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# LOCAL-GLOBAL COMPATIBILITY IN THE $p$-ADIC LANGLANDS PROGRAMME FOR $\mathrm{GL}_{2/\mathbb{Q}}$

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## 1. Introduction

The aim of this paper is to establish (under certain technical hypotheses) the local-global compatibility conjecture introduced in [38]. As one application, we obtain results related to the Fontaine–Mazur conjecture on two-dimensional Galois representations.

### 1.1. The local-global compatibility conjecture.

In [38] we made the following local-global compatibility conjecture (Conjecture 1.1.1 of that reference):

1.1.1. **Conjecture.** *If $V$ is an odd irreducible continuous two-dimensional representation of $G_{\mathbb{Q}}$, defined over a finite extension $E$ of $\mathbb{Q}_p$, and unramified outside of a finite set of primes, then the $\mathrm{GL}_2(\mathbb{A}_f)$-representation $\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_E)$ decomposes as a restricted tensor product*

$$\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_E) \xrightarrow{\sim} B(V_{|G_{\mathbb{Q}_p}}) \otimes \bigotimes_{\ell \neq p}{}' \, \pi_\ell(V).$$

In the statement of the conjecture, $\widehat{H}^1_E$ denotes the $p$-adically completed cohomology of the tower of modular curves (as defined in [35]; see also Subsection 5.1 below), $B(V_{|G_{\mathbb{Q}_p}})$ denotes the admissible unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $V_{|G_{\mathbb{Q}_p}}$ via the $p$-adic local Langlands correspondence, and for each prime $\ell \neq p$, $\pi_\ell(V)$ is the admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ associated to $V_{|G_{\mathbb{Q}_\ell}}$ via (a modified version of) the classical local Langlands correspondence. (The representation that we here denote $\pi_\ell(V)$ was denoted by $\pi_\ell^{\mathrm{m}}(V)$ in [38]. It coincides with the representation attached to $V_{|G_{\mathbb{Q}_\ell}}$ by the classical local Langlands correspondence with respect to the Tate normalization, except in those cases in which the latter representation is not generic (i.e. not infinite-dimensional). See Subsection 4.2 below for a precise specification of $\pi_\ell(V)$.)

1.1.2. **Remark.** In fact, Conjecture 1.1.1 is likely false as stated in one particular case — namely, when the local Galois representation $V_{|G_{\mathbb{Q}_p}}$ is a twist of an extension of the cyclotomic character by the trivial character; see Remark 6.1.23 below.

At the time that [38] was written, the existence of a $p$-adic local Langlands correspondence for two-dimensional $p$-adic representations of $G_{\mathbb{Q}_p}$ was still conjectural. However, such a correspondence has now been constructed by Colmez and Kisin [25, 61]. We rely heavily on the deformation-theoretic formulation of the correspondence due to Kisin [61], not only to give Conjecture 1.1.1 a precise sense, but also as a key tool in our arguments. For our application to the Fontaine–Mazur conjecture, we also rely crucially on Colmez's results on the non-vanishing of locally algebraic vectors [25, Thm. VI.6.18].

1.2. **Statement of results.** We work in the context of promodular Galois representations. Let $E$ be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$, residue field $k$, and uniformizer $\varpi$. Recall that a continuous irreducible two-dimensional representation $V$ of $G_{\mathbb{Q}}$ over $E$ is called promodular if it is isomorphic to the Galois representation attached to a cuspidal $p$-adic modular eigenform of some (possibly non-integral) weight, or equivalently, if the pseudo-representation attached to $V$ lies in the Zariski closure, in an appropriate deformation space, of the set of $p$-adic pseudo-representations attached to classical cuspforms. (See e.g. [38, §7.3].)

The following theorem is our main result in the direction of Conjecture 1.1.1. It is proved in Subsection 6.2 below.

1.2.1. **Theorem.** *Let $V$ be a continuous, irreducible, promodular two-dimensional representation of $G_{\mathbb{Q}}$ over $E$, unramified outside of a finite set of primes. Let $\overline{V}$ denote the residual representation attached to $V$, and assume that the following conditions are satisfied:*

(a) *$\overline{V}$ is absolutely irreducible.*

(b) *$\overline{V}_{|G_{\mathbb{Q}_p}} \not\sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$ for any $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$ (where $*$ may or may not be zero, and $\overline{\varepsilon}$ denotes the mod $p$ cyclotomic character).*

*The following conclusions then hold:*

(1) *For some finite set of primes $\Sigma$ containing the primes at which $V$ is ramified, together with $p$, there is a non-zero $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^{\Sigma})$-equivariant map*
$$B(V_{|G_{\mathbb{Q}_p}}) \otimes \bigotimes_{\ell \notin \Sigma}{}' \pi_\ell(V) \to \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_E).$$
*If, furthermore, $V_{|G_{\mathbb{Q}_p}}$ is neither the direct sum of two characters, nor an extension of a character by itself, then any such map is an embedding.*

(2) *If $\overline{V}$ is $p$-distinguished, i.e. $\overline{V}_{|G_{\mathbb{Q}_p}} \not\sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ for any $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$ (where $*$ may or may not be zero), then there is a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant isomorphism*
$$B(V_{|G_{\mathbb{Q}_p}}) \otimes \bigotimes_{\ell \neq p}{}' \pi_\ell(V) \xrightarrow{\sim} \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_E).$$

In fact, in the situation of part 2 of the theorem, we prove a stronger result, which describes not only the multiplicity spaces $\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_E)$ for those $V$ lifting $\overline{V}$, but the structure of the summand of $\widehat{H}^1_E$ itself (and even its unit ball $\widehat{H}^1_{\mathcal{O}}$) obtained by localizing at $\overline{V}$. (See Theorem 6.2.13 below for the precise statement.) As well as the $p$-adic local Langlands correspondence of [25, 61], this description involves the local Langlands correspondence in families developed in [43] (and recalled in Section 4 below).

As explained in [38], Theorem 1.2.1 has the following corollary (proved in detail as Theorems 7.1.1 and 7.2.1 below). It is here that we use Colmez's result on the non-vanishing of locally algebraic vectors, a result which was stated as one of the conjectural properties of the $p$-adic local Langlands correspondence in [38].

1.2.2. **Corollary.** *Let $V$ be a continuous, irreducible, promodular two-dimensional representation of $G_{\mathbb{Q}}$ over $E$, unramified outside of a finite set of primes. Let $\overline{V}$*

*denote the residual representation attached to $V$, and assume that the following conditions are satisfied:*

(a) $\overline{V}$ *is absolutely irreducible.*

(b) $\overline{V}_{|G_{\mathbb{Q}_p}} \not\sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$ *for any $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$ (where $*$ may or may not be zero).*

*The following conclusions then hold:*

(1) *If $V_{|G_{\mathbb{Q}_p}}$ is trianguline, then $V$ is a twist of the Galois representation attached to an overconvergent $p$-adic cuspidal eigenform of finite slope.*

(2) *If $V_{|G_{\mathbb{Q}_p}}$ is de Rham with distinct Hodge–Tate weights, and if $V_{|G_{\mathbb{Q}_p}}$ is* not *the direct sum of two characters that have isomorphic reductions modulo $\varpi$, then $V$ is a twist of the Galois representation attached to a classical cuspidal eigenform of weight $k \geq 2$.*

Our requirement that $V$ be promodular is little restriction, in light of the following theorem, which is a consequence of the results and methods of the papers [4, 31, 53, 54, 55, 60]. (See Subsection 7.3 below for the proof.)

1.2.3. **Theorem** (Böckle, Diamond–Flach–Guo, Khare–Wintenberger, Kisin)**.** *Let $V$ be a continuous, irreducible, odd, two-dimensional representation of $G_{\mathbb{Q}}$ over $E$, unramified outside of a finite set of primes $\Sigma$. Assume that the following conditions hold:*

(a) $p > 2$.

(b) $\overline{V}_{|G_{\mathbb{Q}(\zeta_p)}}$ *is absolutely irreducible.*

(c) $\overline{V}_{|G_{\mathbb{Q}_p}} \not\sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ *or* $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$*, for any $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$ (where $*$ may or may not be zero).*

*Then $V$ is promodular, of tame level divisible only by the primes in $\Sigma \setminus \{p\}$.*

Combining this result with corollary 1.2.2, we obtain the following result.

1.2.4. **Theorem.** *Let $V$ be a continuous, irreducible, odd, two-dimensional representation of $G_{\mathbb{Q}}$ over $E$, unramified outside of a finite set of primes, with associated residual representation $\overline{V}$. Assume that the following conditions hold:*

(a) $p > 2$.

(b) $\overline{V}_{|G_{\mathbb{Q}(\zeta_p)}}$ *is absolutely irreducible.*

(c) $\overline{V}_{|G_{\mathbb{Q}_p}} \not\sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ *or* $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$*, for any $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$ (where $*$ may or may not be zero).*

*The following conclusions then hold:*

(1) *If $V_{|G_{\mathbb{Q}_p}}$ is trianguline, then $V$ is a twist of the Galois representation attached to an overconvergent $p$-adic cuspidal eigenform of finite slope.*

(2) *If $V_{|G_{\mathbb{Q}_p}}$ is de Rham with distinct Hodge–Tate weights, then $V$ is a twist of the Galois representation attached to a classical cuspidal eigenform of weight $k \geq 2$.*

The first of these results confirms (under the given hypotheses) a conjecture of Kisin. (See [56, Conj. 11.8], as well as the discussion in note (2) on p. 450 of the same paper.) The second of these results confirms (again for $V$ satisfying the given hypotheses) a conjecture of Fontaine and Mazur [46, Conj. 3c].

1.2.5. **Remark.** A result similar to part 2 of Theorem 1.2.4, but with slightly different hypotheses on $\overline{V}$, has been proved by Kisin [59]. In Kisin's result, hypothesis (c) is replaced by the condition that $\overline{V}_{|G_{\mathbb{Q}_p}} \not\sim \chi \otimes \begin{pmatrix} \overline{\varepsilon} & * \\ 0 & 1 \end{pmatrix}$ for any $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$ (where $*$ may or may not be zero). As a corollary of his result, Kisin has deduced Conjecture 11.8 of [56] (under the same set of hypotheses on $\overline{V}$), a result which is related to, but somewhat weaker than, part 1 of our Theorem 1.2.4.

In fact, what Kisin established in [59] is that the Fontaine–Mazur conjecture for two-dimensional representations of $G_{\mathbb{Q}}$ (under the additional technical hypotheses just described) follows from the existence of a $p$-adic local Langlands correspondence for two-dimensional de Rham representations of $G_{\mathbb{Q}_p}$ with distinct Hodge–Tate weights, provided that this correspondence is compatible in a suitable manner with the classical correspondence. Such a correspondence, satisfying the requisite compatibility, had already been constructed by Berger and Breuil in the crystabelline setting [3] and by Colmez in the semi-stable (up to a twist) setting [22]. As we have already remarked, the $p$-adic local Langlands correspondence for two-dimensional representations of $G_{\mathbb{Q}_p}$ has now been constructed in full generality by Colmez and Kisin. In the case of two-dimensional representations that are potentially crystalline, but not crystabelline, we establish the compatibility with the classical correspondence in Theorem 3.3.22 below, building on Colmez's results in [25]. This completes the proof of the properties of the $p$-adic local Langlands correspondence required to deduce the Fontaine–Mazur conjecture in such cases (i.e. potentially crystalline, but not crystabelline, locally at $p$) via the arguments of [59].

Some further discussion of the relationship between our approach and Kisin's approach to the Fontaine–Mazur conjecture is given in Subsection 1.4 below.

Our local-global compatibility result (in the strong form provided by Theorem 6.2.13) has as a corollary the following theorem, which describes the multiplicities with which a mod $p$ Galois representation can appear in the mod $p$ étale cohomology of a modular curve of *arbitrary* level.

Write $H^1_k := \varinjlim_N H^1_{\text{ét}}(Y(N), k)$, the inductive limit over all levels $N$ of the étale cohomology, with coefficients in $k$, of the full (geometrically disconnected) modular curve of level $N$. This is naturally a representation of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$.

1.2.6. **Theorem.** *Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ be absolutely irreducible and modular (or equivalently, odd* [53, 54, 55, 60]*). Suppose furthermore that $p > 2$, and that $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is not equivalent to either $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ or $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$, for any $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$ (where $*$ may or may not be zero). Then there is a $k$-linear and $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant isomorphism*

$$\overline{\pi} \otimes {\bigotimes_{\ell \neq p}}' \, \overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}}) \xrightarrow{\sim} \mathrm{Hom}_{k[G_{\mathbb{Q}}]}(\overline{\rho}, H^1_k),$$

*where the representation* $\overline{\pi}$ *of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *is attached to* $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ *via the mod p local Langlands correspondence of Theorem* 3.3.2 *below, and for each* $\ell \neq p$*, the representation* $\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}})$ *of* $\mathrm{GL}_2(\mathbb{Q}_\ell)$ *is attached to* $\overline{\rho}_{|G_{\mathbb{Q}_\ell}}$ *via the mod p local Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_\ell)$ *given in* [43] (*and recalled in Theorem* 4.3.1 *below*).

