May 21: Bertrand Toën: Secondary K-theory: Lecture III

End of lecture 2:

∞-cat = Segal categories

$$(\mathcal{C}, W) \rightsquigarrow L_W\mathcal{C} \in \infty\text{-cat}$$

cat. weak equiv.

Example: $(\cdot)\,(Perf(k), \text{q-isom}) \rightsquigarrow LPerf(k) \in \infty\text{-cat}$ $\quad L_{pe}(k) \in dgcat$ $\ni$

$(\cdot)\,(dgcat/k, \text{Morita eq.}) \rightsquigarrow \mathbb{D}_g^{mor}(k) \stackrel{=}{} \infty$-cat of dg-cats

$[L\,Perf(k)] \cong D_{perf}(k)$

$[\mathbb{D}_g^{mor}(k)] \cong Ho^{mor}(dgcat/k) \longleftarrow$ Hom $= D(T \otimes T')$ + conditions

Description of Hom:

- $E, F \in LPerf(k)$

  $\pi_i(LPerf(k)(E,F), \cdot) \cong Ext^{-i}_{D(k)}(E,F)$

- $T, T' \in D_g^{mor}(k)$

  $\pi_1(\mathbb{D}_g^{mor}(k)(T,T'), E) \cong aut(E) \qquad E \in D(T \otimes T')$

  $\pi_i(\mathbb{D}_g^{mor}(k)(T,T'), E) \cong \cancel{Ext^{1-i}_{D(T\otimes T')}(E,E)}\ Ext^{1-i}_{D(T\otimes T')}(E,E)$

$$\Longrightarrow \pi_i(\mathbb{D}_g^{mor}(k)(T,T)) \cong HH^{1-i}(T) = \text{Hochschild cohomology}$$

$$\| $$

$$Ext^{1-i}_{D(T\otimes T')}(T,T)$$

$(\cdot)\ (\infty\text{-cat}, \text{equiv}) \rightsquigarrow \underline{\infty\text{-Cat}} = \infty$-cat of ∞-cats

Adjunctions and limits

$f: A \to B$ in ∞-Cat.

Def f has a right adjoint if $\exists g: B \to \overbrace{RA \hookleftarrow^{i} A}^{\text{fibrant replacement}}$ and $u \in \underline{Hom}(A, RA)(i, gf)$ s.t. $\forall a \in A,\ b \in B$

$$B(f(a), b) \xrightarrow{g} RA(gf(a), g(b)) \longrightarrow RA(i(a), g(b))$$

with the composite $B(f(a),b) \xrightarrow[\text{w.e.}]{\cong} RA(i(a), g(b))$.

This gives a notion of limit and colimit in a given ∞-cat:

$A \in \infty\text{-cat}$, $I \in \infty\text{-cat}$

$$\begin{tikzcd} A \arrow[r, "c_*"] & \mathbb{R}\underline{\mathrm{Hom}}(I,A) \arrow[l, bend right, "\lim = \text{right adjoint}"'] \arrow[l, bend left, "\mathrm{colim} = \text{left adjoint}"] \end{tikzcd}$$

If M is a (nice) model category, then $L_w M$ has all limits and colimits, and they "are" the holim and hocolim defined in model category theory.

$$\begin{tikzcd} & \mathbb{R}\underline{\mathrm{Hom}}(I, L_w M) \arrow[dl, bend right, "\lim"'] \arrow[d, "\cong"] \\ L_w M \arrow[r, phantom, "\circlearrowright"] & L_w(M^I) \arrow[ul, bend left, "\mathrm{holim}"] \end{tikzcd}$$

Lecture 3

(1) $\mathbb{D}_g^{mor}(X)$, $\mathbb{D}_g^{ct}(X)$ (continuous), $\mathbb{D}_g^{sat}(X)$ (saturated)

X is now a scheme or an algebraic stack (e.g., $[X/G]$ for scheme X) or any ∞-stack.

$$\boxed{\mathbb{D}_g^{mor}(X) = \lim_{(\mathrm{Spec}\,k \to X)} \mathbb{D}_g^{mor}(k)} \in \infty\text{-}\underline{\mathrm{Cat}}$$

$$\shortparallel$$

$$\Gamma(X, \underline{\mathbb{D}_g^{mor}})$$

Similarly for $\mathbb{D}_g^{ct}$, $\mathbb{D}_g^{sat}$

__Remark__ • $L\mathrm{Perf}(X) = \lim_{(\mathrm{Spec}\,k \to X)} L\mathrm{Perf}(k)$, ∞-cat model for $[L\mathrm{Perf}(X)] \cong D_{parf}(X)$

