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ON THE EFFACEABILITY OF CERTAIN δ-FUNCTORS

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1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$ and let $\mathfrak{o}$ be its ring of integers. Let $G := \mathrm{GL}_2(F)$, let $K := \mathrm{GL}_2(\mathfrak{o})$, and let Z be the centre of G. Let A be a finite local Artinian $\mathbb{Z}_p$-algebra with maximal ideal $\mathfrak{m}$ and residue field k **containing the residue field of** F. Recall that a representation V of G on an A-module is said to be *smooth* if for all $v \in V$ the stabilizer of v is an open subgroup of G. Let $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ denote the category of smooth A-representations. Further recall that a smooth A-representation V is *admissible* if for every open subgroup J of G the space V^J of J-invariants is finite dimensional. Let $\mathrm{Mod}^{\mathrm{adm}}_G(A)$ denote the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ consisting of admissible representations. The categories $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ and $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ are abelian. In practice, one is interested in admissible representations, but $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ does not have enough injectives. The category $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ has enough injectives, but it is too big. To remedy this the first author, in [2], [3], has introduced an intermediate category of locally admissible representations $\mathrm{Mod}^{\mathrm{l.adm}}_G(A)$. We recall the definition: If V is a smooth A-representation of G, a vector $v \in V$ is called *locally admissible* if the $A[G]$-submodule of V generated by v is admissible; a smooth representation V of G over A is then called *locally admissible* if every $v \in V$ is locally admissible. We let $\mathrm{Mod}^{\mathrm{l.adm}}_G(A)$ denote the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ consisting of locally admissible representation. The category $\mathrm{Mod}^{\mathrm{l.adm}}_G(A)$ is abelian and has enough injectives [2, Prop. 2.2.15], [3, Prop. 2.1.1].

We introduce some variants of the preceding categories:

If $\zeta : Z \to A^\times$ is a smooth character, then we denote by $\mathrm{Mod}^{\mathrm{adm}}_{G,\zeta}(A)$, $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$, and $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$ the full subcategories of $\mathrm{Mod}^{\mathrm{adm}}_G(A)$, $\mathrm{Mod}^{\mathrm{l.adm}}_G(A)$, and $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ respectively, consisting of representations admitting ζ as a central character. We also let $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(A)$ denote the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_K(A)$ consisting of K-representations admitting $\zeta_{|Z\cap K}$ as a central character. The categories $\mathrm{Mod}^{\mathrm{adm}}_{G,\zeta}(A)$, $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$, $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$, and $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(A)$ are abelian, and the last two have enough injectives. (See Lemma 2.5 (1) below.)

Let ϖ be a uniformizer of F, and view ϖ as an element of Z via the isomorphism $Z \cong F^\times$. If $u \in A^\times$, then we let $\mathrm{Mod}^{\mathrm{adm}}_{G,\varpi=u}(A)$, $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$, and $\mathrm{Mod}^{\mathrm{sm}}_{G,\varpi=u}(A)$ denote the full subcategories of $\mathrm{Mod}^{\mathrm{adm}}_G(A)$, $\mathrm{Mod}^{\mathrm{l.adm}}_G(A)$, and $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ respectively, consisting of representations on which ϖ acts via u. Again these categories are abelian and the last two have enough injectives. (See Lemma 2.5 (2) below.)

In this note we show that the restriction to K of an injective object in $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$ (resp. $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$ or $\mathrm{Mod}^{\mathrm{l.adm}}_G(A)$) is an injective object in $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(A)$ (resp.

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$\mathrm{Mod}_K^{\mathrm{sm}}(A)$). (Here $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$ has the obvious meaning, namely it is the full subcategory of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$ consisting of K-representations admitting $\zeta_{|Z\cap K}$ as a central character.) This implies that certain δ-functors defined in [3] are effaceable, and remain effaceable when restricted to $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(A)$ or $\mathrm{Mod}_{G,\varpi=u}^{\mathrm{sm}}(A)$. In particular, it proves Conjecture 3.7.2 of [3] for $\mathrm{GL}_2(F)$.

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2. Injectives

We establish some simple results about injective objects in various contexts.