1.2.7. **Remark.** I expect the same result to hold when $p = 2$, but our imperfect understanding of the mod 2 and 2-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_\ell)$ ($\ell \neq 2$) leads to a less definitive result in this case; see Remark 6.1.21.

1.2.8. **Remark.** Theorem 1.2.6 is related to, and indeed generalizes, classical "mod $p$ multiplicity one" results due to Mazur, Ribet, Wiles, and others (see e.g. [64, 72, 65, 48, 33, 86]). These results describe the multiplicity with which $\overline{\rho}$ appears in the cohomology of modular curves of level $\Gamma_1(N)$, when $N$ is prime-to-$p$, or divisible by at most a single power of $p$. They may be recovered from Theorem 1.2.6 (for those $\overline{\rho}$ to which the theorem applies) by passing to the invariants under a suitably chosen compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$; we should point out, though, that these results are among the tools that we use in the proof of Theorem 1.2.6. (Note, however, that we only use the more straightforward results of [64, 72, 65, 48, 33]; we do not use either the existence of companion forms [48] or [86, Thm. 2.1 (ii)] in the more difficult case when (in the notation of that reference) $\Delta_{(p)}$ is trivial mod $\mathfrak{m}$. Indeed, Theorem 1.2.6 gives a new proof of these results, for those $\overline{\rho}$ to which it applies; see Remark 6.2.15 below.)

1.2.9. **Remark.** We conjecture that Theorem 1.2.6 continues to hold in the case when $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is equivalent to $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, for some $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$ (with $p > 3$, so that $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is not equivalent to $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$, for any $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$). In the remaining cases, when $\overline{\rho}$ *is* equivalent to $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$, for some $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$, we also expect an analogous result to hold. However, the correct definition of $\overline{\pi}$ in this case is more problematic (and in particular, is not provided by Theorem 3.3.2 below). See Remark 6.1.23 below for a further discussion of this point.

1.2.10. **Remark.** In the paper [79] Serre stated his conjecture that any odd continuous irreducible representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ should be modular, of minimal weight and prime-to-$p$ level determined by the local properties of $\overline{\rho}$ at $p$ and away from $p$ respectively. This conjecture has now been proved [53, 54, 55, 60]. In the same paper, Serre raised the question as to whether his conjecture might admit a reformulation in terms of the mod $p$ representation theory of the adèlic $\mathrm{GL}_2$ [79, Question 3.4 (2)]. The preceding theorem gives a positive answer to Serre's question (for those $\overline{\rho}$ to which it applies). Indeed, from this representation-theoretic statement one can determine the minimal weight of $\overline{\rho}$ (which is determined by the socle of $\overline{\pi}$ as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation — see Lemma 3.5.5 below, as well as the discussion in the proof of [14, Thm. 3.15]) as well as the minimal prime-to-$p$-level of $\overline{\rho}$ (which is equal to the conductor of the smooth $\mathrm{GL}_2(\mathbb{A}_f^p)$-representation $\bigotimes_{\ell \neq p} \overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}})$).

As was already intimated in the discussion of Remark 1.2.5, as another application of our results we prove Theorem 3.3.22 below, which for any de Rham representation with distinct Hodge–Tate weights $V$ of $G_{\mathbb{Q}_p}$ over $E$, gives a precise description of the locally algebraic vectors in the Banach space representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $V$ via the $p$-adic local Langlands correspondence. This completes the results of Colmez in [25, §§VI.6], where this question is investigated, and completely solved in the case when $V$ becomes crystalline or semi-stable over an abelian extension of $\mathbb{Q}_p$.

1.3. **The organization of the paper.** In Section 2 we introduce some notation which will be in force throughout the paper. The next three sections are preparatory in nature. In Section 3 we recall various recent results (primarily due to Colmez and Kisin) that construct the mod $p$ and $p$-adic local Langlands correspondences, and establish some additional related results. In Section 4, we recall some of the results of [43] regarding the local Langlands correspondence at primes $\ell \neq p$ in characteristic $p$ or over complete local rings of residue characteristic $p$. In Section 5 we recall the construction of (and introduce notation for) the $p$-adically completed cohomology of modular curves, as well as some related constructions and results (in particular, for the mod $p$ cohomology of modular curves). We give representation-theoretic interpretations of several results from the literature in this context, including the mod $p$ multiplicity one results of [64, 72, 65, 48, 33], as well as some of the results from the theory of $p$-adically ordinary modular forms.

Section 6 presents our main results. As already noted above, in addition to Theorem 1.2.1, which describes the multiplicities with which various Galois representations appear in $p$-adically completed cohomology, we prove (in many cases) a stronger result (Theorem 6.2.13), which gives a complete description (as a representation of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$) of the space of $p$-adically completed cohomology itself (after localizing at an appropriately chosen two-dimensional mod $p$ Galois representation). Finally, in Section 7, we deduce the various modularity results stated above.

In the appendices we develop some basic functional analysis that is required in the main body of the text.

1.3.1. **Remark.** It may be useful, for those readers who are primarily interested in the proof of Theorem 1.2.4, to indicate those parts of the paper that are necessary for the proof of this result. The proof of the theorem (other than part 2 of the theorem in the case when $V_{|G_{\mathbb{Q}_p}}$ is the direct sum of two characters) requires the material of Subsections 3.1–3.4 and 5.1–5.4; the following material from Section 6: Definition 6.1.1, Theorem 6.2.1, Proposition 6.2.2, the proof of Theorem 1.2.1 (which is given in Subsection 6.2), Subsection 6.3 up to and including Corollary 6.3.17 (always taking $\theta = \pi_{\Sigma}^{\mathfrak{m}}$), and Subsection 6.4 up to Remark 6.4.3; Subsections 7.1–7.3; and the material of the appendices. The same material (but substituting Subsection 7.4 in place of Subsections 7.1–7.3) serves to give the proof of Theorem 3.3.22.

The remainder of the paper (including all the material in Section 4) is devoted to proving the precise local-global compatibility of Theorem 1.2.1 (2). The most difficult case of the argument is when $\overline{V}_{|G_{\mathbb{Q}_p}}$ is a direct sum of two characters, and a large part of the paper is devoted solely to dealing with this case; this is true in particular of Subsections 3.6, 3.7, 5.5, 5.6, and 6.5. Our proof of part 2 of Theorem 1.2.4 (i.e. the Fontaine–Mazur conjecture) in the case when $V_{|G_{\mathbb{Q}_p}}$ is the direct sum of two characters requires the full strength of all this material; on the other hand, from the point of view of the Fontaine–Mazur conjecture (as opposed to local-global compatibility), this case is little interest, since Skinner and Wiles have proved much stronger results in the direction of the Fontaine–Mazur conjecture in the ordinary case [80, 81].

1.4. **Background and additional remarks.** In this section we describe some of the previous work on the problems considered in this paper, and compare our approach to that of the earlier work. We will begin by discussing the Fontaine-Mazur conjecture [46, Conj. 3c]. In fact, the version of the conjecture that we address in this paper is the following:

1.4.1. **Conjecture.** *If $V$ is an odd irreducible continuous two-dimensional representation of $G_{\mathbb{Q}}$, defined over a finite extension $E$ of $\mathbb{Q}_p$, unramified outside of a finite set of primes, and de Rham at $p$ with distinct Hodge–Tate weights, then $V$ is a twist of the Galois representation attached to a classical modular form of weight $k \geq 2$.*

1.4.2. **Remark.** It is believed that the condition of being odd can be omitted from this conjecture, i.e. that it is a consequence of the other hypotheses (see [46, Conj. 3c]). Recently, this has been verified by Calegari [15] in the case when $V$ is ordinary at $p$ (under the technical assumptions that $p \geq 7$, and that the associated residual representation has image containing $\mathrm{SL}_2(\mathbb{F}_p)$).

The original breakthrough in the direction of Conjecture 1.4.1 was made in the work of Wiles [86] and Taylor–Wiles [83]. Further progress was made in the papers [30], [26], [11], [32], [31], [82], [57], [58], [59], [60]. Each of these papers, other than [59], establishes a result in the direction of Conjecture 1.4.1 of the following form: if $V$ is as in the statement of the theorem, and if furthermore $p > 2$ (except in [32, 60], which treat the case $p = 2$), $\overline{V}$ is modular and $\overline{V}_{|G_{\mathbb{Q}(\zeta_p)}}$ is irreducible (where $\overline{V}$ denotes the residual Galois representation attached to $V$),[1] and $V$ satisfies some additional restrictions, both on the Hodge–Tate weights of $V_{|G_{\mathbb{Q}(\zeta_p)}}$ and (with the exception of [57] and [60]) on the amount of ramification occurring in the type of the de Rham representation $V_{|G_{\mathbb{Q}_p}}$, then $V$ is modular.[2] (For a discussion of the result of [59], see Remark 1.2.5 above, as well as the remarks below.) Each of these papers also follows (in broad outline) the strategy originally implemented in [86] and [83], namely, to consider a deformation ring $R$ parametrizing deformations of $\overline{V}$ of the same Hodge–Tate weights and type as $V$, as well as a Hecke algebra $\mathbb{T}$ parametrizing modular deformations of $\overline{V}$ of these weights and type, and then to prove that the natural map $R \to \mathbb{T}$ is an isomorphism.

In their paper [47], Gouvêa and Mazur prove that, under sufficiently strong hypotheses on $\overline{V}$, *every* deformation of $\overline{V}$ is promodular, i.e. is attached to a $p$-adic modular form. More precisely, Gouvêa and Mazur considered an unrestricted

[1] In each of the papers [32] and [60], both of which consider the case $p = 2$, the irreducibility condition on $\overline{V}_{|G_{\mathbb{Q}(\zeta_p)}}$ — the so-called Taylor–Wiles non-degeneracy condition — is replaced by a slight variant.

[2] We should mention that in the case when $V$ is assumed to be ordinary, or nearly ordinary, at $p$, it is not necessary to impose any restrictions on the Hodge–Tate weights of $V$, essentially because Hida theory [50] provides a very tight link between the $p$-ordinary eigenforms in an arbitrary weight $k \geq 2$ and the $p$-ordinary eigenforms in weight 2. The papers [80] and [81] prove very strong results in the direction of Conjecture 1.4.1 in the $p$-ordinary case; in particular, they succeed in eliminating the Taylor–Wiles non-degeneracy condition in this case.

deformation ring $R$ associated to $\overline{V}$, as well as a Hecke algebra $\mathbb{T}$ acting on an appropriate space of $p$-adic modular forms, and proved that the natural map $R \to \mathbb{T}$ is an isomorphism. A little later, Böckle [4] was able to combine the results of [86, 83, 30] with the argument of [47] to prove in some generality that an arbitrary deformation of a modular $\overline{V}$ is necessarily pro-modular. (Theorem 1.2.3 above provides a generalization of Böckle's result.) The results of [47] and [4] are sometimes called "big $R$ equals big $\mathbb{T}$" theorems, to distinguish them from the "small $R$ equals small $\mathbb{T}$" theorems discussed in the preceding paragraph.

When they were first proved, the "big $R$ equals big $\mathbb{T}$" result of [47, 4] appeared to have no bearing on Conjecture 1.4.1, since one seemed no closer to proving that conjecture in the case of $V$ associated to $p$-adic modular forms than in the case of arbitrary $V$.[3] However, in [38] we showed that Conjecture 1.1.1 for $V$ implies Conjecture 1.4.1 for $V$, provided that the $p$-adic local Langlands correspondence exists, and has sufficiently nice properties. This suggested the approach to Conjecture 1.4.1 that we take in this paper, namely to prove Conjecture 1.1.1 for all promodular deformations of a given modular $\overline{V}$, and then to combine this result with a "big $R$ equal big $\mathbb{T}$" theorem to deduce Conjecture 1.4.1 for all deformations of $\overline{V}$. In fact, one can hope to implement this strategy for any odd $\overline{V}$, since, as we recalled above, Serre's celebrated conjecture on the modularity of such a $\overline{V}$ [79] has now been proved [53, 54, 55, 60].

Of course, this strategy depends on the existence of a $p$-adic local Langlands correspondence with the required properties. The original idea of the $p$-adic local Langlands correspondence for two-dimensional representations of $G_{\mathbb{Q}_p}$ is due to Breuil [9], who conjectured the form that such a correspondence should take in the case of de Rham representations with distinct Hodge–Tate weights. In [10], Breuil also formulated a version of Conjecture 1.1.1, for those $V$ associated to classical modular forms of weight $k \geq 2$. Some results in the direction of this conjecture were obtained in [10], [12], and [38].

To the best of my knowledge, the suggestion that the $p$-adic local Langlands correspondence should exist for arbitrary two-dimensional representations of $G_{\mathbb{Q}_p}$ (rather than just de Rham representations with distinct Hodge–Tate weights, as considered by Breuil) was first made by Colmez (in his talk at the Durham symposium in the summer of 2004). This in turn suggested our generalization of Breuil's local-global compatibility conjecture (i.e. Conjecture 1.1.1).

