

May 21: Bertrand Toën: Secondary K-theory: Lecture III

End of lecture 2:

∞ -cat = Segal categories

$$(\mathcal{C}, W) \rightsquigarrow L_w \mathcal{C} \in \infty\text{-cat}$$

cat: weak equiv.

$L_{\text{pe}}(k) \in \text{dgcat} \exists$

Example $(\circ) (\text{Perf}(k), q\text{-isom}) \rightsquigarrow L\text{Perf}(k) \in \infty\text{-cat}$

$(\circ) (\text{dgcat}/k, \text{Morita eq.}) \rightsquigarrow \mathbb{D}_g^{\text{mor}}(k) \cong \infty\text{-cat of dg-cats}$

$$[L\text{Perf}(k)] \cong \mathbb{D}_{\text{perf}}(k)$$

$$[\mathbb{D}_g^{\text{mor}}(k)] \cong \text{Ho}^{\text{mor}}(\text{dgcat}/k) \longleftarrow \text{Hom} = \mathbb{D}(T \otimes T') + \text{conditions}$$

Description of Hom:

- $E, F \in L\text{Perf}(k)$

$$\pi_i(L\text{Perf}(k)(E, F), \circ) \cong \text{Ext}_{\mathbb{D}(k)}^{-i}(E, F)$$

- $T, T' \in \mathbb{D}_g^{\text{mor}}(k)$

$$\pi_i(\mathbb{D}_g^{\text{mor}}(k)(T, T'), E) \cong \text{aut}(E) \quad E \in \mathbb{D}(T \otimes T')$$

$$\pi_i(\mathbb{D}_g^{\text{mor}}(k)(T, T'), E) \cong \text{Ext}_{\mathbb{D}(T \otimes T')}^{-i}(E, E) \quad \text{Ext}_{\mathbb{D}(T \otimes T')}^{-i}(E, E)$$

$$\Rightarrow \pi_i(\mathbb{D}_g^{\text{mor}}(k)(T, T') \cong \text{HH}^{-i}(T) = \text{Hochschild cohomology}$$

$$\parallel$$

$$\text{Ext}_{\mathbb{D}(T \otimes T')}^{-i}(T, T')$$

$(\circ) (\infty\text{-cat, equiv}) \rightsquigarrow \underline{\infty\text{-Cat}} = \infty\text{-cat of } \infty\text{-cats}$

Adjunctions and limits

$f: A \rightarrow B$ in $\infty\text{-Cat}$.

fibrant repluzement

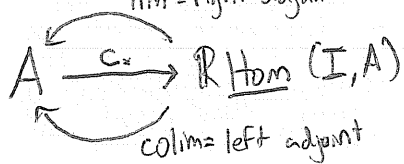
Def f has a right adjoint if $\exists g: B \rightarrow RA \xrightarrow{\text{fib}} A$ and

$$u \in \text{Hom}(A, RA)(i, gf) \xrightarrow{\text{w.e.} \cong} \text{s.t. } \forall a \in A, b \in B$$

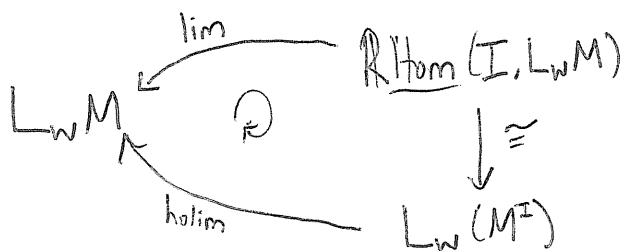
$$B(f(a), b) \xrightarrow{g} RA(gf(a), g(b)) \rightarrow RA(i(a), g(b))$$

This gives a notion of limit and colimit in a given ∞ -cat.

$A \in \infty$ -cat, $I \in \infty$ -cat
 $\lim = \text{right adjoint}$



If M is a (nice) model category, then $L_w M$ has all limits and colimits, and they "are" the holim and hocolim defined in model category theory.



Lecture 3

continuous saturated

(1) $\mathbb{D}_g^{\text{mor}}(X)$, $\mathbb{D}_g^{\text{ct}}(X)$, $\mathbb{D}_g^{\text{sat}}(X)$

X is now a scheme or an algebraic stack (e.g., $[X/G]$ for scheme X) or any ∞ -stack.

$$\boxed{\mathbb{D}_g^{\text{mor}}(X) = \lim_{(\text{Spec } k \rightarrow X)} \mathbb{D}_g^{\text{mor}}(k)} \in \infty\text{-Cat}$$

\parallel
 $\Gamma(X, \mathbb{D}_g^{\text{mor}})$

Similarly for \mathbb{D}_g^{ct} , $\mathbb{D}_g^{\text{sat}}$

Remark • $L\text{Perf}(X) = \lim_{(\text{Spec } k \rightarrow X)} L\text{Perf}(k)$, ∞ -cat model for $[L\text{Perf}(X)] \cong \mathbb{D}_{\text{perf}}(X)$

• $\text{End}_{\mathbb{D}_g^{\text{ct}}(X)}(\mathbb{1}) \cong LQ\text{Coh}(X)$ (what David calls $Q(X)$)

Idea of construction of Ch

$T \in \mathbb{D}_g^{\text{ct}}(X)$ T is a "sheaf of dgcat / X "

We consider the loop space of X , $\int X \xrightarrow{\pi} X$.