2.1. **Lemma.** *If H is a compact p-adic analytic group, if V is an injective object of $\mathrm{Mod}_H^{\mathrm{sm}}(k)$, and if W is an injective envelope of V in $\mathrm{Mod}_H^{\mathrm{sm}}(A)$, then the inclusion $V \hookrightarrow W$ induces an isomorphism $V \xrightarrow{\sim} W[\mathfrak{m}]$.*

Proof. Certainly the inclusion $V \hookrightarrow W$ factors through an inclusion $V \hookrightarrow W[\mathfrak{m}]$. Since the source is injective, this inclusion splits. If C denotes a complement to the inclusion, then $V \cap C = 0$, and thus $C = 0$ (as W is an essential extension of V). This proves the lemma. ☐

2.2. **Lemma.** *Let H be a finite index open subgroup of G*

1. *An object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ is admissible (resp. locally admissible) as a G-representation if and only if it is so as an H-representation.*
2. *If V is an object of $\mathrm{Mod}_H^{\mathrm{sm}}(A)$, so that $\mathrm{Ind}_H^G V$ ($\xrightarrow{\sim} A[G] \otimes_{A[H]} V$) is an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, then $\mathrm{Ind}_H^G V$ is admissible (resp. locally admissible) as a G-representation if and only if V is admissible (resp. locally admissible) as an H-representation.*

Proof. The admissibility claim of part (1) is clear, since H contains a cofinal collection of open subgroups of G. Since H has finite index in G, the group ring $A[G]$ is finitely generated as an $A[H]$-module, and thus an $A[G]$-module is finitely generated if and only if it is finitely generated as an $A[H]$-module. The local admissibility claim of part (1) follows from this, together with the admissibility claim, since an $A[G]$-module (resp. $A[H]$-module) is locally admissible if and only if every finitely generated submodule is admissible.

To prove the if direction of claim (2), suppose first that V is an admissible H-representation. If we write G as a union of finitely many left H-cosets, say $G = \bigcup_{i=1^n g_i H}$, if H' is an open subgroup of H, and if we write $H'' := H' \cap \bigcap_{i=1}^n g_i H g_i^{-1}$, then

$$(\mathrm{Ind}_H^G V)^{H'} \subset (\mathrm{Ind}_H^G V)^{H''} \xrightarrow{\sim} (A[G] \otimes_{A[H]} V)^{H''} \xrightarrow{\sim} \oplus_{i=1}^n (g_i V)^{H''} = \oplus_{i=1}^n g_i V^{g_i^{-1} H'' g_i}.$$

Since $g_i^{-1} H'' g_i$ is an open subgroup of H, each of the summands appearing on the right-hand side is finite dimensional, and thus so is their direct sum. Thus $\mathrm{Ind}_H^G V$

is admissible as claimed. If we suppose that V instead is locally admissible, or equivalently, is the inductive limit of its admissible subrepresentations, we see that the same is true of $\mathrm{Ind}_H^G V$, since Ind_H^G commutes with the formation of induction limits (being naturally isomorphic to $A[G] \otimes_{A[H]} -$).

To prove the other direction of (2), note first that the inclusion $A[H] \subset A[G]$ gives rise to an H-equivariant embedding $V \hookrightarrow A[G] \otimes_{A[H]} V \xrightarrow{\sim} \mathrm{Ind}_H^G V$. Thus if $\mathrm{Ind}_H^G V$ is (locally) admissible as a G-representation, and hence also (locally) admissible as an H-representation, by part (1), the same is true of its H-subrepresentation V. □

2.3. **Proposition.** *If H is an open subgroup of G of finite index, then an object V of the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ (resp. $\mathrm{Mod}_G^{\mathrm{adm}}(A)$, resp. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$) is injective if and only if $\mathrm{Ind}_H^G V$ is injective as an object of the same category.*

Proof. Consider the sequence of adjunction isomorphisms

$$\mathrm{Hom}_{A[G]}(U, \mathrm{Ind}_H^G V) \xrightarrow{\sim} \mathrm{Hom}_{A[H]}(U, V) \xrightarrow{\sim} \mathrm{Hom}_{A[G]}(\mathrm{Ind}_H^G U, V).$$

Since the composite of Ind_H^G (which is naturally equivalent to $A[G] \otimes_{A[H]} -$) and the forgetful functor induces an exact functor from $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ (resp. $\mathrm{Mod}_G^{\mathrm{adm}}(A)$, resp. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$) to itself (here we are taking into account Lemma 2.2), the proposition follows. □

2.4. **Definition.**

(1) If $\zeta : Z \to A^\times$ is a smooth character and V is a representation of G (resp. K) over A, then we let

$$V^{Z=\zeta} := \{v \in V \mid z \cdot v = \zeta(z)v \text{ for all } z \in Z\}$$

(resp. $V^{Z \cap K=\zeta} := \{v \in V \mid z \cdot v = \zeta(z)v \text{ for all } z \in Z \cap K\}$.)