Prior to the workshop on $p$-adic representations held in Montréal in September 2005, Conjecture 1.1.1 seemed inaccessible; indeed, the $p$-adic local Langlands correspondence was not yet defined in general. After the Montréal workshop, the situation changed completely. Colmez, in his talk at the workshop, explained the construction of a functor from admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representations over Artinian $\mathbb{Z}_p$-algebras to $(\varphi, \Gamma)$-modules (and hence to local Galois representations); see [25] and Subsection 3.2 below. Soon after the workshop, Kisin explained to Colmez how his functor could be used to provide a deformation-theoretic construction of the $p$-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, provided that a certain map on Exts was injective. Colmez quickly saw how to prove this injectivity result [25]

[3]The only prior result that we are aware of that considers Conjecture 1.4.1 specifically for Galois representations arising from $p$-adic modular forms is the paper [56] of Kisin, which essentially proves the conjecture in the case of representations associated to overconvergent $p$-adic eigenforms of finite slope.

(which has since been proved in greater generality by Paškūnas [67]), and as a consequence he and Kisin were able to construct the previously hypothetical $p$-adic local Langlands correspondence. After learning of this construction of Colmez and Kisin, I realized that using it (and especially, exploiting its deformation-theoretic formulation), one could deduce (many cases of) local-global compatibility for promodular Galois representations (i.e. Theorem 1.2.1 above), and hence Theorem 1.2.4 in the direction of the Fontaine–Mazur conjecture.

Already in his talk at the Montréal conference, Kisin had announced a result on the Fontaine–Mazur conjecture, essentially identical to part 2 of Theorem 1.2.4 above, in the case when $\overline{V}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible and $V_{|G_{\mathbb{Q}_p}}$ is crystabelline. His argument also relied on results from the $p$-adic local Langlands correspondence, in particular, the results of [3]. Indeed, as we already noted in Remark 1.2.5, it is not the triangulanity of $V_{|G_{\mathbb{Q}_p}}$ which is crucial to Kisin's method, but rather the compatibility between the $p$-adic and classical local Langlands correspondences for $V_{|G_{\mathbb{Q}_p}}$, which at the time of the Montréal conference was known only in the trianguline context [3, 22]. Soon after the conference, Kisin was able to simplify his arguments through the use of Colmez's functor, while simultaneously generalizing them so as to include the case when $\overline{V}_{|G_{\mathbb{Q}_p}}$ is reducible and (now that the compatibility between the $p$-adic and classical local Langlands for two-dimensional representations of $G_{\mathbb{Q}_p}$ has been completely proved) $V_{|G_{\mathbb{Q}_p}}$ is not necessarily trianguline. (See [59, Thm., p. 642] and the discussion following the statement of that theorem.)

Despite the similarity of several ingredients, the approach to the Fontaine–Mazur conjecture taken in this paper is quite different from that of [59]. The argument of [59] involves a "small $R$ equals small $\mathbb{T}$" theorem in the spirit of the original approach of Wiles [86] and Taylor–Wiles [83]; Colmez's functor and the $p$-adic local Langlands correspondence appear as tools for obtaining control over the complicated singularities of the deformation spaces that can arise when one makes no restrictions on the weight and type. On the other hand, the argument of this paper relies on a "big $R$ equals big $\mathbb{T}$" theorem. It is local-global compatibility, rather than a Taylor–Wiles-type argument, that then allows one to characterize the classical points among the promodular ones in terms of their $p$-adic Hodge theoretic properties.

1.5. **Acknowledgments.** The debt that this paper owes to the ideas of Christophe Breuil, Pierre Colmez, and Mark Kisin regarding the $p$-adic local Langlands correspondence will be evident to the reader. In particular, I would like to thank Pierre Colmez and Mark Kisin for sharing their ideas and results with me. I would also like to thank Mark Kisin, as well as Christophe Breuil, Kevin Buzzard, Frank Calegari, Brian Conrad, Toby Gee, James Newton, and Robert Pollack, for their helpful conversations and correspondence related to this article.

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## 2. Notation and conventions

Throughout the paper, we fix a prime $p$, as well as a finite extension $E$ of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$. We let $k$ denote the residue field of $\mathcal{O}$, and $\varpi$ a choice of uniformizer of $\mathcal{O}$. Let $\mathrm{Art}(\mathcal{O})$ denote the category of local Artinian $\mathcal{O}$-algebras whose residue fields are a finite extension of $k$ (or equivalently, which are of finite length as $\mathcal{O}$-modules), and let $\mathrm{Comp}(\mathcal{O})$ denote the category of complete Noetherian local $\mathcal{O}$-algebras whose residue fields are a finite extension of $k$.

We let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and for each prime $\ell$ we choose an algebraic closure $\overline{\mathbb{Q}}_\ell$. As usual, we write $G_{\mathbb{Q}}$ to denote the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $G_{\mathbb{Q}_\ell}$ to denote the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. For each prime $\ell$ we also fix an isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ (whose choice, however, will play no overt role). This determines an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, and hence an embedding $G_{\mathbb{Q}_\ell} \hookrightarrow G_{\mathbb{Q}}$, via which we identify $G_{\mathbb{Q}_\ell}$ with a decomposition group at $\ell$ in $G_{\mathbb{Q}}$. We also fix an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ (the choice of which, again, will play no overt role). Given this choice, together with our choice of isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$, it makes sense to speak of modular forms being defined over $E$, or of a $\mathbb{C}$-linear representation of a group being defined over $E$.

We let $\widehat{\mathbb{Z}}$ denote the profinite completion of $\mathbb{Z}$, so $\widehat{\mathbb{Z}} \xrightarrow{\sim} \prod_{\ell \text{ prime}} \mathbb{Z}_\ell$. We let $\mathbb{A}_f$ denote the ring of finite adèles over $\mathbb{Q}$; so $\mathbb{A}_f = \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. More generally, for any finite set of primes $\Sigma$ we write $\widehat{\mathbb{Z}}^\Sigma := \prod_{\ell \notin \Sigma} \mathbb{Z}_\ell$ and $\mathbb{A}_f^\Sigma := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^\Sigma$.

We will frequently employ the following notation related to finite sets of primes: we will let $\Sigma_0$ denote a finite set of primes that does not contain $p$, and then write $\Sigma := \Sigma_0 \cup \{p\}$.

For each prime $\ell$, local class field theory gives an embedding $\mathbb{Q}_\ell^\times \hookrightarrow G_{\mathbb{Q}_\ell}^{\mathrm{ab}}$ (the local Artin map), which we normalize by mapping $\ell$ to a lift of geometric Frobenius. Global class field theory gives an isomorphism

$$\widehat{\mathbb{Z}}^\times \xrightarrow{\sim} \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times \xrightarrow{\sim} G_{\mathbb{Q}}^{\mathrm{ab}}$$

(the first isomorphism being induced by the embedding $\widehat{\mathbb{Z}}^\times \hookrightarrow \mathbb{A}_f^\times$), which we normalize to be compatible with our chosen normalization of the local Artin maps.

We let $\varepsilon : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ denote the $p$-adic cyclotomic character, and $\overline{\varepsilon} : G_{\mathbb{Q}_p} \to \mathbb{F}_p^\times$ the mod $p$ cyclotomic character (i.e. the mod $p$ reduction of $\varepsilon$). We also regard $\varepsilon$ and $\overline{\varepsilon}$ as characters of $\mathbb{Q}_p^\times$, via local class field theory.

For any prime $\ell$ we let $|\ \ |_\ell$ denote the $\ell$-adic absolute value on $\mathbb{Q}_\ell$, normalized so that $|\ell|_\ell = \ell^{-1}$. Note that the if $\ell \neq p$, then the $p$-adic cyclotomic character of $G_{\mathbb{Q}_\ell}$ corresponds to the character $|\ \ |_\ell$ of $\mathbb{Q}_\ell^\times$ via local class field theory.

If $V$ is any $G_{\mathbb{Q}_p}$-representation, then we write $V^{\mathrm{ab}}$ to denote the maximal subrepresentation of $V$ on which $G_{\mathbb{Q}_p}$ acts through its maximal abelian quotient. In other words, $V^{\mathrm{ab}}$ denotes the set of fixed points of the commutator subgroup of $G_{\mathbb{Q}_p}$.

We say that a continuous finite-dimensional representation of $G_{\mathbb{Q}_p}$ is crystabelline if it becomes crystalline after restricting to the Galois group of an abelian extension of $\mathbb{Q}_p$.

We write $G := \mathrm{GL}_2(\mathbb{Q}_p)$. If $\Sigma_0$ is a finite set of primes not containing $p$, then we write $G_{\Sigma_0} := \prod_{\ell\in\Sigma_0} \mathrm{GL}_2(\mathbb{Q}_\ell)$. We let $T$ denote the diagonal torus in $G$, namely

$$T := \{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{Q}_p^\times \} \subset G,$$

and let $S$ denote the following subtorus of $T$:

$$S := \{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}_p^\times \} \subset T.$$

We will identify $S$ with $\mathbb{Q}_p^\times$ in the evident manner, i.e. via the isomorphism $\mathbb{Q}_p^\times \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in S$. We let $B$ denote the upper triangular Borel subgroup of $G$, with unipotent radical $N$ (so $B = TN$), and we write $P = SN \subset B$. We let $\overline{B}$ denote the lower triangular Borel subgroup of $G$, with unipotent radical $\overline{\mathrm{N}}$ (so $\overline{B} = T\overline{\mathrm{N}}$, and $T = B\bigcap\overline{B}$). We typically denote a character of $T$ via $\chi_1 \otimes \chi_2$, where the $\chi_i$ are characters of $\mathbb{Q}_p^\times$ (and $\chi_1 \otimes \chi_2 : \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$).

If $H$ is any locally compact group, and $A$ is a ring, then we say that a representation of $H$ on an $A$-module $M$ is smooth if every element $m \in M$ is fixed by some open subgroup of $H$. We say that the representation is smooth admissible if for every open subgroup $H'$ of $H$, the $A$-submodule $M^{H'}$ of $M$, consisting of $H'$-invariant elements, is finitely generated. In the case when $H = G$ ($:= \mathrm{GL}_2(\mathbb{Q}_p)$), we will apply some additional terminology in the case of non-smooth representations, based in part on that of [39]. For precise definitions and terminology, the reader is referred to Subsection 3.1.

For any profinite group $H$, and any object $A$ of $\mathrm{Comp}(\mathcal{O})$, having maximal ideal $\mathfrak{m}$, we let $A[[H]]$ denote the completed group ring

$$A[[H]] := \varprojlim_{H'} A[H/H'],$$

where $H'$ runs over the open subgroups of $H$, each of the rings $A[H/H']$ is equipped with its $\mathfrak{m}$-adic topology, and $A[[H]]$ is equipped with the projective limit topology. The ring $A[[H]]$ is then a profinite topological ring.

If $V$ is any $E$-vector space, by a lattice (or $\mathcal{O}$-lattice) in $v$, we will always mean a $\varpi$-adically separated $\mathcal{O}$-submodule of $V$ that spans $V$ over $E$. Note that if $V$ is an $E$-Banach space, then an $\mathcal{O}$-submodule of $V$ is $\varpi$-adically separated if and only if it is bounded, and is furthermore $\varpi$-adically complete if and only if it is closed in $V$ (or equivalently, open, since any Banach space is barrelled).

If $V$ is a representation of $G$, then we let $V_{\mathrm{l.alg}}$ denote the $G$-subrepresentation of $V$ consisting of the locally algebraic vectors (as defined in [34, Prop.-Def. 4.2.6]). If we let $\hat{G}$ denote the set of isomorphism classes of irreducible finite-dimensional algebraic representations of $G$ over $E$, then $V_{\mathrm{l.alg}}$ is the image of the natural evaluation map $\bigoplus_{W\in\hat{G}} \varinjlim_{H} \mathrm{Hom}_H(W, V) \otimes_E W \to V$ (the inductive limit being taken over the directed set of all open subgroups $H$ of $G$); see [34, Prop. 4.2.4, Cor. 4.2.7]. If $H$ is a fixed open subgroup of $G$, then we write $V_{H-\mathrm{alg}}$ to denote the $E$-subspace of $V$ consisting of $H$-algebraic vectors, which we define to be the image of the natural evaluation map $\bigoplus_{W\in\hat{G}} \mathrm{Hom}_H(W, V) \otimes_E W \to V$.

We normalize the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_\ell)$ (for any prime $\ell$) in the following way: if $\pi = \mathrm{Ind}_{\overline{B}}^{G} \chi_1 \otimes \chi_2$ is an irreducible parabolically induced representation (with the $\chi_i$ being smooth characters of $\mathbb{Q}_p^\times$ with values in the multiplicative group of some characteristic zero field), then we define $\mathrm{WD}(\pi)$, the associated Weil–Deligne representation (which in the case is simply a representation of the Weil group) to be $\chi_1 \oplus \chi_2 | \ |_\ell^{-1}$, where we now regard the $\chi_i$ as characters of $\mathrm{W}_p$, via the reciprocity isomorphism $\mathbb{Q}_p^\times \xrightarrow{\sim} \mathrm{W}_p^{\mathrm{ab}}$ of local class field theory. (See Subsection 4.2 for a further explication of our normalization.)