(Think of as "Hom(S^1, X)", will define later.)

$\pi^*(T) \in \mathbb{D}_g^{ct}(\mathbb{L}X)$. This comes with a natural autoequivalence

$$M: \pi^*(T) \xrightarrow{\cong} \pi^*(T), \quad \begin{cases} M = \text{monodromy along the loop} \\ \mathbb{L}X \times S^1 \xrightarrow{ev} X \Rightarrow \text{self-homotopy equivalence of } \pi \\ \Rightarrow \text{self-equivalence of } \pi^* \end{cases}$$

$\mathbb{D}_g^{ct}(Y)$ is a rigid monoidal ∞ -cat ~~$\Rightarrow \text{Tr}(M) \in \mathbb{D}_g^{ct}(\mathbb{L}X)$~~

$\Rightarrow \text{Tr}(M) \in \mathbb{D}_g^{ct}(\mathbb{L}X)(\mathbb{1}, \mathbb{1}) \cong \text{LQCoh}(ZX) \quad \text{"} = \mathbb{Q}(\mathbb{L}X) \text{"}$

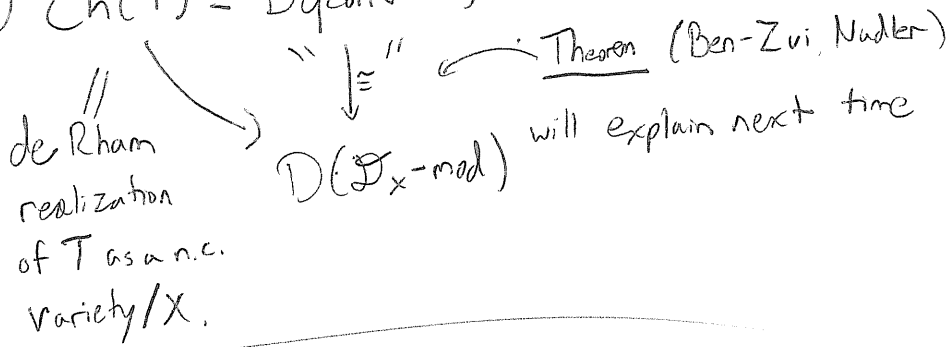
Key result $\text{Tr}(M)$ is S^1 -invariant. This follows from Lurie's theorem on $\widetilde{\text{Bord}}$.

Remark (1) We can replace $\mathbb{D}_g^{ct}(-)$ by any functor "Schemes" $\xrightarrow{A} \left\{ \begin{array}{l} \text{rigid monoidal} \\ \infty\text{-cats} \end{array} \right\}$

$$A(X) \xrightarrow{ch} \text{End}(\mathbb{1})^{S^1} \xrightarrow{A(X)} \mathbb{R}$$

IF $A = \text{LPerf}$, then $\text{HC}^-(X)$ and $ch = \text{usual Chern character}$.

(2) $Ch(T) = \text{Dqcoh}^{S^1}(\mathbb{L}X)$



\otimes - ∞ -cat

A \otimes - ∞ -cat is a "monoid object in ∞ -cat."

A commutative monoid in Set is a Γ -object in Set .

$\Gamma = \text{category of pointed finite sets } n = \{0^s, \dots, n\}$

$$\Gamma \xrightarrow{M} \text{Sets} \quad \begin{cases} M_0 = * \\ M_n \cong M_1^n \end{cases}$$

A \otimes - ∞ -cat is ($\otimes = \text{Symmetric monoidal}$)

$$A: \Gamma \rightarrow \infty\text{-cat} \quad \text{s.t.} \quad \begin{cases} A_0 \xrightarrow{\cong} * \\ A_n \xrightarrow{\cong} A_1 \times \dots \times A_1 \end{cases} \quad \text{equiv. of } \infty\text{-cats.}$$

\otimes - ∞ -cat form an ∞ -cat: $\infty\text{-Cat}^{\otimes}$.

Exercise If A is a \otimes - ∞ -cat, then $[A_i]$ is endowed with a natural \otimes -structure.

Remark \otimes - ∞ -cat can be obtained by localization of a \otimes -cat with respect to a set of maps.

e.g. $\text{dgcats}/k, \otimes_k, \text{Morita}$