(2) If $u \in A^\times$ and V is a representation of G over A, then we let

$$V^{\varpi=u} := \{v \in V \mid \varpi \cdot v = uv\}.$$

Since the subrepresentation of a smooth admissible (resp. smooth locally admissible, resp. smooth) representation is again smooth admissible (resp. smooth locally admissible, resp. smooth), we see, in the context of the preceding definition, that the construction $V \mapsto V^{Z=\zeta}$ induces a functor $\mathrm{Mod}_G^{\mathrm{adm}}(A) \to \mathrm{Mod}_{G,\zeta}^{\mathrm{adm}}(A)$ (resp. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A) \to \mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$, resp. $\mathrm{Mod}_G^{\mathrm{sm}}(A) \to \mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(A)$) that is right adjoint to the forgetful functor, that the construction $V \mapsto V^{Z \cap K=\zeta}$ induces a functor $\mathrm{Mod}_K^{\mathrm{sm}}(A) \to \mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$ that is right adjoint to the forgetful functor, and that the construction $V \mapsto V^{\varpi=u}$ induces a functor $\mathrm{Mod}_G^{\mathrm{adm}}(A) \to \mathrm{Mod}_{G,\varpi=u}^{\mathrm{adm}}(A)$ (resp. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A) \to \mathrm{Mod}_{G,\varpi=u}^{\mathrm{l.adm}}(A)$, resp. $\mathrm{Mod}_G^{\mathrm{sm}}(A) \to \mathrm{Mod}_{G,\varpi=u}^{\mathrm{sm}}(A)$) that is right adjoint to the forgetful functor. In particular, the functors $V \mapsto V^{Z=\zeta}$, $V \mapsto V^{Z \cap K=\zeta}$, and $V \mapsto V^{\varpi=u}$ all preserve injectives.

2.5. **Lemma.**

(1) *If $\zeta : Z \to A^\times$ is a smooth character, then each of the categories $\mathrm{Mod}_{G,\zeta}^{\mathrm{adm}}(A)$, $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$, $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(A)$, $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$ are abelian, and the last three have enough injectives.*

(2) *If $u \in A^\times$, then each of the categories $\mathrm{Mod}_{G,\varpi=u}^{\mathrm{adm}}(A)$, $\mathrm{Mod}_{G,\varpi=u}^{\mathrm{l.adm}}(A)$, and $\mathrm{Mod}_{G,\varpi=u}^{\mathrm{sm}}(A)$ is abelian, and the last two have enough injectives.*

Proof. The abelianess claims are evident. To establish the claim of (1) regarding injectives, let V be an object of $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$ (resp. $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$, resp. $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(A)$), and let $V \hookrightarrow W$ be an $A[G]$-linear embedding into an injective object in $\mathrm{Mod}^{\mathrm{l.adm}}_{G}(A)$ (resp. $\mathrm{Mod}^{\mathrm{sm}}_{G}(A)$, resp. $\mathrm{Mod}^{\mathrm{sm}}_{K}(A)$). This embedding then factors through an embedding $V \hookrightarrow W^{Z=\zeta}$ (resp. $V \hookrightarrow W^{Z\cap K=\zeta}$ in the $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}$ case), and the latter object is injective in $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$ (resp. $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$, resp. $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(A)$), as was noted above. The claim of (2) regarding injective is proved analogously, utilizing the functor $V \mapsto V^{\varpi=u}$. $\square$

3. Main result

We introduce notation for some subgroups of G that we will need to consider, namely: we write $G^+ := \{g \in G : \mathrm{val}_F(\det g) \equiv 0 \pmod 2\}$, write $I := \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \varpi\mathfrak{o} & \mathfrak{o}^\times \end{pmatrix}$ (an Iwahori subgroup of K), let $N_G(I)$ denote the normalizer in G of I, set $\mathcal{G} := N_G(I)/\varpi^{\mathbb{Z}}$, and write $N_0 := \begin{pmatrix} 1 & \mathfrak{o} \\ 0 & 1 \end{pmatrix}$.

We recall the following result, proved by Christophe Breuil and the second author in [1], which is the main input into the theorems of this note.

3.1. **Theorem.** *Let V be an object in $\mathrm{Mod}^{\mathrm{adm}}_{G,\varpi=1}(k)$, and if $p = 2$ then assume that $V \cong \mathrm{Ind}^{G}_{G^+} V'$ for some representation V' of G^+. Then there exists a smooth admissible representation Ω of G in $\mathrm{Mod}^{\mathrm{adm}}_{G,\varpi=1}(k)$ and a G-equivariant injection $V \hookrightarrow \Omega$ such that $V|_K \hookrightarrow \Omega|_K$ is an injective envelope of $V|_K$ in $\mathrm{Mod}^{\mathrm{sm}}_{K}(k)$.*

Proof. [1] Corollary 9.11. If $p = 2$ then [1] Proposition 9.2 implies that V satisfies the conditions of [1] Corollary 9.11. $\square$

The next two propositions will let us bootstrap the preceding theorem from the case of coefficients in a field to the more general case of coefficients in an Artinian ring.