• $\underline{\mathrm{End}}_{\mathbb{D}_g^{ct}(X)}(\mathbb{1}) \cong LQCoh(X)$ (what David calls $Q(X)$)

Idea of construction of Ch

$T \in \mathbb{D}_g^{ct}(X)$ $\quad T$ is a "sheaf of dgcat$/X$"

We consider the __loop space__ of X, $\mathcal{L}X \xrightarrow{\pi} X$. (Think of as "$\underline{\mathrm{Hom}}(S^1, X)$", will define later.)

$\pi^*(T) \in \mathbb{D}_g^{ct}(\mathcal{L}X)$. This comes with a natural autoequivalence

$M: \pi^*(T) \xrightarrow{\simeq} \pi^*(T)$,
$$\begin{cases} M = \text{monodromy along the loop} \\ \mathcal{L}X \times S^1 \xrightarrow{ev} X \Rightarrow \text{self-homotopy equivalence of } \pi \\ \qquad \Rightarrow \text{self-equivalence of } \pi^* \end{cases}$$

$\mathbb{D}_g^{ct}(Y)$ is a rigid monoidal ∞-cat ~~$\Rightarrow Tr(M) \in \mathbb{D}_g^{ct}(\mathcal{L}X)$~~

$\Longrightarrow Tr(M) \in \mathbb{D}_g^{ct}(\mathcal{L}X)(\mathbb{1},\mathbb{1}) \simeq LQCoh(\mathcal{L}X)$ "$= \widetilde{Q(\mathcal{L}X)}$"

__Key result__ $Tr(M)$ is __S^1-invariant__. This follows from Lurie's theorem on 1-Bord.

__Remark__ (1) We can replace $\mathbb{D}_g(-)$ by any functor "Schemes" $\xrightarrow{\mathcal{A}} \left\{ \begin{matrix} \text{rigid monoidal} \\ \infty\text{-cats} \end{matrix} \right\}$

$$\mathcal{A}(X) \xrightarrow{ch} \operatorname{End}_{\mathcal{A}(\mathcal{L}X)}(\mathbb{1})^{S^1}$$

If $\mathcal{A} = LPerf$, then $\operatorname{End}_{\mathcal{A}(\mathcal{L}X)}(\mathbb{1})^{S^1} \simeq HC^-(X)$ and $ch =$ usual Chern character.

(2)
$$\begin{tikzcd} Ch(T) = Dqcoh^{S^1}(\mathcal{L}X) \arrow[d, "\simeq"] \arrow[dr] \\ D(\mathcal{D}_X\text{-mod}) \end{tikzcd}$$

"$\downarrow \simeq$" ← __Theorem__ (Ben-Zvi, Nadler) will explain next time

$//$ de Rham realization of T as a n.c. variety$/X$.

__$\otimes$-∞-cat__

A $\otimes$-∞-cat is a "monoid object in ∞-cat."

A commutative monoid in Set is a Γ-object in Set.

$\Gamma =$ category of pointed finite sets $n = \{0^*, \dots, n\}$

$$\Gamma \xrightarrow{M} \text{Sets} \qquad \begin{cases} M_0 = * \\ M_n \xrightarrow{\simeq} M_1^n \end{cases}$$

A $\otimes$-∞-cat is ($\otimes$ = symmetric monoidal)

$A : \Gamma \to \infty$-cat s.t. $\begin{cases} A_0 \xrightarrow{\simeq} * \\ A_n \xrightarrow{\simeq} A_1 \times \cdots \times A_1 \end{cases}$ equiv. of ∞-cats.

$\otimes$-∞-cat form an ∞-cat : ∞-$\underline{Cat}^{\otimes}$.

Exercise If A is a $\otimes$-∞-cat, then $[A_1]$ is endowed with a natural $\otimes$-structure.

Remark $\otimes$-∞-cat can be obtained by localization of a $\otimes$-cat with respect to a set of maps.

e.g. dgcat$/k$, $\otimes_k$, Morita