ALGEBRAIC GEOMETRY — FOURTH HOMEWORK (DUE MONDAY MAR 9)

Please complete all the questions. For each question, please provide examples/graphs/pictures illustrating the ideas behind the question and your answer. Throughout you should presume given a field k contained in an algebraically closed field K, work with K-valued points, and feel free to use the Nullstellensatz for K if necessary.

1. Consider the ideal $I = (XY - sZ^2, Y - tX, stXZ) \subseteq k[s,t,X,Y,Z]$. Think of s,t as the coordinates on $\mathbb{A}^2$, and X,Y,Z as the homogeneous coordinates on $\mathbb{P}^2$. Note that the generators of I are homogeneous in X,Y,Z, and so I cuts out a zero locus $Z_I(K) \subseteq (\mathbb{A}^2 \times \mathbb{P}^2)(K)$. What is the image of $Z_I(K)$ in $\mathbb{A}^2(K)$ under the projection?

2. (a) Prove that $\varphi : t \mapsto (t^2, t^3)$ defines a morphism from $\mathbb{A}^1(K)$ to $Z_I(K) \subseteq \mathbb{A}^2(K)$, where $I \subseteq k[x,y]$ is the ideal $(y^2 - x^3)$.

(b) Show that φ induces a bijection on points, but is not an isomorphism.

(c) Prove that there is a unique extension of φ to a morphism $\widetilde{\varphi} : \mathbb{P}^1(K) \to Z_{\widetilde{I}}(K) \subseteq \mathbb{P}^2$, where $\widetilde{I} \subseteq k[X,Y,Z]$ is the homogeneous ideal $(X^3 - Y^2 Z)$.

3. Let x,y be coordinates on $\mathbb{A}^2$, and let X,Y be homogeneous coordinates on $\mathbb{P}^1$. Consider the ideal $I = (xY - yX) \subseteq k[x,y,X,Y]$, which cuts out an algebraic set $Z := Z_I(K) \subseteq (\mathbb{A}^2 \times \mathbb{P}^1)(K)$. Let $\pi : Z \to \mathbb{A}^2(K)$ be the projection.

(a) Let $U = \mathbb{A}^2(K) \setminus \{0\}$; note that U is a Zariski open subset of $\mathbb{A}^2(K)$. Show that the restriction of π to $\pi^{-1}(U)$ induces an isomorphism $\pi^{-1}(U) \xrightarrow{\sim} U$.

(b) Prove that $\pi^{-1}\big((0,0)\big)$ is a Zariski closed subset of Z that is isomorphic to $\mathbb{P}^1(K)$.

[The algebraic set Z is called the *blow-up* of $\mathbb{A}^2$ at the origin; the process of forming Z from $\mathbb{A}^2$ is called *blowing up*. This question shows that blowing up at the origin leaves $\mathbb{A}^2$ unchanged away from the origin, but replaces the origin by a copy of $\mathbb{P}^1$ (which more canonically can be thought of the space of directions of lines passing through the origin).]

4. Maintain the notation Z, π of the previous question. Let $C \subseteq \mathbb{A}^2(K)$ be the curve cut out by $y^2 = x^3$ (i.e. $C = Z_J(K)$, where $J = (y^2 - x^3) \subseteq k[x,y]$).

(a) Prove that $C \setminus \{(0,0)\}$ is isomorphic to $\mathbb{A}^1(a) \setminus \{0\}$. (This is a variation on Question 2 above.)

(b) Let $\widetilde{C}$ denote the Zariski closure in Z of $\pi^{-1}(C \setminus \{(0,0)\})$. Prove that your isomorphism of (a) extends to an isomorphism between $\mathbb{A}^1(K)$ and $\widetilde{C}$.

[We call $\widetilde{C}$ the *proper transform* of C in the blow-up of $\mathbb{A}^2$ at the origin. This question shows that although C is singular at the origin, its proper transform $\widetilde{C}$ is non-singular (since it is isomorphic to the non-singular curve $\mathbb{A}^1$). A big part of the interest in blowing-up is that it provides a tool for resolving singularities.]

5. Let C be a smooth curve in $\mathbb{P}^2(K)$, thought of now as a Zariski closed subset. Let ℓ be a line in $\mathbb{P}^2(K)$, and suppose that ℓ is *not* tangent to C at any point.

If P is a point of C, let t_P denote the tangent line to C at P. Define a map $f : C \to \ell$ via

$$P \mapsto \text{ the unique point of intersection of } t_P \text{ and } \ell.$$

Prove that φ is a morphism.

6. Let $Q \subset \mathbb{P}^3(K)$ be the quadric cut out by the homogeneous equation

$$X^2 + Y^2 - Z^2 - W^2 = 0.$$

Find an injective morphism from $\mathbb{A}^2(K)$ to Q. Can you extend this to a morphism from $\mathbb{P}^2(K)$? If not, what is the largest open subset of $\mathbb{P}^2(K)$ containing $\mathbb{A}^2(K)$ to which you *can* extend it?

7. If X is a topological space, recall that a subset Y of X is called *locally closed* if every point P in Y admits a neighbourhood U in X such that $U \cap Y$ is closed in U (when U is equipped with the induced topology). Show that Y is locally closed if and only if it can be written as the intersection of an open and a closed subset of X.

8. If X is a topological space and Z is a subset of X, show that the following three conditions are equivalent:

(i) Every point $P \in X$ admits a neighbourhood U such that $U \cap Z$ is closed in U (when we equip U with the induced topology).

(ii) There exists an open cover $\{U_i\}_{i \in I}$ of X such that $Z \cap U_i$ is closed in U_i for all I (when U_i is given the induced topology).

(iii) Z is a closed subset of X.

What is the difference between condition (i) and the condition appearing in the definition of *locally closed*?