3.2. **Proposition.** *Suppose that p is odd. If V is an object of $\mathrm{Mod}^{\mathrm{adm}}_{G,\varpi=1}(A)$ with the property that $V[\mathfrak{m}]$ is injective as an object of $\mathrm{Mod}^{\mathrm{sm}}_{K}(k)$, and if*

$$V \hookrightarrow W \tag{3.3}$$

is an injective envelope of V in the category $\mathrm{Mod}^{\mathrm{sm}}_{K}(A)$, then the K-action on W may be extended to a G-action in such a way that W becomes an object of $\mathrm{Mod}^{\mathrm{adm}}_{G,\varpi=1}(A)$, and the embedding (3.3) is G-equivariant.

Proof. We adapt the method of proof of [4, Thm. 6.1]. First, since V is an essential extension of $V[\mathfrak{m}]$ (even as an A-module), we see that W is equally well the K-injective envelope of $V[\mathfrak{m}]$. It then follows from Lemma 2.1 that the embedding $V[\mathfrak{m}] \hookrightarrow W[\mathfrak{m}]$ is an isomorphism. Thus W is an essential extension (even as an A-module) of $V[\mathfrak{m}]$, and since by [3, Prop. 2.1.2] it is injective as a smooth I-representation, it is an I-injective envelope of $V[\mathfrak{m}]$, and hence also of V.

Since I is open in K, it follows from [3, Prop. 2.1.2] that $V[\mathfrak{m}]$ is injective as a smooth I-representation over k. Since p is odd and $\mathcal{G}/I$ has order two, in fact $V[\mathfrak{m}]$ is injective as a smooth $\mathcal{G}$-representation over k. Thus an analogous argument to that of the preceding paragraph shows that if $V \hookrightarrow W_1$ denotes an injective envelope of V as an $\mathcal{G}$-representation over A, then W_1 is also an I-injective envelope.

Thus we may find an I-equivariant isomorphism $W \xrightarrow{\sim} W_1$, respecting the given embeddings of V into source and target. We use this isomorphism to transport the $\mathcal{G}$-action on W_1 to a corresponding action on W. Since G is the amalgam of KZ and $N_G(I)$ along IZ, we may then glue the K-action and $\mathcal{G}$-action on W to obtain a G-action on W with respect to which $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ acts trivially, which by construction is compatible with the G-action on V. Finally, since V is admissible as a G-representation (and so, equivalently, as a K-representation), [3, Prop. 2.1.9] shows that W is admissible as a K-representation (and so, equivalently, as a G-representation). This completes the proof of the proposition. $\square$

3.4. **Proposition.** *Suppose that* $p = 2$*. If* V *is an object of* $\mathrm{Mod}^{\mathrm{adm}}_{G,\varpi=1}(A)$ *with the property that* $V[\mathfrak{m}]$ *is injective as an object of* $\mathrm{Mod}^{\mathrm{sm}}_K(k)$*, and if*

$$\mathrm{Ind}_{G^+}^G V \hookrightarrow W \tag{3.5}$$

is an injective envelope of $\mathrm{Ind}_{G^+}^G V$ *in the category* $\mathrm{Mod}^{\mathrm{sm}}_K(A)$*, then the* K*-action on* W *may be extended to a* G*-action in such a way that* W *becomes an object of* $\mathrm{Mod}^{\mathrm{adm}}_{G,\varpi=1}(A)$*, and the embedding* (3.5) *is* G*-equivariant.*

Proof. Let $V \hookrightarrow W_1$ denote an injective envelope of V in the category $\mathrm{Mod}^{\mathrm{sm}}_K(A)$. Note that $\mathrm{Ind}_{G^+}^G V \xrightarrow{\sim} V \otimes_A \mathrm{Ind}_{G^+}^G \underline{1}$ (where here $\underline{1}$ denotes the free rank one trivial representation of G^+ over A). Since $\mathrm{Ind}_{G^+}^G \underline{1}$ is trivial when restricted to K, we find that there is a corresponding $A[K]$-linear isomorphism $W \xrightarrow{\sim} W_1 \otimes_A \mathrm{Ind}_{G^+}^G \underline{1}$.