In Sections 5 and 6 below, we consider the action of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$ on the inductive limit of the cohomology of modular curves, and on its $p$-adic completion. We follow the same conventions for this action as in the reference [38]. Unfortunately, these conventions do not agree with those of [16]: if $g \in \mathrm{GL}_2(\mathbb{A}_f)$, then the automorphism of the cohomology of modular curves induced by $g$ that we consider in this paper is equal to the automorphism induced by the inverse transposed element $(g^{-1})^t$ under the action considered in [16].

## 3. Mod $p$ and $p$-adic local Langlands

Our goal in this section is to describe certain relationships between, on the one hand, the mod $p$ and $p$-adic representation theory of $G$, and, on the other, two-dimensional mod $p$ and $p$-adic representations of the local Galois group $G_{\mathbb{Q}_p}$, relationships which are known respectively as the mod $p$ and $p$-adic local Langlands correspondence. In the remainder of these introductory remarks we briefly recall the history of these two correspondences.

The semi-simple mod $p$ local Langlands correspondence (so-called because it does not take into account possible extension classes) was constructed by Breuil [8, Déf 1.1]. The possible existence of a $p$-adic local Langlands correspondence was originally suggested by Breuil in [8, 9]. Substantial progress on its development was made in the papers [22, 3, 23, 2], after Colmez observed a relationship between some of the constructions introduced by Breuil in [9] (in the context of the representation theory of $G$) and certain constructions in the theory of $(\varphi, \Gamma)$-modules (which provide a way of describing $G_{\mathbb{Q}_p}$-representations).

In his talk at the Montréal conference in September 2005, Colmez further developed this relationship by defining an actual functor from $G$-representations to $G_{\mathbb{Q}_p}$-representations. Upon seeing Colmez's talk, Kisin suggested that this functor might be combined with a certain deformation-theoretic argument to construct the $p$-adic local Langlands correspondence in general. Kisin's suggestion was predicated upon the construction of a mod $p$ correspondence that enhanced the correspondence considered in [8, 2] by taking extensions into account. Such an enhanced mod $p$ correspondence has now been constructed by Colmez (in most cases — see Theorem 3.3.2 below), and building on this and Kisin's deformation-theoretic strategy, the $p$-adic local Langlands correspondence has now been constructed – see [25, 61], as well as Theorem 3.3.13, Definition 3.3.15, and Remark 3.3.19 below.

In fact, in the paper [25] Colmez goes significantly further than he went in his Montréal lecture, and as well as defining a functor from $G$-representations to $G_{\mathbb{Q}_p}$-representations, he gives a construction going in the opposite direction, from two-dimensional $G_{\mathbb{Q}_p}$-representations to $G$-representations. In this way he is able to construct the $p$-adic local Langlands correspondence without directly using deformation theory (although the construction of the functor still employs a version of Kisin's deformation-theoretic strategy). However, for the purposes of this paper, we require Kisin's full-blown deformation-theoretic formulation of the correspondence.

3.1. **Representations of $G$.** Let $A$ be an object of $\mathrm{Comp}(\mathcal{O})$, with maximal ideal $\mathfrak{m}$. In the paper [39], we introduced various categories of representations of the group $G$ over $A$.[4] For the purposes of this paper, the most important of these is the category of $\varpi$-adically admissible $G$-representations over $A$, whose definition we will recall here.

3.1.1. **Definition.** An $A[G]$-module $\pi$ is called a $\varpi$-adically admissible representation of $G$ over $A$ if the following conditions are satisfied:

(1) $\pi$ is $\varpi$-adically complete and separated.
(2) The $\mathcal{O}$-torsion subspace $\pi[\varpi^\infty]$ is of bounded exponent, i.e. $\pi[\varpi^i] = \pi[\varpi^\infty]$ for sufficiently large values of $i$.
(3) For any $i \geq 0$, the induced $A[G]$-action on $\pi/\varpi^i\pi$ satisfies the following condition: for any element $\overline{v} \in \pi/\varpi^i\pi$, there is some compact open subgroup $H$ of $G$ and some $j \geq 0$ such that $\overline{v}$ is fixed by $H$ and annihilated by $\mathfrak{m}^j$.
(4) The $A/\mathfrak{m}$-vector space $(\pi/\varpi\pi)[\mathfrak{m}]$ forms an admissible smooth representation of $G$ under the induced $G$-action.

We denote by $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ the category of $\varpi$-adically admissible representations of $G$ over $A$. If $A$ is in fact an object of $\mathrm{Art}(\mathcal{O})$, then an $A[G]$-module is $\varpi$-adically admissible if and only if it is a smooth admissible representation of $G$ in the usual sense, and hence we will also denote this category simply by $\mathrm{Mod}_G^{\mathrm{adm}}(A)$. Examples of objects in $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ for non-Artinian $A$ are provided by Lemma 3.1.16 below.

3.1.2. **Caution.** The reader who consults [39] should be aware that, in the case when $A$ is not Artinian, the definitions used in [39] of a smooth $G$-representation over $A$, and of an admissible smooth $G$-representation over $A$, are not the standard ones. Throughout this paper, we will always use the standard terminology, as explained in Section 2.

If $\pi$ is a $\varpi$-adically admissible representation of $G$ over $A$, then the tensor product $E \otimes_{\mathcal{O}} \pi$ has a natural structure of $E$-Banach space (given by taking the norm whose unit ball is the image of $\pi$ in $E \otimes_{\mathcal{O}} \pi$). The actions of each of $A$ and $G$ on $E \otimes_{\mathcal{O}} \pi$ will then be jointly continuous. (We equip $A$ with its $\mathfrak{m}$-adic topology.)

3.1.3. **Proposition.** *If $\pi_1$ and $\pi_2$ are two objects of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ and $\pi_1 \to \pi_2$ is an $A[G]$-linear morphism, then the induced map on $E$-Banach spaces*

$$E \otimes_{\mathcal{O}} \pi_1 \to E \otimes_{\mathcal{O}} \pi_2 \tag{3.1.4}$$

*necessarily has closed image.*

*Proof.* Replacing $\pi_1$ and $\pi_2$ by their maximal $\mathcal{O}$-torsion free quotients (which again lie in the category $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, by [39, Lem. 2.4.8]), we may assume that each $\pi_i$ is $\mathcal{O}$-torsion free, and via the natural map $\pi_i \to E \otimes_{\mathcal{O}} \pi_i$, may identify $\pi_i$ with the unit ball in the Banach space $E \otimes_{\mathcal{O}} \pi_i$. Furthermore, replacing $\pi_1$ by its image in $\pi_2$ (which lies in the category $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, by [39, Prop. 2.4.12]), we may assume that the map $\pi_1 \to \pi_2$, and hence also the map (3.1.4), is injective. From [39, Prop. 2.4.13] we see that the map $\pi_1 \to \pi_2$ has closed image, and that the

[4] Although we will only apply the results of this subsection in the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, in fact they make sense and are valid for any locally $p$-adic analytic group $G$.

$\varpi$-adic topology on $\pi_2$ induces the $\varpi$-adic topology on $\pi_1$. This implies that the map (3.1.4) has closed image, as required. □

3.1.5. **Remark.** In the case when $A = \mathcal{O}$, the preceding proposition is a consequence of the results of [77].

3.1.6. **Lemma.** *If $\pi_1$ and $\pi_2$ are two object of* $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ *that are torsion free as $\mathcal{O}$-modules, and if*

$$\pi_1 \to \pi_2 \tag{3.1.7}$$

*is an $A[G]$-linear morphism for which the induced map on $E$-Banach spaces* (3.1.4) *is injective, then the given map* (3.1.7) *is an isomorphism if and only if the map*

$$(\pi_1/\varpi\pi_1)[\mathfrak{m}] \to (\pi_2/\varpi\pi_2)[\mathfrak{m}], \tag{3.1.8}$$

*obtained by reducing* (3.1.7) *modulo $\varpi$ and then passing to the $\mathfrak{m}$-torsion parts of its source and target, is injective.*

*Proof.* Clearly the given condition on (3.1.8) is necessary. We will show that it is also sufficient.

The assumptions of the lemma imply that (3.1.7) is injective, with $\mathcal{O}$-torsion cokernel. If we denote this cokernel by $\pi_3$, then since $\pi_1$ and $\pi_2$ are both $\varpi$-adically admissible smooth representations of $G$ over $A$, it follows from [39, Prop. 2.4.13] that the same is true of $\pi_3$.

The map (3.1.7) sits in the short exact sequence

$$0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0,$$

which induces the exact sequence

$$0 \to \pi_3[\varpi] \to \pi_1/\varpi\pi_1 \to \pi_2/\varpi\pi_2. \tag{3.1.9}$$

Since $\pi_3$ is a $\varpi$-adically admissible representation of $G$ over $A$ that is torsion as an $\mathcal{O}$-module, we see that every element of $\pi_3$ is killed by some power of $\mathfrak{m}$. On the other hand, if (3.1.8) is injective, then we see from a consideration of (3.1.9) that $\pi_3[\mathfrak{m}] = \pi_3[\varpi, \mathfrak{m}] = 0$. It then follows that in fact $\pi_3 = 0$, and thus that (3.1.7) is an isomorphism, as claimed. □

We next recall the definition of a finitely generated augmented representation of $G$ over $A$ [39, §2.1].

3.1.10. **Definition.** By a finitely generated augmented representation of $G$ over $A$ we mean an $A[G]$-module $M$ equipped with a finitely generated $A[[H]]$-module structure for some (equivalently, any) compact open subgroup $H$ of $G$, such that the two induced $A[H]$-actions (the first induced by the inclusion $A[H] \subset A[[H]]$ and the second by the inclusion $A[H] \subset A[G]$) coincide.

We let $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ denote the category of finitely generated augmented representations of $G$ over $A$, with morphisms being maps which are simultaneously $A[G]$-linear and $A[[H]]$-linear. Each object of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ is equipped with its canonical topology [39, Prop. 2.1.2], making it a profinite topological $A[[H]]$-module for each compact open subgroup $H$ of $G$.

We now introduce introduce a class of representations that plays an important role in the deformation theory of $G$-representations.

3.1.11. **Definition.** We say that an $A[G]$-module $\pi$ is an orthonormalizable admissible representation of $G$ over $A$ if the following conditions are satisfied:

(1) $\pi$ is an orthonormalizable $A$-module, in the sense of Definition B.1.
(2) The induced $G$-action on $\pi/\mathfrak{m}^i\pi$ makes this quotient an admissible smooth representation of $G$ over $A/\mathfrak{m}^i$, for each $i \geq 0$.

3.1.12. **Proposition.** *The functor* $\pi \mapsto \mathrm{Hom}_A(\pi, A)$ *induces an equivalence of categories between the category of orthonormalizable admissible representations of* $G$ *over* $A$ (*with morphisms taken to be* $A[G]$*-linear maps*), *and the full subcategory of* $\mathrm{Mod}_G^{\mathrm{fg\ aug}}(A)$ *consisting of finitely generated augmented representations of* $G$ *on pro-free* $A$*-modules.* (*See Definition* B.10 *for the notion of a pro-free* $A$*-module.*)

*Proof.* Proposition B.11 shows that $\pi \mapsto \mathrm{Hom}_A(\pi, A)$ induces an equivalence of categories between the category of orthonormalizable $A$-modules and the category of pro-free profinite $A$-modules, and also that there is an isomorphism

$$\mathrm{Hom}_A(\pi, A)/\mathfrak{m}^i \mathrm{Hom}_A(\pi, A) \xrightarrow{\sim} \mathrm{Hom}_{A/\mathfrak{m}^i}(\pi/\mathfrak{m}^i\pi, A/\mathfrak{m}^i),$$

for each $i \geq 0$. From this isomorphism, one sees that if $\pi$ is equipped with an $A$-linear $G$-action, then this action induces an admissible smooth action on $\pi/\mathfrak{m}^i\pi$ if and only if the contragredient action on $\mathrm{Hom}_A(\pi, A)$ makes $\mathrm{Hom}_A(\pi, A)/\mathfrak{m}^i$ a finitely generated augmented $G$-representation over $A/\mathfrak{m}^i$. Since this latter condition holds for all $i > 0$ if and only if $\mathrm{Hom}_A(\pi, A)$ is a finitely generated augmented $G$-representation over $A$, the proposition follows. □

We note for the record that there is a broader class of representations containing the orthonormalizable ones, which we now define, although we will have no need to consider representations in this class that are not orthonormalizable.

3.1.13. **Definition.** We say that an $A[G]$-module $\pi$ is an $\mathfrak{m}$-adically admissible representation of $G$ over $A$ if the following conditions are satisfied:

(1) $\pi$ is $\mathfrak{m}$-adically complete.
(2) The action map $G \times \pi \to \pi$ is continuous, when $\pi$ is given its $\mathfrak{m}$-adic topology. (Equivalently, the induced action of $G$ on $\pi/\mathfrak{m}^i\pi$ is smooth for each $i \geq 0$.)
(3) The induced representation of $G$ on $\pi/\mathfrak{m}^i$ (which is smooth, by (2)) is admissible. (It is in fact enough to assume this for $\pi/\mathfrak{m}\pi$, since a smooth extension of admissible representations is again admissible [39, Prop. 2.2.13].)

