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ON A CLASS OF COHERENT RINGS, WITH APPLICATIONS TO THE SMOOTH REPRESENTATION THEORY OF $\mathrm{GL}_2(\mathbb{Q}_p)$ IN CHARACTERISTIC p

MATTHEW EMERTON

In this short note, we introduce a certain class of coherent rings, namely non-commutative polynomial rings of the form $A[F]$, where A is a commutative Noetherian ring and F is a variable satisfying the commutation relation $F \cdot a = F(a)F$, where we use the same notation F to stand for some given flat endomorphism of A. (See Proposition 1.3 for the proof that such a ring is coherent.)

If we take A to be a smooth finite type k-algebra, where k is a field of characteristic p, and F to be the absolute Frobenius endomorphism of A, then (sheafified versions of) the ring $A[F]$ play a prominent role in the paper [7]; in particular, it follows from the results of this note that the sheaf of rings $\mathcal{O}_{F,X}$ on a smooth finite type k-scheme X, introduced in [7], is coherent. (See Example 1.4 below.) This sheds a new light on some of the results of [7].

Suppose instead that we take $A = k[[t]]$, where k is a finite field of characteristic p, and take F to be the relative Frobenius endomorphism of A over k. If we let P denote the Borel subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, then any smooth P-representation over k is naturally an $A[F]$-module. (See Section 4 below.) In particular, this is true of a smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation over k. Suppose that V is a finitely generated admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation over k, and let V_0 be finite-dimensional k-subspace invariant under KZ (where, as usual, $K := \mathrm{GL}_2(\mathbb{Z}_p)$ and Z denotes the centre of $\mathrm{GL}_2(\mathbb{Q}_p)$) which generates V over $\mathrm{GL}_2(\mathbb{Q}_p)$. We show that the $A[F]$-submodule of V generated by V_0 is finitely presented over $A[F]$, and is admissible as an A-module – i.e. is Pontrjagin dual to a finitely generated A-module. (See Theorem 4.7 below.) This result is essentially due to Colmez (see [4] and Remark 4.8 below). However, his proof is more involved than ours, and relies in particular on the classification of irreducible admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representations over k due to Barthel-Livné [1] and Breuil [3]. Our proof is independent of this classification, and indeed can be used to rederive it. (See Section 5 below.)

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1. A class of coherent rings

Let A be a Noetherian commutative ring, and let $F : A \to A$ be a flat endomorphism. We let $A[F]$ denote the non-commutative polynomial ring in the variable "F" over A, with the commutation relation

$$F \cdot a = F(a)F.$$

If M is an A-module, we write $F^*M := A \otimes_A M$, the tensor product being taken with respect to the map $F : A \to A$. If M is an $A[F]$-module, then the action

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of the element F on M induces an A-linear map $\phi_M : F^*M \to M$, defined via $\phi_M : a \otimes m \mapsto aF \cdot m$. Conversely, if M is an A-module, and $\phi_M : F^*M \to M$ is an A-linear map, then there is a unique extension of the A-module structure on M to an $A[F]$-module structure such that the map ϕ_M arises from this structure via the preceding construction.

If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $A[F]$-modules, then we may form the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^*M' & \longrightarrow & F^*M & \longrightarrow & F^*M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_{M'}} & & \downarrow{\scriptstyle \phi_M} & & \downarrow{\scriptstyle \phi_{M''}} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0, \end{array}$$

whose top row is again exact (since F is flat). The snake lemma gives rise to an exact sequence

$$(1) \quad 0 \to \ker \phi_{M'} \to \ker \phi_M \to \ker \phi_{M''} \to \operatorname{coker} \phi_{M'} \to \operatorname{coker} \phi_M \to \operatorname{coker} \phi_{M''} \to 0.$$

We remark that the ring $A[F]$ is typically non-Noetherian. Our main goal in this section is to prove that it is nevertheless left coherent (i.e. that finitely generated left submodules of $A[F]$ are finitely presented).

1.1**. Lemma.** *If M is a left $A[F]$-submodule (i.e. left ideal) of $A[F]$, then M is finitely generated over $A[F]$ if and only if* $\operatorname{coker} \phi_M$ *is finitely generated over A.*

Proof. If $A[F]^n \to M$ is surjective, then so is the induced map

$$A^n \xrightarrow{\sim} \operatorname{coker} \phi_{A[F]^n} \to \operatorname{coker} \phi_M.$$

This proves the "only if" direction.

Suppose, conversely, that $\operatorname{coker} \phi_M$ is finitely generated over A. For any $d \geq 0$, write $M^{\leq d} := M \cap (\bigoplus_{i=0}^d AF^i) \subset M$. Since A is Noetherian, each $M^{\leq d}$ is finitely generated over A, and clearly $M = \bigcup_{d \geq 0} M^{\leq d}$. If we choose $d \geq 0$ such that the natural map $M^{\leq d} \to \operatorname{coker} \phi_M$ is surjective, then for any $d' > d$, we find that $M^{\leq d'} \subset M^{\leq d} + FM$. On the other hand, clearly $FM \bigcap M^{\leq d'} = FM^{\leq d'-1}$. Thus

$$M^{\leq d'} \subset M^{\leq d} + FM^{\leq d'-1}.$$

Proceeding by recursion on d', we find that $M^{\leq d'} \subset A[F]M^{\leq d}$, and thus (since $d' > d$ was arbitrary) that $M = A[F]M^{\leq d}$. This proves the "if" direction. □