There is a natural isomorphism $\mathcal{G}/I \xrightarrow{\sim} G/G^+$, and hence natural $\mathcal{G}$-equivariant isomorphisms $\mathrm{Ind}_{G^+}^G \underline{1} \xrightarrow{\sim} \mathrm{Ind}_I^{\mathcal{G}} \underline{1}$ and $\mathrm{Ind}_{G^+}^G V \xrightarrow{\sim} \mathrm{Ind}_I^{\mathcal{G}} V$. Thus there is a commutative diagram of isomorphism

$$\begin{array}{ccccccc}
\mathrm{Ind}_{G^+}^G V & & \xrightarrow{\quad\quad\sim\quad\quad} & & & & \mathrm{Ind}_I^{\mathcal{G}} V \\
\big\downarrow & & & & & & \big\downarrow \\
W & \xrightarrow{\sim} & W_1 \otimes_A \mathrm{Ind}_{G^+}^G \underline{1} & \xrightarrow{\sim} & W_1 \otimes_A \mathrm{Ind}_I^{\mathcal{G}} \underline{1} & \xrightarrow{\sim} & \mathrm{Ind}_I^{\mathcal{G}} W_1,
\end{array}$$

in which the top horizontal arrow is $\mathcal{G}$-equivariant, the left-hand vertical arrow is K-equivariant, the right hand vertical arrow is $\mathcal{G}$-equivariant, and the bottom three arrows are, reading from left to right, K-equivariant, I-equivariant, and I-equivariant respectively. We may use the bottom row of this diagram (whose composite is I-equivariant) to transport the $\mathcal{G}$-action on $\mathrm{Ind}_I^{\mathcal{G}} W_1$ to a corresponding action on W. Since G is the amalgam of KZ and $N_G(I)$ along IZ, we see this action glues with the K-action on W to induced a G-action on W with respect to which $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ acts trivially. A consideration of the diagram shows that the embedding (3.5) is both K-equivariant and $\mathcal{G}$-equivariant, and hence is G-equivariant. Finally, since V is admissible as a G-representation (and so, equivalently, as a K-representation), the same is true of $\mathrm{Ind}_{G^+}^G V$, by Lemma 2.2. Thus [3, Prop. 2.1.9] shows that W is admissible as a K-representation (and so, equivalently, as a G-representation), and the proof of the proposition is complete. $\square$

We can now prove our first main result.

3.6. **Theorem.** *If $u \in A^\times$, then any injective object of* $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$ *is injective as a smooth K-representation over A.*

Proof. We begin with a straightforward reduction to the case $u = 1$. Namely, let $B = A[x]/(x^2 - u)$; note that $B \cong A \oplus A$ as an A-module. Thus if V is an injective object of $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$, then V is a direct summand of $B \otimes_A V$ when the latter is regarded as an object of $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$, and so it suffices to verify that the latter is injective as a smooth K-representation over A. A consideration of the natural isomorphism $\mathrm{Hom}_{A[K]}(U, B \otimes_A V) \xrightarrow{\sim} \mathrm{Hom}_{B[K]}(B \otimes_A U, B \otimes_A V)$, for objects U of $\mathrm{Mod}^{\mathrm{sm}}_K(A)$, shows in turn that it suffices to prove that $B \otimes_A V$ is injective as an object of $\mathrm{Mod}^{\mathrm{sm}}_K(B)$.

Now B is Gorenstein over A, i.e. there is an isomorphism of B-modules

$$B \xrightarrow{\sim} \mathrm{Hom}_A(B, A),$$

and hence there is an isomorphism of $B[G]$-modules

$$B \otimes_A V \xrightarrow{\sim} \mathrm{Hom}_A(B, V).$$

A consideration of the resulting isomorphism

$$\mathrm{Hom}_{B[G]}(W, B \otimes_A V) \xrightarrow{\sim} \mathrm{Hom}_{B[G]}\big(W, \mathrm{Hom}_A(B, V)\big) \xrightarrow{\sim} \mathrm{Hom}_{A[G]}(W, V),$$

for objects W of $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(B)$, shows that $B \otimes_A V$ is injective as an object of $\mathrm{Mod}^{\mathrm{l.adm}}_G(B)$. Thus the claim of the theorem for $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$ follows from the corresponding claim for $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(B)$.

If $\chi_u : F^\times \to B^\times$ denotes the character defined by $\chi_u(a) = u^{\mathrm{val}_F(a)}$, then twisting by $\chi_u \circ \det$ induces an equivalence of categories between $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=1}(B)$ and $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(B)$ (with an essential inverse given via twisting by χ_u^{-1}). Thus the claim of the theorem for $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(B)$ follows from the corresponding claim for $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=1}(B)$. Altogether, we have reduced to a consideration of the case when $u = 1$, and we assume that indeed $u = 1$ from now on.