Related to this definition, we can define the class of admissible unitarizable Banach representations of $G$ on modules over certain localizations of $A$.

3.1.14. **Definition.** If $\mathfrak{p} \in \mathrm{Spec}\, A$ is a prime ideal of $A$, then an admissible unitary Banach representation of $G$ over $A_{\mathfrak{p}}$ consists of a Banach module $\pi$ over $A_{\mathfrak{p}}$ (in the sense of Definition A.3), equipped with an action of $G$, such that $A_{\mathfrak{p}}$ admits a choice of unit ball $\pi^0$ which is $G$-invariant, and which is $\mathfrak{m}$-adically admissible as a $G$-representation over $A$.

3.1.15. **Remark.** Other than in Subsection 3.6, we will have no cause to consider the preceding definition other than in the case when $A = \mathcal{O}$ and $\mathfrak{p}$ is the zero ideal, so that $A_{\mathfrak{p}} = E$. In this case, the above definition reduces to the usual notion of an admissible unitary Banach representation of $G$ over $E$.

We close this subsection with some technical lemmas.

3.1.16. **Lemma.** *If $\pi$ is an orthonormalizable admissible $A[G]$-module, and if $X$ is a cofinitely generated $A$-module (in the sense of Definition* C.1*), then $\pi \hat{\otimes}_A X$ is a $\varpi$-adically admissible $G$-representation over $A$.*

*Proof.* Let $M := \mathrm{Hom}_A(\pi, A)$ be the object of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ corresponding to $\pi$ under the anti-equivalence of Proposition 3.1.12. Proposition C.5 shows that $\mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ is finitely generated over $A$, and hence $M \otimes_A \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ is again an $\mathcal{O}$-torsion free object of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$. It follows from [39, Prop. 2.4.11] and its proof that the continuous $\mathcal{O}$-dual of $M \otimes_A X$ is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$. One easily verifies that this dual is isomorphic to $\pi \hat{\otimes}_A X$. □

3.1.17. **Lemma.** *If $M$ is a finitely generated $A$-module, if $\pi$ is an orthonormalizable admissible $A[G]$-module, and if $X$ is a cofinitely generated $A$-module (in the sense of Definition* C.1*), then the natural map*

$$\mathrm{Hom}_A(M, X)\hat{\otimes}_A \pi \to \mathrm{Hom}_A(M, X\hat{\otimes}_A \pi)$$

*is an isomorphism.*

*Proof.* Part 3 of Lemma B.6 shows that the natural map

$$\mathrm{Hom}_A(M, X) \otimes_A \pi \to \mathrm{Hom}_A(M, X \otimes_A \pi)$$

is an isomorphism. Passing to $\varpi$-adic completions, we obtain an isomorphism

$$\mathrm{Hom}_A(M, X)\hat{\otimes}_A \pi \to \mathrm{Hom}_A(M, X \otimes_A \pi)^{\wedge}.$$

Thus it remains to show that the natural map

$$\mathrm{Hom}_A(M, X \otimes_A \pi)^{\wedge} \to \mathrm{Hom}_A(M, X\hat{\otimes}_A \pi)$$

is an isomorphism.

If we choose a presentation $A^s \to A^r \to M \to 0$ of $A$, then we may form the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_A(M, X \otimes_A \pi) & \longrightarrow & (X \otimes_A \pi)^r & \longrightarrow & (X \otimes_A \pi)^s \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}_A(M, X\hat{\otimes}_A \pi) & \longrightarrow & (X\hat{\otimes}_A \pi)^r & \longrightarrow & (X\hat{\otimes}_A \pi)^s \end{array} \tag{3.1.18}$$

whose rows are exact. Thus it suffices to show that the bottom row is obtained from the top row by passing to $\varpi$-adic completions. Lemma 3.1.16 shows that $X\hat{\otimes}_A \pi$ is a $\varpi$-adically admissible $G$-representation over $A$, and so [39, Prop. 2.4.12] shows that the $\mathcal{O}$-torsion part of the cokernel of the upper right arrow in (3.1.18) has exponent bounded independently of $i$. Since

$$(X \otimes_A \pi)/\varpi^i (X \otimes_A \pi) \xrightarrow{\sim} (X\hat{\otimes}_A \pi)/\varpi^i (X\hat{\otimes}_A \pi)$$

for any $i$, we conclude that the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_A(M, X \otimes_A \pi)/\varpi^i \mathrm{Hom}_A(M, X \otimes \pi) &\to \big((X \otimes_A \pi)/\varpi^i (X \otimes_A \pi)\big)^r \\ &\to \big((X \otimes_A \pi)/\varpi^i (X \otimes_A \pi)\big)^s \end{aligned}$$

induced by the top row of (3.1.18) is exact modulo $A$-modules that are torsion over $\mathcal{O}$, of exponent bounded independently of $i$. Passing to the projective limit in $i$, we thus obtain an exact sequence

$$0 \to \mathrm{Hom}_A(M, X \otimes_A \pi)^{\wedge} \to (X\hat{\otimes}_A \pi)^r \to (X\hat{\otimes}_A \pi)^s,$$

which must then coincide with the bottom row of (3.1.18). This completes the proof of the lemma. □

3.1.19. **Lemma.** *Suppose that $A$ is finite and flat over $\mathcal{O}$, and reduced, and that $\pi$ is an orthonormalizable admissible $A[G]$-module with the property that for every minimal prime $\mathfrak{p}$ of $A$, the tensor product $\kappa(\mathfrak{p})\otimes_A \pi$ is an absolutely topologically irreducible Banach space representation of $G$ over $\kappa(\mathfrak{p})$. Then if $M$ is any finitely generated $A$-module that is flat over $\mathcal{O}$, the natural map $M \to \mathrm{Hom}_{A[G]}(\pi, M\otimes_A \pi)$ is an isomorphism.*

*Proof.* Lemma B.7 shows that the natural map

$$M\otimes_A \mathrm{End}(\pi) \to \mathrm{Hom}_A(\pi, M\otimes_A \pi)$$

is an isomorphism, and hence so is the natural map

$$M\otimes_A \mathrm{End}_{A[G]}(\pi) \to \mathrm{Hom}_{A[G]}(\pi, M\otimes_A \pi).$$

Thus it suffices to prove that $A \xrightarrow{\sim} \mathrm{End}_{A[G]}(\pi)$.

Our assumption on $\pi$ implies that $E\otimes_{\mathcal{O}} A \xrightarrow{\sim} \mathrm{End}_{A[G]-\mathrm{cont}}(E\otimes_A \pi)$ (where the target denotes the space of continuous $A[G]$-module endomorphisms of the $E$-Banach space $E\otimes_A\pi$). Since by Lemma B.8 we have $A = E\otimes_{\mathcal{O}} A \bigcap \mathrm{End}_A(\pi)$ (the intersection taking place in $\mathrm{End}_{E\otimes_{\mathcal{O}}A}(E\otimes_A \pi)$), the present lemma follows. □

3.1.20. **Lemma.** *If $\pi$ is a smooth $G$-representation over $A$, then a finitely generated $A$-submodule of $\pi$ is $B$-invariant if and only if it is $G$-invariant.*

*Proof.* Any finitely generated $A$-submodule of $\pi$ is fixed by some compact open subgroup $H$ of $G$. Since $H$ and $B$ together generate $G$, the lemma follows. □

3.2. **Colmez's magical functor.** Let $A$ be an object of $\mathrm{Art}(\mathcal{O})$. As was already mentioned, Colmez [25] has defined an exact functor

$$\begin{aligned}\mathrm{MF} : \text{full subcategory of } \mathrm{Mod}_G^{\mathrm{adm}}(A) &\text{ consisting of objects of finite length}^5\\ \longrightarrow \quad &\text{category of continuous } G_{\mathbb{Q}_p}\text{-representations}\\ &\qquad\text{on finitely generated } A\text{-modules.}\end{aligned}$$

We sketch the definition of MF; for more details, the reader should consult [25, §IV]. (See also [41].) Let $A$ be an object of $\mathrm{Art}(\mathcal{O})$, and let $\pi$ be an object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ which is of finite length. The action of $\begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}$ on $\pi$ makes $\pi$ an $A[[\mathbb{Z}_p]]$-module, and hence (via the isomorphism $A[[\mathbb{Z}_p]] \xrightarrow{\sim} A[[X]]$ defined by $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \mapsto X+1$) an $A[[X]]$-module. The actions of $\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}\mathbb{Z}_p^\times & 0\\ 0 & 1\end{pmatrix}$ furthermore give $\pi$ the structure of a $(\varphi,\Gamma)$-module over $A[[X]]$.

Let $W$ be a finitely generated $A$-submodule of $\pi$, which generates $\pi$ over $A[G]$, and is invariant under $\mathrm{GL}_2(\mathbb{Z}_p)Z$. Then $W$ is a $\Gamma$-invariant $A[[X]]$-submodule of $\pi$, and we define $M(V,W)$ to be the minimal $\varphi$-invariant $A[[X]]$-submodule of

[5] Equivalently [39, Thm. 2.3.8], objects that are finitely generated as $A[G]$-modules.

$\pi$ containing $W$. (Equivalently, $M(V,W)$ is the minimal $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$- and $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$-invariant subspace of $\pi$ containing $W$.) We see that $M(V,W)$ is in fact a $(\varphi,\Gamma)$-invariant $A[[X]]$-submodule of $\pi$.

The $\varphi$-action on $M(V,W)$ induces a $\Gamma$-equivariant, $A[[X]]$-linear map

$$\varphi^* M(V,W) \to M(V,W), \tag{3.2.1}$$

and one can show that the kernel and cokernel of this map are both finitely generated $A$-modules. If $M(V,W)^*$ denotes the Pontrjagin dual of $M(V,W)$, then we set $\mathbf{D}(\pi) := A((X)) \otimes_{A[[X]]} M(V,W)^*$. We equip $M(V,W)^*$, and hence $\mathbf{D}(\pi)$, with the contragredient $\Gamma$-action. The map (3.2.1) gives rise to a $\Gamma$-equivariant dual map

$$M(V,W)^* \to \varphi^* M(V,W)^*$$

whose source and target are again finitely generated over $A$. This latter map thus induces a $\Gamma$-equivariant isomorphism

$$\mathbf{D}(\pi) \xrightarrow{\sim} \varphi^* \mathbf{D}(\pi).$$

The inverse of this isomorphism equips $\mathbf{D}(\pi)$ with an étale $\varphi$-structure, so that $\mathbf{D}(\pi)$ becomes an étale $(\varphi,\Gamma)$-module over $A((X))$. One shows that $\mathbf{D}(\pi)$ is independent (up to natural isomorphism) of the choice of $W$, and that it is finitely generated over $A((X))$. It thus corresponds to a representation of $G_{\mathbb{Q}_p}$ on a finitely generated $A$-module. We define $\mathrm{MF}(\pi)$ to be the Pontrjagin dual of this $G_{\mathbb{Q}_p}$-representation.[6] Note that with these definitions, the functor $\pi \mapsto \mathbf{D}(\pi)$ is contravariant, while the functor $\pi \mapsto \mathrm{MF}(\pi)$ is covariant.

It follows directly from the definition of MF that there is an isomorphism of $(\varphi,\Gamma)$-modules

$$\mathbf{D}(\pi) \xrightarrow{\sim} \mathbf{D}(\mathrm{MF}(\pi)^*). \tag{3.2.2}$$

From the construction of $\mathbf{D}(\pi)$, one also easily deduces the existence of a canonical $P$-equivariant map

$$\big(\psi^{-\infty}(\mathbf{D}(\pi))\big)^* \to \pi \tag{3.2.3}$$

(see [25, Prop. IV.3.2][7]).

The functor MF is compatible with change of scalars. We give a table of values of MF on the standard absolutely irreducible representations of $G$ over $k$ (as computed in [25]):[8]

$$\begin{aligned}
\mathrm{MF}(\chi \circ \det) &= 0 \\
\mathrm{MF}((\chi \circ \det) \otimes \mathrm{St}) &= \chi \\
\mathrm{MF}(\mathrm{Ind}_B^G \chi_1 \otimes \chi_2) &= \chi_1 \\
\mathrm{MF}\big(c\text{-}\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)Z}^G (\mathrm{Sym}^r k)^\vee / T(c\text{-}\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)Z}^G (\mathrm{Sym}^r k)^\vee)\big) &= \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \psi_2^{-(r+1)}
\end{aligned}$$

[6] The notation MF is chosen to stand for "magical functor", "mysterious functor", or "Montréal functor". Colmez denotes his functor by $\pi \mapsto \mathbf{V}(\pi)$. Note, though, that the definition of $\mathbf{V}$ involves a twist by the cyclotomic character (see [25, §IV.2.3]), which does not appear in the definition of MF.

[7] We use the notation $\psi^{-\infty}(\mathbf{D})$ of Colmez's earlier papers, e.g. [22]. This corresponds to the notation $\mathbf{D}^\sharp \boxtimes \mathbb{Q}_p$ of [25].