1.2**. Lemma.** *If M is a finitely generated left $A[F]$-module, then M is finitely presented if and only if* $\ker \phi_M$ *is finitely generated over A.*

Proof. Since M is finitely generated over $A[F]$, we may choose a presentation of M of the form $0 \to M' \to A[F]^n \to M \to 0$. Applied to this short exact sequence, the exact sequence (1) reduces to

$$0 \to \ker \phi_M \to \operatorname{coker} \phi_{M'} \to A^n \to \operatorname{coker} \phi_M \to 0.$$

In particular, we see that $\ker \phi_M$ is finitely generated over A if and only if the same is true of $\operatorname{coker} \phi_{M'}$. On the other hand, lemma 1.1 shows that $\operatorname{coker} \phi_{M'}$ is finitely generated over A if and only if M' is finitely generated over $A[F]$. The lemma follows. □

1.3. **Proposition.** *The ring $A[F]$ is left coherent.*

Proof. If M is a finitely generated left submodule of $A[F]$, then the inclusion $M \subset A[F]$ induces an inclusion $\ker \phi_M \subset \ker \phi_{A[F]} = 0$. Thus $\ker \phi_M$ vanishes, and so by the preceding lemma, M is finitely presented over $A[F]$. □

As a corollary, the kernel and image (as well as the cokernel – although this doesn't require coherence) of any map between finitely presented left $A[F]$-modules is again finitely presented.

1.4. **Example.** If A is a smooth algebra over a field k of characteristic $p > 0$, then both the absolute Frobenius endomorphism F_A of A and the relative Frobenius endomorphism $F_{A/k}$ of A are flat maps. Thus the rings $A[F_A]$ and $A[F_{A/k}]$ are left coherent. More generally, if X is a smooth k-scheme, then we may form sheaves of rings $\mathcal{O}_X[F]$ and $\mathcal{O}_X[F_{X/k}]$, both of which are then seen to be left coherent. (The first of these coincides with the sheaf denoted $\mathcal{O}_{F,X}$ in [7].)

1.5. **Example.** Continuing to suppose that X is a smooth scheme over a field k of characteristic $p > 0$, a quasi-coherent sheaf of left $\mathcal{O}_X[F_X]$-modules $\mathcal{M}$ is said to be *unit* if the map $\phi_{\mathcal{M}} : F_X^* \mathcal{M} \to \mathcal{M}$ is an isomorphism. In particular, the kernel of $\phi_{\mathcal{M}}$ then vanishes, and so it follows from Lemma 1.2 that a locally finitely generated unit $\mathcal{O}_X[F_X]$-module is in fact locally finitely presented as a left $\mathcal{O}_X[F_X]$-module, a result which was proved by a different method in [7]. (See Remark 6.1.2 of that reference.)

2. A homological application

Let A and F be as in the preceding section. Suppose given a map $A \to B$, with B again taken to be Noetherian, and a flat endomorphism of B (which we will again denote by F) compatible with the endomorphism F of A, in the sense that the diagram

$$\begin{array}{ccc} A & \xrightarrow{F} & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{F} & B \end{array}$$

commutes. The map $A \to B$ then extends naturally to a map of rings $A[F] \to B[F]$.

2.1. **Lemma.** *There is a natural B-linear isomorphism of δ-functors*

$$\mathrm{Tor}_\bullet^{A[F]}(B[F], –) \xrightarrow{\sim} \mathrm{Tor}_\bullet^A(B, –).$$

Proof. Clearly $A[F]$ is free, and so flat, as a left A-module. Thus if M is a left $A[F]$-module, and $P_\bullet$ is a left-resolution of M by free $A[F]$-modules, then $P_\bullet$ is also a resolution of M by free A-modules. Furthermore, there is an evident isomorphism of $(B, A[F])$-bimodules $B \otimes_A A[F] \to B[F]$, which gives rise to a B-linear isomorphism

$$B[F] \otimes_{A[F]} P_\bullet \xrightarrow{\sim} B \otimes_A P_\bullet.$$

Passing to homology on the source and target of this isomorphism gives the required isomorphism of Tor-functors. □

In particular, if M is a left $A[F]$-module, then the B-module structure on each $\mathrm{Tor}_\bullet^A(B, M)$ extends in a natural way to a left $B[F]$-module structure.

2.2. **Proposition.** *If M is a finitely presented left $A[F]$-module, then each of the* Tor*-modules* $\mathrm{Tor}^A_\bullet(B, M)$ *is finitely presented as a left $B[F]$-module.*

Proof. Since $A[F]$ is left coherent, we may choose a resolution $P^\bullet$ of M by free $A[F]$-modules, each member of which is finitely generated over $A[F]$. The preceding lemma shows that we may compute the modules $\mathrm{Tor}^A_\bullet(B, M)$, with their natural $B[F]$-module structure, as the homology of the complex $B[F] \otimes_{A[F]} P^\bullet$. This is a complex of finite rank free left $B[F]$-modules, and so, by the left coherence of $B[F]$, has finitely presented homology modules. □

2.3. **Example.** Let $f : Y \to X$ be a morphism of smooth k-schemes, where k is a field of characteristic $p > 0$. In [7, §2] a functor $f^!$ from complexes of left $\mathcal{O}_X[F_X]$-modules to complexes of left $\mathcal{O}_Y[F_Y]$-modules is defined. Up to a shift, it coincides with the sheaf-theoretic pull-back f^{-1}, followed by the left-derived tensor product with $\mathcal{O}_Y[F_Y]$ over $f^{-1}(\mathcal{O}_X[F_X])$. As Lemma 2.1 shows, on the underlying $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules, this functor coincides (up to a shift) with the left-derived functor of the usual pull-back f^*. Proposition 2.2 shows that $f^!$ takes complexes with finitely presented cohomology sheaves to complexes with finitely presented cohomology sheaves. (In the case of complexes with unit cohomology sheaves, the fact that $f^!$ preserves the property of having finitely generated, or equivalently, by Examples 1.5, finitely presented, cohomology sheaves, is proved in [7, Prop. 6.7].)

