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ON THE VANISHING OF CERTAIN EXTENSIONS OF ADMISSIBLE SMOOTH REPRESENTATIONS IN CHARACTERISTIC p

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Let k be a finite field of characteristic p and G be a p-adic analytic group. We work in the category $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(k)$ [2], which is antiequivalent to the category of admissible smooth G-representations over k.

For any $i \geq 0$, we write $E^i(-) := \mathrm{Ext}^i_{k[[H]]}(-, k[[H]])$, for some fixed compact open subgroup H of G. Up to natural isomorphism, these functors are independent of the choice of H, and taken together form a δ functor from $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(k)$ to itself. We recall that an object M of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(k)$ is said to be of codimension $\geq c$ (resp. codimension c) if $E^i(M) = 0$ for $i < c$ (and in addition $E^c(M) \neq 0$).

For any object M of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(k)$, there is a double duality fourth quadrant spectral sequence

$$E_2^{i,j} := E^{-i}E^{-j}(M) \implies M,$$

which is natural in M. In particular, if M is non-zero of codimension c then there is a natural non-zero double duality map $M \to E^cE^c(M)$. (In particular, $E^c(M)$ is again of codimension c.)

Proposition 1. *If M is an object of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(k)$ of codimension c for which $E^c(M)$ is irreducible, and for which the double duality map $M \to E^cE^c(M)$ is an isomorphism, then $\mathrm{Ext}^1(N, M) = 0$ for all objects N of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(k)$ of codimension $>\ c$.*

Proof. Consider an extension $0 \to M \to E \to N \to 0$, where N has codimension $>\ c$. Applying the delta functor $E^\bullet$, we obtain the exact sequence

$$0 \longrightarrow E^c(E) \to E^c(M) \xrightarrow{\delta} E^{c+1}(N).$$

Since $E^c(M)$ is irreducible (by assumption) of codimension c, while $E^{c+1}(N)$ is of codimension $>\ c$, we see the connecting homomorphism δ necessarily vanishes, and so $E^c(E) \xrightarrow{\sim} E^c(M)$. Applying E^c to this isomorphism, we obtain an isomorphism $M \xrightarrow{\sim} E^cE^c(M) \xrightarrow{\sim} E^cE^c(E)$, the first isomorphism holding by assumption. The double duality map $E \to E^cE^c(E)$ thus yields a splitting of the given extension. $\square$

Example 2. Let G be a p-adic reductive group and B be a Borel subgroup of G, and write $\alpha^u : B \to \mathbb{F}_p^\times$ to denote the mod p reduction of the unit part of the character α describing the action of B on $\det \mathfrak{n}$, where $\mathfrak{n}$ is Lie algebra of the unipotent radical of B. If χ is a character of B, then $(\mathrm{Ind}_B^G \chi)^\vee$ has codimension equal to $c := \dim B$, and $E^c\big((\mathrm{Ind}_B^G \chi)^\vee\big) = \big(\mathrm{Ind}_B^G \chi^{-1}\alpha^u\big)^\vee$, which is irreducible if χ is chosen generically. Thus for a generic character χ, we have $\mathrm{Ext}^1\big(N, (\mathrm{Ind}_B^G \chi)^\vee\big) = 0$ when N has codimension $> \dim B$.

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Example 3. We consider the previous example in the particular case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, so that $\dim B = 3$. (To fix ideas, we take B to the Borel subgroup of upper triangular matrices.) An object of $\mathrm{Mod}_G^{\mathrm{fg\ aug}}(k)$ has codimension 4 if and only if it is finite-dimensional over k, and so the previous result shows that for generic characters χ, there are no non-trivial extensions of $\mathrm{Ind}_B^G \chi$ by a finite dimensional representation of G (a result which can also be proved by a computation with ordinary parts [3, Prop. 4.3.13]).

Of course, this is not true for every character χ. For example, there is a non-trivial extension of $\mathrm{Ind}_B^G \varepsilon \otimes \varepsilon^{-1}$ by the trivial representation. This does not contradict the preceding result, because $E^3\big((\mathrm{Ind}_B^G \varepsilon \otimes \varepsilon^{-1})^\vee\big) = (\mathrm{Ind}_B^G \underline{1} \otimes \underline{1})^\vee$ (the character $\varepsilon \otimes \varepsilon^{-1}$ is precisely the character α^u considered above) and the representation $\mathrm{Ind}_B^G \underline{1} \otimes \underline{1}$ is not irreducible.

Example 4. Suppose that $f \geq 2$, and let $\rho : G_{\mathbb{Q}_{p^f}} \to \mathrm{GL}_2(k)$ be reducible but indecomposable, and suitably generic. Breuil and Paškūnas [1] have constructed a family of admissible smooth representations $\pi(\rho)$ of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ which are associated to ρ in a certain sense, as well as a family of admissible smooth representations $\pi(\rho^{\mathrm{ss}})$ which are similarly associated to the semi-simplification ρ^{ss} of ρ. Furthermore, $\pi(\rho^{\mathrm{ss}})$ can be taken to be the direct sum of two principal series representations (each of which is the induction of a generic character, since ρ was assumed to be generic) and $f - 1$ supersingular representations.

It does not seem to be known, but is generally hoped, that $\pi(\rho)$ and $\pi(\rho^{\mathrm{ss}})$ can be furthermore chosen so that $\pi(\rho^{\mathrm{ss}}) = \pi(\rho)^{\mathrm{ss}}$. If this is the case, then at least one of the principal series constituents of $\pi(\rho^{\mathrm{ss}})$ will admit an extension by at least one of the supersingular constituents. Our proposition implies that the dual of any such supersingular constituent has codimension at most f.

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References

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