In the remainder of the proof we treat the cases when p is odd and $p = 2$ separately. Thus suppose first that p is odd, and let V be an injective object of $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=1}(A)$. If U is an admissible G-subrepresentation of V, then by Theorem 3.1 there is a G-equivariant map $U[\mathfrak{m}] \hookrightarrow \Omega$, where Ω is an object of $\mathrm{Mod}^{\mathrm{adm}}_G(k)$ which is also an injective envelope of $U[\mathfrak{m}]$ as a smooth K-representation over k. By the injectivity of V, we may extend the inclusion $U[\mathfrak{m}] \subset V$ to a G-equivariant map $\Omega \to V$. Since Ω is an essential extension of $U[\mathfrak{m}]$ (even as a K-representation) this latter map must again be an embedding; we thus regard Ω as a G-subrepresentation of V containing $U[\mathfrak{m}]$.

If we write $X = U + \Omega \subset V$, then X is an admissible G-subrepresentation of V containing U, and $X[\mathfrak{m}] = \Omega$ is injective as a smooth K-representation over k. Thus by Proposition 3.2, we may find a G-equivariant embedding $X \hookrightarrow W$, where W is an object of $\mathrm{Mod}^{\mathrm{adm}}_{G,\varpi=1}(A)$ which is an injective envelope of X as a smooth K-representation over A. By the injectivity of V we may extend the inclusion $X \subset V$ to a G-equivariant map $W \to V$, which must again be an inclusion, since W is an essential extension of X (even as a K-representation). Thus we see that U

is contained in an admissible G-subrepresentation W of V which is injective when regarded as a smooth K-representation over A.

Since V is locally admissible, it is isomorphic to the inductive limit of its admissible G-subrepresentations. The result of the preceding paragraph shows that it is in fact the inductive limit of those of its admissible G-subrepresentations that are injective as smooth K-representations over A. It follows from [3, Prop. 2.1.3] that V itself is injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$.

Suppose now that $p = 2$. In this case, the preceding argument breaks down, and we must use the technique of passing from V to $\mathrm{Ind}_{G^+}^G V$. (In fact, we apply it twice.) To this end, we first note that since V is an injective object of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$, the same is true of $\mathrm{Ind}_{G^+}^G V$, by Proposition 2.3. Now the natural embedding $V \hookrightarrow \mathrm{Ind}_{G^+}^G V$ of injective objects in $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ must split, and thus V is a direct summand of $\mathrm{Ind}_{G^+}^G V$. Hence it suffices to verify that $\mathrm{Ind}_{G^+}^G V$ is injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$. Iterating this argument, it suffices in fact to show that $\mathrm{Ind}_{G^+}^G \mathrm{Ind}_{G^+}^G V$ is injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$, and this is what we will do.

Let U be an admissible subrepresentation of V. By Theorem 3.1 there is a G-equivariant map $\mathrm{Ind}_{G^+}^G U[\mathfrak{m}] \hookrightarrow \Omega$, where Ω is an object of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$ which is also an injective envelope of $\mathrm{Ind}_{G^+}^G U[\mathfrak{m}]$ as a smooth K-representation over k. By the injectivity of $\mathrm{Ind}_{G^+}^G V$, we may extend the inclusion $\mathrm{Ind}_{G^+}^G U[\mathfrak{m}] \subset \mathrm{Ind}_{G^+}^G V$ to a G-equivariant map $\Omega \to \mathrm{Ind}_{G^+}^G V$. Since Ω is an essential extension of $\mathrm{Ind}_{G^+}^G U[\mathfrak{m}]$ (even as a K-representation) this latter map must again be an embedding; we thus regard Ω as a G-subrepresentation of $\mathrm{Ind}_{G^+}^G V$ containing $\mathrm{Ind}_{G^+}^G U[\mathfrak{m}]$.

Write $X = \mathrm{Ind}_{G^+}^G U + \Omega \subset V$, so that X is an admissible G-subrepresentation of $\mathrm{Ind}_{G^+}^G V$ containing $\mathrm{Ind}_{G^+}^G U$, with $X[\mathfrak{m}] = \Omega$ an injective smooth K-representation over k. Since V is the inductive limit of its admissible subrepresentations U, the induction $\mathrm{Ind}_{G^+}^G V$ is the inductive limit of its admissible subrepresentations $\mathrm{Ind}_{G^+}^G U$, and hence also is the inductive limit of its admissible subrepresentations X for which $X[\mathfrak{m}]$ is injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(k)$. The representation $\mathrm{Ind}_{G^+}^G \mathrm{Ind}_{G^+}^G V$ is thus the inductive limit of its corresponding subrepresentations $\mathrm{Ind}_{G^+}^G X$.