3. The case when A is a discrete valuation ring

As indicated in the section title, we now suppose that A is a discrete valuation ring. We fix a uniformizer t of A, and we write $k := A/tA$ to denote the residue field of A. We also suppose that the flat map $F : A \to A$ is local, and that the induced endomorphism of k is the identity. Thus $k[F]$ is the usual commutative polynomial ring over k.

We may apply the discussion of the preceding section to the surjection $A \to k$, and conclude that if M is any $A[F]$-module, then the modules $\mathrm{Tor}^A_\bullet(k, M)$ have a natural $k[F]$-module structure. Of course, since A is a discrete valuation ring, it has projective dimension 1, and the modules $\mathrm{Tor}^A_\bullet(k, M)$ admit the following explicit descriptions, for any A-module M:

$$\mathrm{Tor}_0(k, M) \xrightarrow{\sim} M/tM, \qquad \mathrm{Tor}_1(k, M) \xrightarrow{\sim} M[t],$$

where $M[t]$ denotes the submodule of M consisting of elements annihilated by t.

If M is a left $A[F]$-module, then the induced $k[F]$-module structures on these Tor-modules are easy to describe. Namely: the action of F on M/tM is given by

$$m \bmod tM \mapsto Fm \bmod tM,$$

while the action of F on $M[t]$ is given by

$$M[t] \ni m \mapsto \frac{F(t)}{t} Fm$$

(where one notes that since F is local, the image $F(t)$ of t does indeed lie in tA, so that $F(t)/t$ is a well-defined element of A).

We make the following definition.

3.1. **Definition.** We say that an A-module M is admissible if M is A-torsion, and if $M[t]$ is finite-dimensional over k.

If we let K denote the field of fractions of A, and $\widehat{A}$ denote the completion of A, then the functor $M \mapsto \mathrm{Hom}_A(M, K/A)$ induces an anti-equivalence of categories between the category of admissible A-modules and the category of finitely generated $\hat{A}$-modules. If $\mathrm{Hom}_A(M, K/A)$ has free rank r over $\hat{A}$, then we say that the admissible A-module M has corank r. Since F is a local endomorphism, we see that if M is an admissible A-module, of corank r, then the same is true of F^*M.

Our goal in this section is to prove some basic results concerning finitely generated left $A[F]$-modules that are admissible as A-modules.

3.2. **Proposition.** *If M is a finitely generated left $A[F]$-module that is admissible as an A-module, then M is finitely presented as a left $A[F]$-module.*

Proof. Consider the map $\phi_M : F^*M \to M$. This is an A-linear map between two admissible A-modules of equal corank. Since $\mathrm{coker}\, \phi_M$ is finitely generated over A (as M is finitely generated over $A[F]$), we conclude that the same is true of $\ker \phi_M$. Lemma 1.2 implies that M is finitely presented over $A[F]$. □

3.3. **Proposition.** *If M is a finitely generated left $A[F]$-module that is admissible as an A-module, then any subquotient of M is again finitely generated over $A[F]$ and admissible as an A-module.*

Proof. Any subquotient of an admissible A-module is again admissible as an A-module, while any quotient of a finitely generated $A[F]$-module is again finitely generated over $A[F]$. It follows that any subquotient of M is admissible as an A-module, and that any quotient M'' of M is finitely generated over $A[F]$. Proposition 3.2 shows that M and M'' are furthermore both finitely presented over $A[F]$. If M' is an $A[F]$-submodule of M, then applying this to $M'' = M/M'$, and recalling that $A[F]$ is left coherent, we conclude that M' is also finitely presented, and so in particular finitely generated, over $A[F]$. Having proved the proposition for quotients and subobjects, it obviously also holds for arbitrary subquotients. □

3.4. **Corollary.** *If M is a finitely generated left $A[F]$-module that is admissible as an A-module, then M is of finite length.*

Proof. Since M is admissible as an A-module, it is Artinian as an A-module, and so also as an $A[F]$-module. Propositions 3.2 and 3.3 show that any $A[F]$-submodule of M is finitely generated, and thus that M is also Noetherian as an $A[F]$-module. The corollary follows. □

The following proposition gives a criterion for recognizing when a finitely generated left $A[F]$-module is admissible over A.

3.5. **Proposition.** *If M is a finitely generated left $A[F]$-module that is torsion over A, then the following conditions are equivalent:*

(1) *M is admissible as an A-module.*
(2) *The quotient M/tM is finite-dimensional over k.*
(3) *The quotient M/tM is torsion over $k[F]$.*

Proof. That 1 implies 2 is valid for any admissible A-module, while any $k[F]$-module that is finite-dimensional over k is certainly $k[F]$-torsion, so that 2 implies 3. It remains to show that 3 implies 1. Thus for the remainder of the proof we assume that M/tM is torsion over $k[F]$. Since M is A-torsion by assumption, we must prove that $M[t]$ is finite-dimensional over k.