[8] When comparing this table of values to the values of the functor $\mathbf{V}$, as computed in [25], the reader should remember that MF and $\mathbf{V}$ differ by a cyclotomic twist.

(where, in the last line, $\psi_2$ denotes a fundamental character of level 2, and $0 \leq r \leq p-1$). Furthermore, MF is compatible with twisting: if $\chi : \mathbb{Q}_p^\times \to A^\times$ is a continuous character, and if $\pi$ is a finite length object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$, then there is a natural isomorphism

$$\mathrm{MF}(\chi \circ \det \otimes_A \pi) \xrightarrow{\sim} \chi \otimes_A \mathrm{MF}(\pi)$$

(where, in the expression on the right hand side, $\chi$ is regarded as a character of $G_{\mathbb{Q}_p}$, via local class field theory).

Let $A$ be an object of $\mathrm{Comp}(\mathcal{O})$. In the paper [39] we defined the functor $\mathrm{Ord}_B : \mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A) \to \mathrm{Mod}_T^{\varpi-\mathrm{adm}}(A)$ from $\varpi$-adically admissible smooth $G$-representations over $A$ to $\varpi$-adically admissible smooth $T$-representations over $A$, right adjoint to the functor $\mathrm{Ind}_{\overline{B}}^G : \mathrm{Mod}_T^{\varpi-\mathrm{adm}}(A) \to \mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ given by parabolic induction. (See [39, Thm. 4.3.2].)

Our next result describes the interaction between the functor $\mathrm{Ord}_B$ and Colmez's functor MF. Let $A$ be an object of $\mathrm{Art}(\mathcal{O})$. Note that, via our identification of $S$ with $\mathbb{Q}_p^\times$, any finite-dimensional smooth representation of $S$ over $A$ may be regarded as such a representation of $\mathbb{Q}_p^\times$ over $A$, and hence, via local class field theory, as a continuous representation of $G_{\mathbb{Q}_p}^{\mathrm{ab}}$ over $A$.

3.2.4. **Proposition.** *If $\pi$ is a finite length representation of $G$ over some object $A$ of $\mathrm{Art}(\mathcal{O})$, then there is a natural $A$-linear $G_{\mathbb{Q}_p}^{\mathrm{ab}}$-equivariant embedding*

$$\mathrm{Ord}_B(\pi)_{|S} \hookrightarrow \mathrm{MF}(\pi)^{\mathrm{ab}}.$$

(*We regard the source of this embedding as a $G_{\mathbb{Q}_p}^{\mathrm{ab}}$-representation in the manner indicated above.*) *If, furthermore, $\pi$ admits a central character, and contains no $G$-invariant finitely generated $A$-submodules, then this embedding is in fact an isomorphism.*

*Proof.* The adjointness property of $\mathrm{Ord}_B$ yields a natural $G$-equivariant map

$$\mathrm{Ind}_{\overline{B}}^G \mathrm{Ord}_B(\pi) \to \pi,$$

whose kernel is finitely generated over $A$. Applying MF to this map thus yields a $G_{\mathbb{Q}_p}$-equivariant injection

$$\mathrm{Ord}_B(\pi)_{|S} \hookrightarrow \mathrm{MF}(\pi)$$

(where the source is regarded as a $G_{\mathbb{Q}_p}^{\mathrm{ab}}$-representation in the manner indicated above), which evidently factors through the inclusion

$$\mathrm{MF}(\pi)^{\mathrm{ab}} \subset \mathrm{MF}(\pi), \tag{3.2.5}$$

yielding a $G_{\mathbb{Q}_p}$-equivariant injection

$$\mathrm{Ord}_B(\pi)_{|S} \hookrightarrow \mathrm{MF}(\pi)^{\mathrm{ab}}. \tag{3.2.6}$$

Conversely, the inclusion (3.2.5) gives rise to a surjection of Pontrjagin duals

$$\mathrm{MF}(\pi)^* \to (\mathrm{MF}(\pi)^{\mathrm{ab}})^*,$$

which in turn gives rise to a $P$-equivariant map

$$\psi^{-\infty}(\mathbf{D}(\pi)) \xrightarrow{\sim} \psi^{-\infty}(\mathbf{D}(\mathrm{MF}(\pi)^*)) \to \psi^{-\infty}(\mathbf{D}((\mathrm{MF}(\pi)^{\mathrm{ab}})^*))$$

(the first isomorphism being provided by (3.2.2)). Passing once more to Pontrjagin duals, we obtain a map

$$\big(\psi^{-\infty}(\mathbf{D}((\mathrm{MF}(\pi)^{\mathrm{ab}})^*))\big)^* \to \big(\psi^{-\infty}(\mathbf{D}(\pi))\big)^*.$$

Composing this with the canonical map (3.2.3) we obtain a $P$-equivariant map

$$\big(\psi^{-\infty}(\mathbf{D}((\mathrm{MF}(\pi)^{\mathrm{ab}})^*))\big)^* \to \pi. \tag{3.2.7}$$

Lemma 3.2.8 below shows that the $P$-representation $\big(\psi^{-\infty}(\mathbf{D}((\mathrm{MF}(\pi)^{\mathrm{ab}})^*))\big)^*$ sits in a short exact sequence of $P$-representations

$$0 \to \varepsilon \circ \det \otimes \mathrm{MF}(\pi)^{\mathrm{ab}} \to \big(\psi^{-\infty}(\mathbf{D}((\mathrm{MF}(\pi)^{\mathrm{ab}})^*))\big)^* \to \mathcal{C}_c(\mathbb{Q}_p, \mathrm{MF}(\pi)^{\mathrm{ab}}) \to 0.$$

Suppose now that $\pi$ admits a central character $\eta$. We extend the $P$-action on each of the members of this short exact sequence to a $B\,(=PZ)$-action, by declaring that $Z$ acts via the character $\eta$. The map (3.2.7) is then $B$-equivariant. If we suppose in addition that $\pi$ contains no non-trivial finitely generated $G$-invariant $A$-submodules, then by Lemma 3.1.20 it similarly contains no non-trivial finitely generated $B$-invariant $A$-submodules, and so the map (3.2.7) factors through $\mathcal{C}_c(\mathbb{Q}_p, \mathrm{MF}(\pi)^{\mathrm{ab}})$. The resulting map $\mathcal{C}_c(\mathbb{Q}_p, \mathrm{MF}(\pi)^{\mathrm{ab}}) \to \pi$ corresponds to a map

$$\mathrm{MF}(\pi)^{\mathrm{ab}} \to \mathrm{Ord}_B(\pi),$$

via the adjunction formula of [39, Cor. 4.2.6], which is immediately checked to provide an inverse to (3.2.6). This completes the proof of the theorem. □

3.2.8. **Lemma.** *If $V$ is a finitely generated $A$-module with a continuous action of $G_{\mathbb{Q}_p}^{\mathrm{ab}}$, then the $P$-representation $\big(\psi^{-\infty}(\mathbf{D}(V))\big)^*$ sits in a short exact sequence of $P$-representations*

$$0 \to (\varepsilon \circ \det) \otimes V^* \to \big(\psi^{-\infty}(\mathbf{D}(V))\big)^* \to \mathcal{C}_c(\mathbb{Q}_p, V^*) \to 0.$$

(*Here we regard the $G_{\mathbb{Q}_p}^{\mathrm{ab}}$-representation $V$ as a $P$-representation via the map $P \to S \xrightarrow{\sim} \mathbb{Q}_p^\times \to G_{\mathbb{Q}_p}^{\mathrm{ab}}$, the first arrow being the canonical surjection and the third being given by local class field theory.*)

*Proof.* Choose $n > 0$ so that $A$ is an $\mathcal{O}/\varpi^n$-algebra. One easily verifies that there is a $P$-equivariant isomorphism $\psi^{-\infty}(\mathbf{D}(V)) \xrightarrow{\sim} V \otimes_{\mathcal{O}/\varpi^n} \psi^{-\infty}(\mathbf{D}(\mathcal{O}/\varpi^n))$, where $\mathcal{O}/\varpi^n$ is equipped with the trivial $G_{\mathbb{Q}_p}$-action, and $V$ is regarded as a $P$-representation in the manner indicated in the statement of the lemma. Thus there is an isomorphism $\big(\psi^{-\infty}(\mathbf{D}(V))\big)^* \xrightarrow{\sim} V^* \otimes_{\mathcal{O}/\varpi^n} \big(\psi^{-\infty}(\mathbf{D}(\mathcal{O}/\varpi^n))\big)^*$. Since there is also an isomorphism $\mathcal{C}_c(\mathbb{Q}_p, V^*) \xrightarrow{\sim} V^* \otimes_{\mathcal{O}/\varpi^n} \mathcal{C}_c(\mathbb{Q}_p, \mathcal{O}/\varpi^n)$, it suffices to verify the lemma in the case when $V = \mathcal{O}/\varpi^n$, i.e. to show that there is a short exact sequence

$$0 \to \varepsilon \circ \det \otimes \mathcal{O}/\varpi^n \to \big(\psi^{-\infty}(\mathbf{D}(\mathcal{O}/\varpi^n))\big)^* \to \mathcal{C}_c(\mathbb{Q}_p, \mathcal{O}/\varpi^n) \to 0. \tag{3.2.9}$$

The inclusion $\mathbf{D}^\sharp(\mathcal{O}/\varpi^n) \subset \mathbf{D}(\mathcal{O}/\varpi^n)$ induces an isomorphism

$$\psi^{-\infty}(\mathbf{D}^\sharp(\mathcal{O}/\varpi^n)) \xrightarrow{\sim} \psi^{-\infty}(\mathbf{D}(\mathcal{O}/\varpi^n))$$

(see [24, p. 6]), while it follows from [22, Ex. 4.32] that there is an exact sequence

$$0 \to \mathcal{C}_c(\mathbb{Q}_p, \mathcal{O}/\varpi^n)^* \to \psi^{-\infty}(\mathbf{D}^\sharp(\mathcal{O}/\varpi^n)) \to \varepsilon^{-1} \circ \det \otimes \mathcal{O}/\varpi^n \to 0.$$

The Pontrjagin dual of this sequence thus yields the required short exact sequence (3.2.9). □

The following proposition is useful for comparing the structure of $\pi$ and $\mathrm{MF}(\pi)$.

3.2.10. **Proposition.** *If $\pi$ is a finite length object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ admitting a central character, and admitting no $G$-invariant subquotients that are finitely generated over $A$, then* MF *induces an isomorphism between the lattice of $G$-invariant $A$-submodules of $\pi$ and the lattice of $G_{\mathbb{Q}_p}$-invariant $A$-submodules of $\mathrm{MF}(\pi)$.*

*Proof.* Since MF is exact, it induces an order preserving map from the lattice of $G$-invariant $A$-submodules of $\pi$ to the lattice of $G_{\mathbb{Q}_p}$-invariant submodules of $\mathrm{MF}(\pi)$. Suppose that $\pi_1, \pi_2 \subset \pi$ are $G$-invariant $A$-submodules for which $\mathrm{MF}(\pi_1)$ and $\mathrm{MF}(\pi_2)$ coincide, when regarded as subobjects of $\mathrm{MF}(\pi)$. If we write $\pi_3 = \pi_1 + \pi_2$, then the exactness of MF implies that $\mathrm{MF}(\pi_3) = \mathrm{MF}(\pi_1) \big(= \mathrm{MF}(\pi_2)\big)$. Again using the exactness of MF, we see that $\mathrm{MF}(\pi_3/\pi_1) = 0$. Since $\pi$ admits no $G$-invariant subquotients that are finitely generated over $A$, we see that $\pi_3/\pi_1 = 0$. Similarly, $\pi_3/\pi_2 = 0$, and thus $\pi_1 = \pi_2$. Hence the map on lattices of subobjects induced by MF is an injection.

We turn to proving that the map on subobjects given by MF is a surjection. Let $M$ be a $G_{\mathbb{Q}_p}$-invariant submodule of $\mathrm{MF}(\pi)$. Choose $\pi' \subset \pi$ maximally so that $\mathrm{MF}(\pi') \subset M$. Replacing $\pi$ by $\pi/\pi'$ and $M$ by $M/\mathrm{MF}(\pi')$, we may assume that $M$ is a subobject of $\mathrm{MF}(\pi)$ with the property that if $\pi' \subset \pi$ is such that $\mathrm{MF}(\pi') \subset M$, then $\pi' = 0$. Our goal is to show that $M = 0$. If in fact $M$ is non-zero, then it is clearly no loss of generality to replace $M$ by an irreducible submodule, and thus we may assume that $M$ is irreducible. In particular, letting $\mathfrak{m}$ denote the maximal ideal of $A$, we may assume that $M \subset \mathrm{MF}(\pi)[\mathfrak{m}]$, the $\mathfrak{m}$-torsion submodule of $\mathrm{MF}(\pi)$. Thus, replacing $\pi$ by $\pi[\mathfrak{m}]$, we may assume that $A = k$.