Given such an X, by Proposition 3.4 we may find a G-equivariant embedding $\mathrm{Ind}_{G^+}^G X \hookrightarrow W$, where W is an object of $\mathrm{Mod}_{G,\varpi=1}^{\mathrm{l.adm}}(A)$ which is an injective envelope of $\mathrm{Ind}_{G^+}^G X$ in the category $\mathrm{Mod}_K^{\mathrm{sm}}(A)$. Since $\mathrm{Ind}_{G^+}^G \mathrm{Ind}_{G^+}^G V$ is injective as an object of $\mathrm{Mod}_{G,\varpi=1}^{\mathrm{l.adm}}(A)$, we may extend the embedding

$$\mathrm{Ind}_{G^+}^G X \hookrightarrow \mathrm{Ind}_{G^+}^G \mathrm{Ind}_{G^+}^G V$$

to a G-equivariant map

$$W \hookrightarrow \mathrm{Ind}_{G^+}^G \mathrm{Ind}_{G^+}^G V,$$

which must again be an embedding, since W is an essential extension of $\mathrm{Ind}_{G^+}^G X$ (even as a K-representation). Thus $\mathrm{Ind}_{G^+}^G \mathrm{Ind}_{G^+}^G V$ is in fact the inductive limit of those of its admissible G-subrepresentations that are injective as objects of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$. It follows from [3, Prop. 2.1.3] that $\mathrm{Ind}_{G^+}^G V \,\mathrm{Ind}_{G^+}^G V$ is an injective object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$. As already observed, this suffices to complete the proof of the theorem. ∎

Our next result removes the hypothesis of having fixed action of ϖ from the preceding theorem.

3.7. **Theorem.** *If V is an injective object of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$, then V is also injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$.*

Proof. Let $B = A[\varpi^{\pm 1}] \xrightarrow{\sim} A[t^{\pm 1}]$. If U is any locally admissible G-representation, then $U = \bigoplus_{\mathfrak{n}} U_{\mathfrak{n}}$, where $\mathfrak{n}$ runs over the maximal ideals of B and $U_{\mathfrak{n}}$ denotes the localization of U at $\mathfrak{n}$. Furthermore,

$$U_{\mathfrak{n}} = U[\mathfrak{n}^\infty] := \bigcup_{i\geq 1} U[\mathfrak{n}^i],$$

where $U[\mathfrak{n}^i]$ denotes the subspace of U consisting of elements annihilated by $\mathfrak{n}^i$. Each maximal ideal $\mathfrak{n}$ is of the form $(\mathfrak{m}, f)$, where $\mathfrak{m}$ is the maximal ideal of A, and $f \in A[t]$ is a monic polynomial. Since A is Artinian, so that $\mathfrak{m}$ is a nilpotent ideal, we see that the powers $\mathfrak{n}^i$ are cofinal with the sequence of principal ideals (f^i). Thus we may equally well write

$$U_{\mathfrak{n}} = \bigcup_{i\geq 1} U[f^i],$$

where of course $U[f^i]$ denotes the subspace of U consisting of elements annihilated by f^i.

Suppose now that V is an injective object of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$. Since, by the discussion of the preceding paragraph, V is the inductive limit of the $V[f^i]$ (where f^i runs over the various powers of the various monic polynomials associated to the various maximal ideals $\mathfrak{n}$ of B), in order to show that V is injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$, it suffices, by [3, Prop. 2.1.3], to show that each $V[f^i]$ is an injective object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$.

If we write $C := B/(f^i)$, and if we let u denote the image of t in C, then $V[f^i]$ is an object of $\mathrm{Mod}_{G,\varpi=u}^{\mathrm{l.adm}}(C)$. Since V is injective as an object of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$, one immediately sees that $V[f^i]$ is injective as an object of $\mathrm{Mod}_{G,\varpi=u}^{\mathrm{l.adm}}(C)$. It then follows from Theorem 3.6 that $V[f^i]$ is injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(C)$. Since C is finite flat over A, we see that $V[f^i]$ is equally well injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$, as required. (The forgetful functor from $\mathrm{Mod}_K^{\mathrm{sm}}(C)$ to $\mathrm{Mod}_K^{\mathrm{sm}}(A)$ is right adjoint to the exact functor $C \otimes_A -$, and so preserves injectives.) □

Finally, we prove an analogue of the preceding results in the context of a fixed central character.