We argue by induction on the number of generators of M as an $A[F]$-module. Suppose first that M is cyclic. Let $m \in M$ be a generator, and let $M_0 \subset M$ denote the A-submodule of M that m generates. By assumption, $M_0 \xrightarrow{\sim} A/t^r A$ for some $r \geq 0$. The inclusion $M_0 \subset M$ induces a surjection $A[F] \otimes_A M_0 \to M$ of $A[F]$-modules, with kernel N, say. Note that there are isomorphisms (of $k[F]$-modules)

$$(A[F] \otimes_A M_0)/t(A[F] \otimes_A M_0) \xrightarrow{\sim} k[F] \tag{2}$$

and

$$(A[F] \otimes_A M_0)[t] \xrightarrow{\sim} k[F]. \tag{3}$$

The short exact sequence $0 \to N \to A[F] \otimes_A M_0 \to M \to 0$ induces an exact sequence

$$N/tN \to (A[F] \otimes_A M_0)/t(A[F] \otimes_A M_0) \to M/tM.$$

Taking into account the isomorphism of (2), and the fact that M/tM is torsion over $k[F]$ by assumption, we see that may choose an element $n \in N$ whose image in $(A[F] \otimes_A M_0)/t(A[F] \otimes_A M_0)$ is non-zero. Let M' denote the left $A[F]$-submodule of $A[F] \otimes_A M_0$ generated by n; since n is non-zero, so is M'. Write $M'' := (A[F] \otimes_A M_0)/M'$.

Now consider the long exact sequence of Tor-modules associated to the short exact sequence $0 \to M' \to A[F] \otimes_A M_0 \to M'' \to 0$. Taking into account the isomorphisms (2) and (3), we may write it in the following form:

$$0 \to M'[t] \to k[F] \to M''[t] \to M'/tM' \to k[F] \to M''/tM'' \to 0. \tag{4}$$

Since M' is cyclic over $A[F]$ by construction, the quotient M'/tM' is cyclic over $k[F]$, while, again by construction, it has non-zero image in $k[F]$. Thus the fifth arrow in (4) must be injective, and so (4) gives rise to a short exact sequence

$$0 \to M'[t] \to k[F] \to M''[t] \to 0.$$

Since M' is a non-zero submodule of the torsion A-module $A[F] \otimes_A M_0$, we see that $M'[t]$ is non-zero. Thus $M''[t]$ is isomorphic to the quotient of $k[F]$ by a non-zero submodule, and hence is finite-dimensional over k. It follows that M'' is admissible over A, and thus so is its quotient M.

Suppose now that M is generated over $A[F]$ by n elements, $m_1, \ldots, m_n$. Let M' denote the $A[F]$-submodule of M generated by m_1, and write $M'' := M/M'$. Clearly M'' is A-torsion, while M''/tM'', being a quotient of M/tM, is $k[F]$-torsion. Since M'' is generated by $n-1$ elements, we conclude by induction on n that M'' is admissible. Now consider the long exact sequence of Tor-modules associated to the short exact sequence $0 \to M' \to M \to M'' \to 0$:

$$0 \to M'[t] \to M[t] \to M''[t] \to M'/tM' \to M/tM \to M''/tM'' \to 0.$$

Since M'' is admissible, we see that $M''[t]$ is finite-dimensional over k, while by assumption M/tM is $k[F]$-torsion. Thus M'/tM' is also $k[F]$-torsion. Since M' is also cyclic over $A[F]$, and A-torsion, we conclude from the case $n = 1$ of the proposition that we have already proved that M' is admissible over A. As M is an extension of admissible A-modules, it is itself admissible over A. □

4. Applications to the representation theory of GL_2

Let E be an unramified finite extension of $\mathbb{Q}_p$, of degree d, with ring of integers $\mathcal{O}$. Write $G := \mathrm{GL}_2(E)$, $K = \mathrm{GL}_2(\mathcal{O})$, $P = \begin{pmatrix} E^\times & E \\ 0 & E^\times \end{pmatrix}$ (the Borel subgroup of upper triangular matrices in G), $N = \begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix}$ (the unipotent radical of P), $N_0 = \begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix}$ (a compact open subgroup of N), $\Gamma = \begin{pmatrix} \mathcal{O}^\times & 0 \\ 0 & 1 \end{pmatrix}$, $F = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in P$, and let Z denote the centre of G.

Let k be a finite field of characteristic p, and write $A = k[[N_0]]$, the completed group ring of N_0 over k. Recall that A is a complete regular local ring of dimension d. We denote its maximal ideal by $\mathfrak{m}$.

Note that N_0 is closed under conjugation by F; indeed, $FnF^{-1} = n^p$ for all $n \in N_0$. Thus the endomorphism of A induced by conjugation by F is equal to the relative Frobenius $F_{A/k}$, and so in particular is a flat endomorphism of A, to which the theory of the preceding sections applies. We write $A[F]$ to denote the non-commutative polynomial ring over A in which F acts via $F_{A/k}$. We also write $A[F,\Gamma]$ to denote the twisted group ring of Γ over $A[F]$: the elements of Γ commute with F and with the field k of scalars, and conjugate the elements of N_0 just as they do in the group G, namely, if $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma$ and $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_0$, then

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}.$$

The endomorphism $F_{A/k}$ is local, and induces the identity endomorphism on k. Thus if M is any $A[F]$-module, then Lemma 2.1 shows that the Tor-modules $\mathrm{Tor}^A_\bullet(k, M)$ are naturally $k[F]$-modules, where $k[F]$ is the usual commutative polynomial ring over k.