If $k'$ is a finite extension of $k$, and if $\pi' \subset k' \otimes_k \pi$ is such that $\mathrm{MF}(\pi') \subset k' \otimes_k M$, then if we let $k' \otimes_k \pi''$ denote the smallest subspace of $k' \otimes_k \pi$ defined over $k$ that contains $\pi'$, then $\pi''$ is $G$-invariant, and $\mathrm{MF}(\pi'') \subset M$. Thus $\pi''$, and hence $\pi'$, vanishes. Consequently, it is no loss of generality to extend scalars. Let $0 = \pi_0 \subset \pi_1 \subset \cdots \subset \pi_{n-1} \subset \pi_n = \pi$ be a Jordan–Hölder filtration of $\pi$. Extending scalars if necessary, we may assume that each of the composition factors $\pi_i/\pi_{i-1}$ is absolutely irreducible. By assumption, each of these composition factors is also infinite-dimensional over $k$, and thus a consideration of our table of values of MF shows that for each $i = 1, \ldots, n$, the $G_{\mathbb{Q}_p}$-representation $\mathrm{MF}(\pi_i/\pi_{i-1})$ is absolutely irreducible, of dimension one or two. In particular, $0 = \mathrm{MF}(\pi_0) \subset \mathrm{MF}(\pi_1) \subset \cdots \subset \mathrm{MF}(\pi_{n-1}) \subset \mathrm{MF}(\pi_n) = \mathrm{MF}(\pi)$ is a Jordan–Hölder filtration of $\mathrm{MF}(\pi)$, and so we may assume that $M$ is absolutely irreducible, of dimension one or two. Our goal is to exhibit a $G$-subrepresentation $\pi'$ of $\pi$ such that $\mathrm{MF}(\pi') = M$.

Suppose first that $M$ is of dimension one, and that $G_{\mathbb{Q}_p}$ acts on $M$ through the character $\chi$. If $\eta$ denotes the central character of $\pi$, then by Proposition 3.2.4, together with the adjointness between $\mathrm{Ord}_B$ and parabolic induction, we may find a $G$-equivariant homomorphism $\mathrm{Ind}_{\overline{B}}^G \chi \otimes \chi^{-1}\eta \to \pi$ with the property that, if $\pi'$ denotes the image of this map, then $\mathrm{MF}(\pi') = M$. Suppose, on the other hand, that $M$ has dimension two. The inclusion $M \subset \mathrm{MF}(\pi)$ induces an injection $\big(\psi^{-\infty}(\mathbf{D}(M^*))\big)^* \to \big(\psi^{-\infty}(\mathbf{D}(\mathrm{MF}(\pi)^*))\big)^*$, which, when composed with the canonical map (3.2.3), yields a $P$-equivariant map

$$\big(\psi^{-\infty}(\mathbf{D}(M^*))\big)^* \to \pi. \tag{3.2.11}$$

If we extend the $P$-action on the source of this map to a $B$-action, by requiring $Z$ to act via the central character $\eta$ of $\pi$, then (3.2.11) becomes $B$-equivariant. Since $M$

is absolutely irreducible of dimension two, the $B$-representation $\big(\psi^{-\infty}(\mathbf{D}(M^*))\big)^*$ is in fact the $B$-representation underlying an absolutely irreducible supersingular representation of $G$ over $k$ [2, Thm. 2.2.1]. The map (3.2.11) is then necessarily $G$-equivariant, by [67, Thm. 1.1]. Thus, if we denote its image by $\pi'$, then again $\pi'$ is a $G$-subrepresentation of $\pi$ with the property that $\mathrm{MF}(\pi') = M$. This completes the proof of the proposition. □

In the remainder of this subsection, we discuss how to extend the domain of MF to certain $G$-representations over more general coefficient rings.

To being with, suppose that $A$ is an object of $\mathrm{Comp}(\mathcal{O})$, with maximal ideal $\mathfrak{m}$. If $\pi$ is an $\mathfrak{m}$-adically admissible representation of $G$ over $A$ (in the sense of Definition 3.1.13), with the additional property that $\pi/\mathfrak{m}\pi$ (and hence $\pi/\mathfrak{m}^i\pi$ for all $i \geq 0$) has finite length as a $G$-representation, then we define

$$\mathrm{MF}(\pi) := \varprojlim_i \mathrm{MF}(\pi/\mathfrak{m}^i\pi).$$

Since MF is exact, it takes admissible smooth $G$-representations on free $A/\mathfrak{m}^i$-modules to $G_{\mathbb{Q}_p}$-representations on free $A/\mathfrak{m}^i$-modules. Thus, with the preceding definition, we see that MF provides an exact covariant functor

$$\begin{aligned}\mathrm{MF} : \{&\text{orthonormalizable admissible } G\text{-representations } \pi \text{ over } A \\ &\text{such that } \pi/\mathfrak{m}\pi \text{ has finite length}\} \\ &\qquad\longrightarrow \{G_{\mathbb{Q}_p}\text{-representations on finite rank free } A\text{-modules}\}.\end{aligned}$$

We may extend the definition of MF further. Namely, let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal of $A$, and suppose that $\pi$ is an admissible unitary Banach representation of $G$ over $A_{\mathfrak{p}}$, in the sense of Definition 3.1.14, satisfying the following assumption:

3.2.12. **Assumption.** The representation $\pi$ contains a $G$-invariant $\mathfrak{m}$-adically admissible unit ball $\pi^0$ with the additional property that $\pi^0/\mathfrak{m}\pi^0$ is of finite length.

We then define

$$\mathrm{MF}(\pi) := A_{\mathfrak{p}} \otimes_A \mathrm{MF}(\pi^0);$$

this evidently depends on $\pi^0$ only up to commensurability.

We will typically apply this construction in the case when $A = \mathcal{O}$ and $\mathfrak{p}$ is the zero ideal, so that $A_{\mathfrak{p}} = E$, and we are considering admissible unitary $E$-Banach representations of $G$ with the additional property that the reduction mod $\varpi$ of a $G$-invariant unit ball is of finite length.

However, in Subsection 3.6, we will have to consider a more general context, and in fact will have to iterate the previous ind-pro extension of MF. Namely, let $\widehat{A}_{\mathfrak{p}}$ denote the $\mathfrak{p}$-adic completion of $A_{\mathfrak{p}}$, and suppose given a $\mathfrak{p}$-adically complete $\widehat{A}_{\mathfrak{p}}$-module $\pi$, equipped with an action of $G$, with the property that for each $i \geq 0$, the quotient $\pi/\mathfrak{p}^i\pi$ is an admissible unitary Banach representation over $A_{\mathfrak{p}}/\mathfrak{p}^iA_{\mathfrak{p}}$ (note that this quotient is isomorphic to the localization $(A/\mathfrak{p}^i)_{\mathfrak{p}}$, so that the notion of admissible unitary Banach representation over this quotient is defined) which satisfies Assumption 3.2.12. We then define

$$\mathrm{MF}(\pi) := \varprojlim_i \mathrm{MF}(\pi/\mathfrak{p}^i\pi).$$

Finally, suppose that $\mathfrak{q}$ is a prime ideal of $\widehat{A}_{\mathfrak{p}}$, and that we are given a Banach module $\pi$ over $(\widehat{A}_{\mathfrak{p}})_{\mathfrak{q}}$, equipped with a $G$-action, and admitting a unit ball $\pi^0$ which is $G$-invariant, with the property that for each $i \geq 0$, the quotient $\pi^0/\mathfrak{p}^i\pi^0$ is an admissible unitary Banach representation over $A_{\mathfrak{p}}/\mathfrak{p}^i A_{\mathfrak{p}}$ which satisfies Assumption 3.2.12. We then define

$$\mathrm{MF}(\pi) := (\widehat{A}_{\mathfrak{p}})_{\mathfrak{q}} \otimes_{\widehat{A}_{\mathfrak{p}}} \mathrm{MF}(\pi^0).$$

This only depends on $\pi^0$ up to commensurability.

3.3. **Mod $p$ and $p$-adic local Langlands.** Let $k$ be finite field of characteristic $p$, and let $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ be a continuous representation. Throughout this subsection we make the following technical assumption regarding $\overline{\rho}$:

3.3.1. **Assumption.** $\overline{\rho}$ is *not* equivalent to a representation of the form

$$\chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$$

(for some $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$, where $*$ may or may not be zero).

The following result is due to Colmez [24].

3.3.2. **Theorem.** *If $\overline{\rho}$ satisfies Assumption 3.3.1, then there is a finite length object $\overline{\pi}$ of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$, unique up to isomorphism, such that:*

(1) $\mathrm{MF}(\overline{\pi}) \cong \overline{\rho}$.
(2) *$\overline{\pi}$ has central character equal to $\det(\overline{\rho})\overline{\varepsilon}$.*
(3) *$\overline{\pi}$ has no finite-dimensional $G$-invariant subobject or quotient.*

Since MF is compatible with extension of scalars, the same is true of the formation of $\overline{\pi}$.

3.3.3. **Remark.** We briefly describe the structure of $\overline{\pi}$ (assuming that $\overline{\pi}$ is either reducible or absolutely irreducible, a condition that can always be attained by making a finite extension of scalars):

(1) Suppose that $\overline{\rho} \sim \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$, and that $\chi_1\chi_2^{-1} \neq \overline{\varepsilon}^{\pm 1}$. Then

$$\overline{\pi} := \big(\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\overline{\varepsilon}\big) \oplus \big(\mathrm{Ind}_B^G \chi_2 \otimes \chi_1\overline{\varepsilon}\big).$$

Our assumption on $\chi_1\chi_2^{-1}$ ensures that both direct summands are irreducible.

(2) Suppose that $\overline{\rho} \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, where $*$ denotes a non-trivial extension. If $\chi_1\chi_2^{-1} \neq \overline{\varepsilon}^{\pm 1}$, then $\overline{\pi}$ is an extension

$$0 \to \mathrm{Ind}_B^G \chi_1 \otimes \chi_2\overline{\varepsilon} \to \overline{\pi} \to \mathrm{Ind}_B^G \chi_2 \otimes \chi_1\overline{\varepsilon} \to 0.$$

(Our hypothesis on $\chi_1\chi_2^{-1}$ implies that each of these induced representations is irreducible.)

(3) Suppose that $\overline{\rho} \sim \begin{pmatrix} \chi & * \\ 0 & \chi\overline{\varepsilon}^{-1} \end{pmatrix}$, where $*$ denotes a non-trivial extension. Suppose also that $p > 3$, so that $\overline{\varepsilon} \neq \overline{\varepsilon}^{-1}$. The $\overline{\pi}$ has a unique Jordan–Hölder filtration

$$0 \subset \overline{\pi}_1 \subset \overline{\pi}_2 \subset \overline{\pi}$$

such that $\overline{\pi}_1 \xrightarrow{\sim} (\chi\circ\det)\otimes\mathrm{St}$, $\overline{\pi}_2/\overline{\pi}_1 \xrightarrow{\sim} \chi\circ\det$, and $\overline{\pi}/\overline{\pi}_2 \xrightarrow{\sim} \mathrm{Ind}_B^G \chi\overline{\varepsilon}^{-1} \otimes \chi\overline{\varepsilon}$. (Our assumption on $p$ ensures that this induced representation is irreducible.)

(4) If $\overline{\rho}$ is absolutely irreducible, then $\overline{\pi}$ is an irreducible supersingular representation of $G$.

3.3.4. **Remark.** Note that the semi-simplification of $\overline{\pi}$ (as a $G$-representation) coincides with the $G$-representation associated to $\overline{\rho}$ via (a suitably normalized version of) the semi-simple mod $p$ local Langlands correspondence of [8].

3.3.5. **Remark.** If $\overline{\rho}_{|I_p}$ is not the twist of an extension of 1 by itself or $\overline{\varepsilon}^{\pm 1}$, then another construction of the associated representation $\overline{\pi}$ is given in [13, §20]. In this reference, $\overline{\pi}$ is not characterized by its behaviour under application of the functor MF, but rather by the structure of its invariants under the principal congruence subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ (together with the fact that it is generated by these invariants).

Our goal now is to describe the deformation-theoretic approach to the construction of the $p$-adic local Langlands correspondence (for two-dimensional $G_{\mathbb{Q}_p}$-representations lifting $\overline{\rho}$) due to Colmez and Kisin.

3.3.6. **Definition.** We let $\mathrm{Def}(\overline{\rho})$ denote the following category fibred in groupoids over $\mathrm{Comp}(\mathcal{O})$: for any object $A$ of $\mathrm{Comp}(\mathcal{O})$, with maximal ideal $\mathfrak{m}$, the groupoid $\mathrm{Def}(A)$ has as objects free rank two $A$-modules $V$ equipped with a continuous action of $G_{\mathbb{Q}_p}$, as well as an $A/\mathfrak{m}$-linear and $G_{\mathbb{Q}_p}$-equivariant isomorphism $\imath : V/\mathfrak{m}V \xrightarrow{\sim} A/\mathfrak{m} \otimes_k \overline{\rho}$, and as morphisms the $A$-linear and $G_{\mathbb{Q}_p}$-equivariant isomorphisms that are compatible with the given maps $\imath$.

