ALGEBRAIC GEOMETRY — SECOND HOMEWORK (DUE FRIDAY JANUARY 31)

Please illustrate your answers with diagrams and examples as appropriate.

1. If k is an infinite perfect field (e.g. any field of char. zero, or any algebraically closed field) and if $f \in k[x, y]$ is non-constant irreducible, cutting out the affine plane curve C, then show that there is a finite extension l of k such that $C(l)$ contains a smooth point. [Hint: To begin with, think of $f(x, y)$ as an irreducible element $k(y)[x]$, and consider its derivative f_x with respect to x. Use known facts about irreducible polynomials in one variable and their derivatives to make some deductions.]

2. Give an example of a non-perfect field k and an irreducible polynomial $f \in k[x, y]$ so that $C(l)$ contains no smooth point for any finite field extension l of k.

3. If k is an algebraically closed field and $f \in k[x, y]$ is non-constant irreducible, show that you can make an affine linear change of coordinates so that f can be written in the form $y^d + f_{d-1}(x)y^{d-1} + \cdots + f_0$, *and* such that f_y is non-zero. Conclude that for all but finitely many choices of $x \in k$, the d solutions to $f(x, y) = 0$ (for this fixed value of x) are *distinct*. [Hint: Use the argument from class about projecting from a point at infinity not lying on the projective closure of C, but be a little bit more careful so that you control the y-derivative as well.]

4. If k is an algebraically closed field and $f \in k[x, y]$ is non-constant irreducible, cutting out the curve C, and if $g \in k[x, y]$ is non-constant, cutting out the curve D, and if $C(k) \subseteq D(k)$, then prove that f divides g in $k[x, y]$. [Hint: Change coordinates as in the previous question, and then use the division algorithm to write $g = qf + r$ in $k[x, y]$, where the degree in y of r is less than the degree in y of f. Now deduce that r must actually vanish.]

5. If k is an algebraically closed field, and $f, g \in k[x, y]$ are non-constant polynomials cutting out curves C and D, show that $C(k) \subseteq D(k)$ iff every irreducible factor of f is an irreducible factor of g.

6. Let C be the *nodal cubic curve* $y^2 = x^3 + x^2$. By considering the intersection of C with the line $y = tx$ of slope t passing through the origin, find a surjective map $\mathbb{A}^1(k) \to C(k)$, defined by a pair of polynomials $\big(x(t), y(t)\big)$, which is even a bijection except at two points.