If V is a smooth P-representation defined over k, then it is in particular smooth as an N_0-representation, and so is naturally a torsion A-module. Combining this A-module structure with the action of F and Γ on V, we see that V is naturally an $A[F,\Gamma]$-module.

4.1. **Definition.** If V is a smooth P-representation over k, and if V_0 is a Γ-subrepresentation of V, then we let $M(V, V_0)$ denote the left $A[F]$-submodule of V generated by V_0 (which we observe is naturally an $A[F,\Gamma]$-module).

Let $\mathcal{C}$ denote the abelian category of left $A[F,\Gamma]$-modules that are torsion as A-modules, and let $\mathcal{C}'$ denote the full subcategory consisting of modules each element of which is annihilated by a power of F. Evidently $\mathcal{C}'$ is a Serre subcategory of $\mathcal{C}$, and we let $\mathcal{C}''$ denote the quotient category $\mathcal{C}'' := \mathcal{C}/\mathcal{C}'$.

4.2. **Definition.** If V is smooth P-representation, then we define $M(V) := V$, regarded as an object of $\mathcal{C}''$.

4.3. **Proposition.** *The formation of $M(V)$ yields an exact and faithful functor from the category of smooth P-representations over k to the category $\mathcal{C}''$.*

Proof. The exactness of $M(V)$ is immediate from the definition. To see that it is faithful, it suffices to note that if V is non-zero, then $M(V)$ is not the zero object in $\mathcal{C}''$. Indeed, since the action of F on V is injective, if V is non-zero then $M(V)$ does not belong to the category $\mathcal{C}'$. □

The following proposition provides a means of studying the functor $M(V)$.

4.4. **Proposition.** *If V is a smooth representation of P over k, and if V_0 is a ΓZ-invariant k-vector subspace of V that generates V over P, then the inclusion $M(V, V_0) \subset V$ induces an isomorphism in the quotient category $\mathcal{C}''$.*

Proof. By assumption $V = k[P]V_0$. Since $P = F^{-\mathbb{N}} N_0 F^{\mathbb{N}} \Gamma Z$, and since V_0 is ΓZ-invariant, we may rewrite this equality as $V = k[F^{-1}]M(V, V_0)$. Thus any element of the quotient $V/M(V, V_0)$ is annihilated by some power of F, i.e. $V/M(V, V_0)$ is an object of $\mathcal{C}'$. This proves the proposition. □

The preceding proposition applies in particular if V is a smooth G-representation, and if V_0 is a KZ-invariant subspace of V that generates V over G. Indeed, $G = PK$, and so $V = k[G]V_0 = k[P]V_0$. Unfortunately, this proposition is not so useful in general, since we can't say much about the $A[F, \Gamma]$-modules $M(V, V_0)$. Indeed, the only general result on their structure that we can prove at the moment is the following:

4.5. **Proposition.** *If V is an admissible smooth representation of G over k, and if V_0 is a finite-dimensional KZ-invariant k-vector subspace of V, then each of the* Tor-*modules* $\mathrm{Tor}^A_\bullet\big(k, M(V, V_0)\big)$ *is a torsion $k[F]$-module.*

Proof. If W denotes the G-subrepresentation of V generated by V_0, then since $M(W, V_0) = M(V, V_0)$, it is no loss of generality to suppose that V is generated by V_0. Proposition 4.4 then shows that $V/M(V, V_0)$ is an object of $\mathcal{C}'$. This is easily seen to imply that each element of any of the Tor-modules $\mathrm{Tor}^A_\bullet\big(k, V/M(V, V_0)\big)$ is annihilated by some power of F. In particular, each of these Tor-modules is $k[F]$-torsion. A consideration of the long exact sequence of Tor-modules associated to the short exact sequence

$$0 \to M(V, V_0) \to V \to V/M(V, V_0) \to 0$$

then shows that to prove the proposition, it suffices (in fact, is equivalent) to prove that each of the Tor-modules $\mathrm{Tor}^A_\bullet\big(k, V\big)$ is a torsion $k[F]$-module. Since $\mathrm{Tor}^A_\bullet\big(k, V\big) \xrightarrow{\sim} H^{d-\bullet}(N_0, V)$, this follows from [6, Thm. 3.2.3(1)]. (Note that under the preceding isomorphism, the action of F on the source corresponds to the action of F, thought of as an element of Z_M^+, in the notation of [6], on the target.) □

The preceding result suggests the following question.

4.6. **Question.** If V is a finitely generated admissible smooth G-representation over k, can we choose a finite-dimensional KZ-invariant k-vector subspace V_0 of V which generates V over G, with the additional property that $M(V, V_0)$ is finitely presented over $A[F]$?

An affirmative answer to the preceding question, together with Propositions 2.2 (applied with $B = k$) and 4.5, would imply that $M(V, V_0)[\mathfrak{m}]$ (which coincides with $\mathrm{Tor}^A_d\big(k, M(V, V_0)\big)$) is a finitely generated torsion $k[F]$-module, and hence is finite-dimensional over k, or equivalently, that $M(V, V_0)$ is an admissible A-module (in an obvious sense, generalizing that given by Definition 3.1 in the case of a discrete valuation ring).

At the moment we can answer the preceding question in the case when $E = \mathbb{Q}_p$, in which case it does indeed have an affirmative answer, as we now show. (Note, though, that we will reverse the logic in the argument of the preceding paragraph: by applying the results of Section 3, we will first show that $M(V, V_0)$ is admissible, and then as a consequence deduce that it is finitely presented over $A[F]$.)