3.8. **Theorem.** *If $\zeta : Z \to A^\times$ is a smooth character, and V is an injective object in $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$, then the restriction of V to K is an injective object in $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$.*

Proof. Write $u = \zeta(\varpi)$. If $V \hookrightarrow W$ is an $A[G]$-linear embedding of V into an injective object W of $\mathrm{Mod}_{G,\varpi=u}^{\mathrm{l.adm}}(A)$, then it factors through $V \hookrightarrow W^{Z\cap K=\zeta}$. The target of this embedding lies in $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$, and so the injectivity of V implies that this embedding is split; i.e. V is a direct summand of $W^{Z\cap K=\zeta}$ as an $A[G]$-module. It thus suffices to note that $W^{Z\cap K=\zeta}$ is an injective object of $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$, since W is injective as an object of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$, by Theorem 3.6, and $U \mapsto U^{Z\cap K=\zeta}$ is right adjoint to the embedding of $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$ into $\mathrm{Mod}_K^{\mathrm{sm}}(A)$. □

3.9. **Corollary.** *If V is injective any of the categories* $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$ *(for some smooth character $\zeta : Z \to A^\times$),* $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$ *(for some $u \in A^\times$), of* $\mathrm{Mod}^{\mathrm{l.adm}}_{G}(A)$*, then $V|_{N_0}$ is an injective object in* $\mathrm{Mod}^{sm}_{N_0}(k)$*.*

Proof. In the latter two cases this follows from Theorems 3.6 and 3.7, together with [3, Prop. 2.1.5]. In the case of $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$, it suffices, by Theorem 3.8, to show that an injective object of $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(A)$ is injective as an N_0-representation. To see this, note first that any injective object of $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(A)$ is the inductive limit of injective modules that are direct summands of $\mathcal{C}^{\mathcal{S}}(K,A)^r$, for some $r \geq 0$. By a consideration of Lemma 2.1.10 and Proposition 2.1.11 of [3] we then see that if suffices to show that $\mathcal{C}^{\mathcal{S}}(K,A)^{Z=\zeta}$ is injective as an N_0-representation. If we write $K_1 := K \bigcap \mathrm{SL}_2(F)$, then clearly $\mathcal{C}^{\mathcal{S}}(K,A)^{Z=\zeta} \xrightarrow{\sim} \mathcal{C}^{\mathcal{S}}(K_1,A)$. The latter is injective as an N_0-representation, by [3, Prop. 2.2.11], and the corollary is proved. □

Let G be the group of $\mathbb{Q}_p$-valued points of a connected reductive linear algebraic group over $\mathbb{Q}_p$. Let P be a parabolic subgroup of G with a Levy subgroup M and let $\overline{P}$ be the parabolic subgroup of G opposite to P with respect to M. In [2], the first author defined a left exact functor $\mathrm{Ord}_P : \mathrm{Mod}^{\mathrm{l.adm}}_{G}(A) \to \mathrm{Mod}^{\mathrm{l.adm}}_{M}(k)$ such that for all U in $\mathrm{Mod}^{\mathrm{l.adm}}_{M})$ and V in $\mathrm{Mod}^{\mathrm{l.adm}}_{G}(A)$ one has

$$\mathrm{Hom}_G(\mathrm{Ind}_{\overline{P}}^G U, V) \cong \mathrm{Hom}_M(U, \mathrm{Ord}_P(V)).$$

Further, for $i \geq 0$ in [3] there are defined functors $H^i \mathrm{Ord}_P : \mathrm{Mod}^{\mathrm{l.adm}}_{G}(A) \to \mathrm{Mod}^{\mathrm{l.adm}}_{M}(A)$ such that $H^0 \mathrm{Ord}_P = \mathrm{Ord}_P$ and $\{H^i \mathrm{Ord}_P : i \geq 0\}$ is a δ-functor. It is conjectured there that for $i \geq 1$ the functors $H^i \mathrm{Ord}_P$ are effaceable, which would imply that they are universal, and hence coincide with the derived functors of Ord_P.

3.10. **Corollary.** *If $G = \mathrm{GL}_2(F)$ and V is an injective object in* $\mathrm{Mod}^{\mathrm{l.adm}}_{G}(A)$ *(resp.* $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$*, resp.* $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$*), then $H^i \mathrm{Ord}_P(I) = 0$ for all $i \geq 1$.*

Proof. Since by Corollary 3.9, $I|_{N_0}$ is an injective object in $\mathrm{Mod}^{sm}_{N_0}(k)$ we have that $H^i(N_0, I) = 0$ for all $i \geq 1$. The claim follows from the definition of $H^i \mathrm{Ord}_P$, see [3, Def.3.3.1]. □

Since $\mathrm{Mod}^{\mathrm{l.adm}}_{G}(A)$, $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$, and $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\varpi=u}(A)$ each have enough injectives, we conclude that the $H^i \mathrm{Ord}_P$ are effaceable for $i \geq 1$ on any of these categories. In particular, we have verified [3, Conj. 3.7.2] in the case $G = \mathrm{GL}_2(F)$.

3.11. **Remark.** The authors of this note strongly believe that an analogue Theorem 3.1 holds for other groups than $\mathrm{GL}_2(F)$. If this is the case than our proof should go through to establish [3, Conj. 3.7.2] for these groups.

References

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