories" $\hookrightarrow$ adjoint is internal Hom
* $\$$-cat $\hookrightarrow$ Pr Se Cat

"is" a Quillen equivalence (not quite)

$\Rightarrow Ho(\$\text{-cat}) \cong \cancel{Ho(SeCat)}\ Ho(\infty\text{-cat})$

<u>Consequence</u> A, B ∞-cat. $\quad R\underline{Hom}(A,B) = \underline{Hom}(A, RB)$,

where RB is a fibrant model for B.

$R\underline{Hom}(A,B)$ is an ∞-cat, the ∞-cat of weak ∞-functors

$SeCat^{fib} + \underline{Hom} \Rightarrow$ Cat. enriched over ∞-categories; example of $(2,\infty)$-category.

<u>Localization</u>

A an ∞-cat, $S \subseteq [A]$ $\qquad A \to L_S A$ of ∞-cat

s.t.

$$\begin{tikzcd} R\underline{Hom}(L_S A, B) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & R\underline{Hom}(A,B) \arrow[d] \\ Hom(S^{-1}[A],[B]) \arrow[r] & Hom([A],[B]) \end{tikzcd}$$

<u>Remark</u> $L_S A$ always exists.

We can start with A a category (non-∞)

<u>Eg</u> $\left.\begin{array}{l} A = M \text{ a model category} \\ S = W \end{array}\right) \Rightarrow L_W M$

<u>Theorem</u> M "nice enough" and simplicial $\Rightarrow$ $L_W M = \underline{M}^{fib\text{-}cof}$

$$L_W M(x,y) \cong \mathrm{Map}_M(x,y)$$

* Same assumption

$\forall A$ a category,

$$\begin{tikzcd} R\underline{\mathrm{Hom}}(A, L_W M) \arrow[r, "\cong"] \arrow[dr, no head, "\cong"'] & L_W(M^A) \arrow[d, "\cong"] \\ & (\underline{M^A})^{fib\,cof} \end{tikzcd}$$

<u>Lecture 3</u>
* limits, colimits of ∞-cat
* ∞-cat of dg-cats
* $\otimes$-∞-cat and rigidity

<u>Lecture 4</u>
- derived alg. geo.
- construct ch.