4.7. **Theorem.** *If $E = \mathbb{Q}_p$, if V is an admissible smooth representation of G over k, and if V_0 is a finite-dimensional k-vector subspace of V, then $M(V, V_0)$ is finitely presented and of finite length as a left $A[F]$-module, and is admissible as an A-module.*

Proof. Since $d = 1$, the complete local ring A is a discrete valuation ring; let t denote a uniformizer. We begin by noting that since $M(V, V_0)$ is finitely generated over $A[F]$ by construction, it follows from Proposition 3.2 and Corollary 3.4 that if $M(V, V_0)$ is admissible over A, then it is both finitely presented and of finite length over $A[F]$. Thus it suffices to prove that $M(V, V_0)$ is admissible.

Write $W := k[G]V_0$ and $V_1 := k[KZ]V_0$. Since V is admissible, we see that V_1 is finite-dimensional over k, while since $G = PK$, we see that V_1 generates W over P. Also, we have that $M(W, V_1) \supset M(W, V_0) = M(V, V_0)$, and hence $M(V, V_0)$ is admissible if $M(W, V_1)$ is. Since Proposition 4.5 shows that $M(W, V_1)/tM(W, V_1)$ is $k[F]$-torsion, it follows from Proposition 3.5 that $M(W, V_1)$ is admissible, as required. □

4.8. **Remark.** The preceding result is essentially due to Colmez [4], although he phrases it differently, as we now explain.

In the context of Theorem 4.7, Colmez writes Π rather than V, and W rather than V_0, and writes $I^+(W, \Pi)$ to denote the $A[F]$-module that we have called $M(V, V_0)$. In the following discussion we will use Colmez's notation, to facilitate the comparison with [4].

The dual $\mathrm{Hom}_k\big(I^+(W, \Pi), k\big)$ is a finitely generated A-module (since $K^+(W, \Pi)$ is admissible as an A-module, by Theorem 4.7), which Colmez denotes $\mathbf{D}^\natural_W(\Pi)$. The cokernel of the map $\phi_{I^+(W,\Pi)} : F^* I^+(W, \Pi) \to I^+(W, \Pi)$ is finitely generated and torsion over A (since $I^+(W, \Pi)$ is finitely generated over $A[F]$ and torsion over A), and thus the the kernel of the induced map $\mathbf{D}^\natural_W(\Pi) \to F^*\mathbf{D}^\natural_W(\Pi)$ is finitely generated and torsion over A. If we write $K := k((t))$ to denote the fraction field of A, then the tensor product $K \otimes_A \mathbf{D}^\natural_W(\Pi)$ is well-defined (up to a canonical isomorphism) independently of the choice of W (as follows from Proposition 4.4, or more accurately, its proof). Colmez denotes this tensor product by $\mathbf{D}(\Pi)$. The induced map $\mathbf{D}(\Pi) \to F^*\mathbf{D}(\Pi)$ is then an isomorphism of finite-dimensional K-vector spaces. Together with the Γ-action on $\mathbf{D}(\Pi)$ induced by the Γ-action on $I^+(W, \Pi)$, the inverse of this isomorphism equips $\mathbf{D}(\Pi)$ with the structure of a finite-dimensional (ϕ, Γ)-module over K. It is the finite-dimensionality of $\mathbf{D}(\Pi)$, rather than Theorem 4.7 itself, which is proved by Colmez [4, Thm. 4.13].

Let us also remark that Colmez in fact assumes that the representation Π is of finite length, and that his proof relies strongly on the classification of irreducible representations (due to Barthel-Livné [1] and Breuil [3]). In [5] we proved, again relying on the classification of irreducibles, that any finitely generated admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation is in fact of finite length. As we observe in the following corollary, this follows directly from Theorem 4.7. We will also show how the theory of $A[F]$-modules can be used to give a simple proof of the classification of irreducible admissible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

It follows from Proposition 4.4 and Theorem 4.7, that on the category of finitely generated admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$ representations, the functor $M(V)$ restricts to a faithful exact functor with image lying in the full subcategory of $\mathcal{C}''$ consisting of finite-length objects.

4.9. **Corollary.** *If $E = \mathbb{Q}_p$, then any finitely generated admissible smooth G-representation over k is of finite length as a P-representation (and so in particular, as a G-representation).*

Proof. If V is a finitely generated admissible smooth G-representation, then Proposition 4.4 and Theorem 4.7 shows that $M(V)$ is a finite length object of $\mathcal{C}''$. Since the functor M is faithful and exact, we conclude that V is of finite length as a P-representation, as claimed. □

We conclude this section by noting the following result, a kind of converse to Theorem 4.7 (which however holds for arbitrary unramified E):

4.10. **Proposition.** *If V is a smooth G-representation over k, and if V_0 is a KZ-invariant subspace of V that generates V_0 over G which has the additional property that $M(V, V_0)$ is admissible over A, then V is admissible as a G-representation.*

Proof. If K_1 denotes the first congruence subgroup of K, then (using, e.g., the Iwahori decomposition of K_1), one sees that $M(V, V_0)$ is K_1-invariant. Since $M(V, V_0)$ is admissible over A by assumption, it is certainly admissible as a K_1-representation. If $g \in K$, then $gM(V, V_0)$ is also K_1-invariant (since K_1 is normal in K), and is again admissible over K_1. The Cartan decomposition of G shows that $G = \coprod_{g\in K/K_1, n\geq 0} gK_1F^nKZ$, and thus that $V = \sum_{g\in K/K_1} gM(V, V_0)$. Hence V is the sum of a finite number of admissible K_1-subrepresentations, and thus is itself admissible as a K_1-representation, and hence also as a G-representation. □

5. Irreducible supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

We maintain the notation of the preceding section, and in addition set $E = \mathbb{Q}_p$. We also let I and I_1 have their usual meanings (so I is the upper triangular Iwahori subgroup of K, and I_1 its pro-p Sylow subgroup).

Up to a twist, any irreducible representation of K over k is of the form $\mathrm{Sym}^r k^2$, where $0 \leq r \leq p - 1$. We regard $\mathrm{Sym}^r k^2$ as a KZ-representation by requiring the element $p \in \mathbb{Q}_p^\times \xrightarrow{\sim} Z$ to act trivially. Recall that the Hecke algebra $\mathcal{H} := \mathrm{End}_G(c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2)$ is isomorphic to a polynomial ring in one generator T over k. The Hecke operator T can be defined in various ways; we recall one useful point of view here. (This description of T was inspired by an argument of Kevin Buzzard.) Recall that for any smooth G-representation V, there is a functorial isomorphism

$$\mathrm{Hom}_G(c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2, V) \xrightarrow{\sim} \mathrm{Hom}_{KZ}(\mathrm{Sym}^r k^2, V).$$

Thus rather than describing T directly as an endomorphism of $c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2$, we instead describe the induced endomorphism of the functor $\mathrm{Hom}_{KZ}(\mathrm{Sym}^r k^2, –)$.

The space of invariants $(\mathrm{Sym}^r k^2)^{I_1}$ is one-dimensional, and I acts on this space through the character $z^r \otimes 1$. Thus there is a surjection

$$\mathrm{Ind}_{IZ}^{KZ} z^r \otimes 1 \to \mathrm{Sym}^r k^2, \tag{5}$$

which fits into a short exact sequence

$$0 \to (\mathrm{Sym}^{p-1-r} k^2) \otimes \det^r \to \mathrm{Ind}_{IZ}^{KZ} z^r \otimes 1 \to \mathrm{Sym}^r k^2 \to 0.$$

Replacing r by $p-1-r$ and twisting, we obtain a corresponding short exact sequence

$$0 \to \mathrm{Sym}^r k^2 \to \mathrm{Ind}_{IZ}^{KZ} 1 \otimes z^r \to (\mathrm{Sym}^{p-1-r} k^2) \otimes \det^r \to 0. \tag{6}$$

If W is any smooth G-representation, we can then construct the following natural map:

$$\begin{aligned} \mathrm{Hom}_{KZ}(\mathrm{Sym}^r k^2, W) &\hookrightarrow \mathrm{Hom}_{KZ}(\mathrm{Ind}_{IZ}^{KZ} z^r \otimes 1, W) \\ &\xrightarrow{\sim} \mathrm{Hom}_{IZ}(z^r \otimes 1, W) \overset{wF}{\xrightarrow{\sim}} \mathrm{Hom}_{IZ}(1 \otimes z^r, W) \\ &\xrightarrow{\sim} \mathrm{Hom}_{KZ}(\mathrm{Ind}_{IZ}^{KZ} 1 \otimes z^r, W) \to \mathrm{Hom}_{KZ}(\mathrm{Sym}^r k^2, W), \end{aligned} \tag{7}$$

in which the first arrow is induced by the surjection (5) (and hence is injective), the first and third isomorphisms are provided by Frobenius reciprocity, w denotes the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K$ (so that $wF = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$), and the final arrow is induced by the injection in (6). The composite (7) induces an endomorphism of the functor $\mathrm{Hom}_{KZ}(\mathrm{Sym}^r k^2, –)$, which as noted above corresponds in turn to an endomorphism of $c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r$. This endomorphism is the Hecke operator T.

Recall that any absolutely irreducible admissible smooth G-representation over k is, up to a twist, a quotient of $\big(c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2\big)/(T - \lambda)\big(c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2\big)$, for some $r \in \{0, \ldots, p-1\}$ and some $\lambda \in k$. In the case when $\lambda \neq 0$, these quotients are easily analyzed [1], and give rise to one-dimensional, principal series, or special representations as their irreducible subquotients. In the case when $\lambda = 0$, Breuil [3] showed that the corresponding quotients are irreducible and admissible. We will give another proof of Breuil's result here, via an analysis of the associated $A[F]$-modules.

5.1. **Theorem.** *If $r \in \{0, \ldots, p-1\}$, then $(c-\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2)/T(c-\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2)$ is an irreducible admissible smooth representation of G.*

Proof. Write $V := \big(c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2\big)/T\big(c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2\big)$, let V_0 denote the KZ-subrepresentation $\mathrm{Sym}^r k^2$ of V, and write v to denote a basis for the one-dimensional subspace $V_0^{N_0}$ of v. Since v is fixed by N_0, we have that

$$tv = 0. \tag{8}$$

(Recall that t denotes the uniformizer of $A := k[[N_0]]$.) Also, Γ acts on v via the character $\overline{\varepsilon}^r$, where $\overline{\varepsilon}$ denotes the $\mathbb{F}_p^\times$-valued character of Γ defined by $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mapsto a \bmod p$.

Since, by assumption, T annihilates the embedding $V_0 \hookrightarrow V$, a consideration of the description of T afforded by (7) shows that under the action of KZ, wFv generates a copy of $V_1 := (\mathrm{Sym}^{p-1-r} k^2) \otimes \det^r$ in V, with wFv spanning the one-dimensional subspace $V_1^{N_0}$. (This is a well-known result, due originally to Breuil.) In particular

$$twFv = 0. \tag{9}$$

Since $w \in K$, we see that Fv also lies in V_1, and (recalling the manner in which w acts on V_1) that

$$t^{p-1-r}Fv = cwFv, \tag{10}$$

for some non-zero scalar $c \in k^\times$. Since $(wF)^2 = 1$, we find that $wFwFv = v$. Recalling the manner in which w acts on V_0, we deduce that $FwFv$ generates V_0 over N_0, or equivalently, over A, and, in particular, that

$$t^r FwFv = dv, \tag{11}$$

for some non-zero scalar $d \in k^\times$. Combining the relations (8), (9), (10), and (11), we find that $M(V, V_0)$ is a cyclic $A[F,\Gamma]$-module, with v as a generator, and that v satisfies (at least) the relations

$$tv = t^{p-r}Fv = (t^{(p-1)(p-r)}F^2 - cd)v = 0, \quad \gamma v = \overline{\varepsilon}(\gamma)^r v \text{ for } \gamma \in \Gamma. \tag{12}$$

These relations imply that $M(V, V_0)$ is A-torsion and that $M(V, V_0)/tM(V, V_0) = 0$, and thus Propositions 3.5 and 4.10 show that V is admissible.

Let M denote the $A[F,\Gamma]$-module cyclically generated by the element v satisfying the relations (12). We will show that M is irreducible. This will imply that $M(V, V_0)$ (which we have shown is a non-zero quotient of M) is isomorphic to M, and hence is irreducible. Since the functor $V \mapsto M(V)$ is exact and faithful, we will conclude that V is irreducible as a G-representation, as claimed, and so will complete the proof of the theorem.

We first observe that $M[t] = k\langle v, t^{p-1-r}Fv\rangle$, as the reader may easily verify. Also, by assumption the group Γ acts on v via the character $\overline{\varepsilon}^r$. The reader may then easily deduce that Γ acts trivially on $t^{p-1-r}v$. Suppose that N is a non-zero $A[F,\Gamma]$-submodule of M. We will show that N contains either v or $t^{p-1-r}Fv$, and so is equal to M. (Indeed, v is a cyclic generator of M, and we have the relation $t^rFt^{p-1-r}Fv = cdv$, with $cd \neq 0$.)

As N is non-zero, its submodule $N[t]$ is a non-zero subspace of $M[t]$, and so contains some non-zero linear combination $\alpha v + \beta t^{p-1-r}Fv$. If $\beta = 0$, then $\alpha \neq 0$, and so M contains v. If $\alpha = 0$, then $\beta \neq 0$, and so M contains $t^{p-1-r}Fv$. Suppose that neither α nor β vanishes. If $r \geq p - r$, then applying t^rF to the element $\alpha v + \beta t^{p-1-r}Fv$, we find that M contains $t^{(p-1)(p-r)}F^2v = cdv$, and thus contains v. If $p-2-r \geq r$ (so that $p-1-r \geq r+1$), then applying $t^{p-1-r}F$ to the element $\alpha v + \beta t^{p-1-r}Fv$, we find that M contains $t^{p-1-r}Fv$. Thus, except in the case $r = (p-1)/2$, we conclude that M is irreducible even as an $A[F]$-module.

Finally, suppose that $r = (p-1)/2$. Note that in this case necessarily $p > 2$ and $0 < r < p-1$. We consider the action of the group Γ on $N[t]$. Since Γ acts on the element v (resp. $t^{(p-1)/2}Fv$) via the character $\overline{\varepsilon}^{(p-1)/2}$ (resp. the trivial character), and since these characters are distinct, we find that since $N[t]$ contains some non-zero linear combination $\alpha v + \beta t^{(p-1)/2}Fv$, it necessarily contains one of v or $t^{(p-1)/2}Fv$. This completes the proof of the theorem. □

5.2. **Remark.** In fact, the proof of the theorem shows that V is even irreducible as a P-representation – a result originally due to Berger [2, Thm. 2.3.1 (1)], and which also follows from the G-irreducibility together with [8, Thm. 4.3].

5.3. **Remark.** If we let $\mathbf{D}$ denote the (ϕ,Γ)-module attached to

$$V := (c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2)/T(c - \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r k^2)$$

as in Remark 4.8 (so $\mathbf{D} = k((t)) \otimes_{k[[t]]} \mathrm{Hom}_k(M(V, V_0), k)$), then the proof of the preceding result shows that $\mathbf{D}$ is irreducible just as a ϕ-module over $k((t))$, except when $r = (p-1)/2$ (in which case one may check that it is in fact reducible as a ϕ-module, though as the preceding theorem shows, it is irreducible as a (ϕ, Γ)-module). The (ϕ, Γ)-module $\mathbf{D}$ corresponds to the representation $\mathrm{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p} \omega_2^{r+1}$ of the Galois group $G_{\mathbb{Q}_p}$ (where ω_2 denotes a fundamental character of level 2, which is a character of $G_{\mathbb{Q}_{p^2}}$), and the irreducibility of $\mathbf{D}$ as a ϕ-module for $r \neq (p-1)/2$ corresponds to the fact that $\mathrm{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p} \omega_2^{r+1}$ remains irreducible when restricted to Galois group $G_{\mathbb{Q}_{p,\infty}}$ of the p-adic cyclotomic extension $\mathbb{Q}_{p,\infty}$ of $\mathbb{Q}_p$, except in the case when $r = (p-1)/2$.

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Mathematics Department, Northwestern University, 2033 Sheridan Rd., Evanston, IL 60208

E-mail address: `emerton@math.northwestern.edu`