We typically regard $\mathrm{Def}(\overline{\rho})$ as a groupoid-valued functor on $\mathrm{Comp}(\mathcal{O})$, and thus refer to $\mathrm{Def}(\overline{\rho})$ as the deformation functor associated to $\overline{\rho}$.

Let $\overline{\pi}$ be the object of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$ attached to $\overline{\rho}$ via Theorem 3.3.2.

3.3.7. **Definition.** We let $\mathrm{Def}(\overline{\pi})$ denote the following category fibred in groupoids over $\mathrm{Comp}(\mathcal{O})$: for any object $A$ of $\mathrm{Comp}(\mathcal{O})$, with maximal ideal $\mathfrak{m}$, the groupoid $\mathrm{Def}(A)$ has as objects orthonormalizable admissible representations $\pi$ of $G$ over $A$ equipped with an $A/\mathfrak{m}$-linear and $G$-equivariant isomorphism $\imath : \pi/\mathfrak{m}\pi \xrightarrow{\sim} A/\mathfrak{m} \otimes_k \overline{\pi}$, and as morphisms the $A$-linear and $G$-equivariant isomorphisms that are compatible with the given maps $\imath$.

As with $\mathrm{Def}(\overline{\rho})$, we typically regard $\mathrm{Def}(\overline{\pi})$ as a groupoid-valued functor on $\mathrm{Comp}(\mathcal{O})$, and thus refer to $\mathrm{Def}(\overline{\pi})$ as the deformation functor associated to $\overline{\pi}$.

The functor MF induces a natural morphism of deformation functors

$$\mathrm{MF} : \mathrm{Def}(\overline{\pi}) \to \mathrm{Def}(\overline{\rho}). \tag{3.3.8}$$

(as follows from the exactness of MF; see [61, §2.2]).

We next introduce some subgroupoids of $\mathrm{Def}(\overline{\pi})$ and $\mathrm{Def}(\overline{\rho})$, and some variants of the morphism (3.3.8).

3.3.9. **Definition.** We define $\mathrm{Def}^*(\overline{\pi})$ to be the subgroupoid of $\mathrm{Def}(\overline{\pi})$ consisting of those deformations $\pi$ such that the centre of $G$ acts on $\pi$ via the $A^\times$-valued character $\det\bigl(\mathrm{MF}(\pi)\bigr)\varepsilon$ (where we regard $\det\bigl(\mathrm{MF}(\pi)\bigr)\varepsilon$ as a character of $\mathbb{Q}_p^\times$ via local class field theory).

The morphism (3.3.8) induces a morphism

$$\mathrm{MF} : \mathrm{Def}^*(\overline{\pi}) \to \mathrm{Def}(\overline{\rho}). \tag{3.3.10}$$

3.3.11. **Definition.** We define $\mathrm{Def}^{\mathrm{crys}}(\overline{\rho})$ to be the full subgroupoid of $\mathrm{Def}(\overline{\rho})$ obtained as the Zariski closure in $\mathrm{Def}(\overline{\rho})$ of the set of crystalline points in the generic fibre of $\mathrm{Def}(\overline{\rho})$. We define $\mathrm{Def}^{\mathrm{crys}}(\overline{\pi})$ to be the fibre product of $\mathrm{Def}^*(\overline{\pi})$ and $\mathrm{Def}^{\mathrm{crys}}(\overline{\rho})$ over $\mathrm{Def}(\overline{\rho})$ (where the former maps to $\mathrm{Def}(\overline{\rho})$ via (3.3.10), and the latter maps to $\mathrm{Def}(\overline{\rho})$ via the natural fully faithful embedding).

The map (3.3.10) induces a tautological map

$$\mathrm{MF} : \mathrm{Def}^{\mathrm{crys}}(\overline{\pi}) \to \mathrm{Def}^{\mathrm{crys}}(\overline{\rho}). \tag{3.3.12}$$

We now state the $p$-adic local Langlands correspondence, in its deformation-theoretic formulation due to Kisin [61].

3.3.13. **Theorem.** *If* $\overline{\rho}$ *satisfies Assumption* 3.3.1*, then* (3.3.10) *is a fully faithful embedding* $\mathrm{Def}^*(\overline{\pi}) \hookrightarrow \mathrm{Def}(\overline{\rho})$, *while its restriction* (3.3.12) *is in fact an equivalence of deformation functors* $\mathrm{Def}^{\mathrm{crys}}(\overline{\pi}) \xrightarrow{\sim} \mathrm{Def}^{\mathrm{crys}}(\overline{\rho})$.

3.3.14. **Remark.** In this remark we discuss some results related to the preceding definitions and theorem which, however, we won't need in the present paper.

We first remark that whenever Assumption 3.3.1 holds, it is known that there is an equality $\mathrm{Def}^{\mathrm{crys}}(\overline{\rho}) = \mathrm{Def}(\overline{\rho})$. Indeed, if $p > 3$, or $p = 3$ and $\overline{\rho}_{I_p}$ is not isomorphic to a twist of $\omega_2^2 \oplus \omega_2^6$, then this has been proved by Kisin [61, Cor. 1.11]; if $p = 3$ and $\overline{\rho}_{I_p}$ *is* isomorphic to a twist of $\omega_2^2 \oplus \omega_2^6$, this has been proved by Böckle (see the discussion following the statement of [5, Thm. 1.1]); and if $p = 2$ this has been proved by Chenevier (see footnote (7) on p. 292 of [25]). However, since we do not need the equality of $\mathrm{Def}^{\mathrm{crys}}(\overline{\rho})$ and $\mathrm{Def}(\overline{\rho})$ in this paper, and since the result for $p = 2$ remains unpublished, we have decided to maintain the *a priori* distinction between $\mathrm{Def}^{\mathrm{crys}}(\overline{\rho})$ and $\mathrm{Def}(\overline{\rho})$ in this paper.

We also mention that if $\mathrm{Def}(\overline{\rho})$ and $\mathrm{Def}(\overline{\pi})$ are representable, i.e. if $\overline{\rho}$ and $\overline{\pi}$ have only scalar endomorphisms (note that the functor MF induces an isomorphism between their endomorphism algebras [61, Lem. 2.1.2]), and if $p \geq 3$ when $\overline{\pi}$ is supersingular, then one finds that $\mathrm{Def}^*(\overline{\pi}) = \mathrm{Def}(\overline{\pi})$, and hence (taking into account the discussion of the preceding paragraph) that (3.3.8) is actually an equivalence $\mathrm{Def}(\overline{\pi}) \xrightarrow{\sim} \mathrm{Def}(\overline{\rho})$. This is most difficult to establish in the supersingular case, where it is due to Paškūnas [69]. (Note that although Paškūnas's results are proved under the assumption that $p \geq 3$, we expect the equality $\mathrm{Def}^*(\overline{\pi}) = \mathrm{Def}(\overline{\pi})$ to continue to hold in the supersingular case when $p = 2$.) More generally, we should mention that the complications caused by the central character in relating the deformations of $\overline{\pi}$ and $\overline{\rho}$ were first observed by Paškūnas (see [69] and the introduction to [61]).

3.3.15. **Definition.** Let $V_E$ be a two-dimensional $E$-vector space equipped with a continuous representation of $G_{\mathbb{Q}_p}$, and suppose that $V_E$ contains a $G_{\mathbb{Q}_p}$-invariant $\mathcal{O}$-lattice $V$ such that $\overline{\rho} := V/\varpi V$ satisfies Assumption 3.3.1, and such that $V$ lies in $\mathrm{Def}^{\mathrm{crys}}(\overline{\rho})$. Theorem 3.3.13 then yields a representation $\pi$ of $G$ on an orthonormalizable $\mathcal{O}$-module lifting $\overline{\pi}$ such that $\mathrm{MF}(\pi) = V$, and we define $B(V_E) := E \otimes_{\mathcal{O}} \pi$. Thus $B(V_E)$ is an $E$-Banach space equipped with an admissible unitary $G$-representation, which is easily seen to be independent of the choice of $V$ (since any two such choices are commensurable).

The association $V_E \mapsto B(V_E)$ *is* the $p$-adic local Langlands correspondence.

3.3.16. **Remark.** The preceding definition applies in particular to any $V_E$ which is absolutely irreducible, provided that $p > 3$. Indeed in this case we have that

$\mathrm{Def}^{\mathrm{crys}}(\overline{\rho}) = \mathrm{Def}(\overline{\rho})$, while $\overline{\varepsilon} \neq \overline{\varepsilon}^{-1}$, so that — applying a suitable analogue of [71, Prop. 2.1] in the case when $\overline{\rho}^{\mathrm{ss}}$ is isomorphic to a twist of $\underline{1} \oplus \overline{\varepsilon}$ — we see that $V_E$ contains a $V$ satisfying the requirements of the definition.

3.3.17. **Remark.** In the paper [25], using his functor from two-dimensional $G_{\mathbb{Q}_p}$-representations to $G$-representations, Colmez has defined the $p$-adic local Langlands correspondence for *every* absolutely irreducible representation of $G_{\mathbb{Q}_p}$.

3.3.18. **Definition.** We say that a continuous two-dimensional representation of $G_{\mathbb{Q}_p}$ over $E$ is exceptional if it is a twist of a crystalline representation whose associated Dieudonné module is not Frobenius semi-simple (or equivalently, for which the characteristic polynomial of the Frobenius endomorphism on the associated Dieudonné module has a double root).

3.3.19. **Remark.** Let $V_E$ be a two-dimensional $E$-vector space equipped with a continuous representation of $G_{\mathbb{Q}_p}$ which is absolutely irreducible and trianguline in the sense of [23], and *not* exceptional in the sense of the preceding definition. The constructions of [3, 22, 23] give rise to an admissible unitary Banach space representation $B(V_E)$ of $G$ associated to $V_E$.[9] It is an important aspect of the $p$-adic local Langlands correspondence of Definition 3.3.15 that it coincides with explicit correspondence of [3, 22, 23] in the case when both are defined. (Indeed, this explicit correspondence is one of the tools used in the proof of Theorem 3.3.13, and hence in the construction of the correspondence of Definition 3.3.15.)

3.3.20. **Remark.** In the case when $V_E$ is a reducible two-dimensional continuous representation of $G_{\mathbb{Q}_p}$, there is an extensive discussion in [38, §6] of the expected structure of the Banach space representation $B(V_E)$ of $G$ that should be attached to $V_E$ by the $p$-adic local Langlands correspondence. Using the results of [40] (and, in particular, applying the methods of computation from §4 of that reference), it is straightforward to verify the various conjectures of [38, §§6.2, 6.3, 6.4, 6.5], and thus $B(V_E)$ can indeed be defined following the prescriptions of [38, §6]. We will not do this here, however, since we do not need this for our purposes. Rather, in Subsection 3.4 below we will establish some somewhat weaker results about $B(V_E)$ in the the reducible case, using the deformation-theoretic view-point adopted here, and the functor MF.

If $V_E$ is a reducible two-dimensional continuous representation of $G_{\mathbb{Q}_p}$ which is de Rham with distinct Hodge–Tate weights, then one expects that the associated $p$-adic Banach representation $B(V_E)$ of $G$ should have the properties originally posited by Breuil in [8, 9]. In this direction, one has the following result, due to Berger, Breuil, and Colmez in the non-exceptional trianguline case, and to Colmez for those $V_E$ that are not trianguline.

3.3.21. **Theorem.** *If $V_E$ is a continuous two-dimensional representation of $G_{\mathbb{Q}_p}$ over $E$ and if $V_E$ is de Rham with distinct Hodge–Tate weights $a > b$, then $B(V_E)$ contains non-zero locally algebraic vectors.*

[9]When $p > 2$, Paškūnas [68] has shown that the results of [3] extend to cover the exceptional cases, and it is expected that they will similarly extend to cover these cases when $p = 2$. Using Paškūnas's result would allow us to simplify some of the arguments of the present paper in the case when $p > 2$, e.g. the proof of Theorem 6.4.7. However, since we wish to include the case $p = 2$ in our arguments, we haven't incorporated these simplifications.

*Proof.* If $V_E$ is absolutely irreducible, trianguline, and not exceptional (in the sense of Definition 3.3.18), then this follows from the explicit construction of $B(V_E)$ given in [3, 22]. If $V_E$ is absolutely reducible, then one may describe $B(V_E)$ explicitly (see e.g. the description of $B(V_E)$ that we give in Subsection 3.4 below), and so verify the theorem. The theorem for arbitrary $V_E$ (satisfying the assumptions of the theorem) has been proved by Colmez [25, Thm. VI.6.18] □

Colmez has shown that, conversely, if $B(V_E)_{\mathrm{l.alg}}$ is non-zero, then $V_E$ is de Rham with distinct Hodge–Tate weights [25, Prop. VI.5.1, Thm. VI.6.13]. Furthermore, we have the following strengthening of Theorem 3.3.21, which precisely describes the locally algebraic vectors in $B(V_E)$ in the case when they are non-zero.

3.3.22. **Theorem.** *If $V_E$ is a continuous two-dimensional representation of $G_{\mathbb{Q}_p}$ over $E$, and if $V_E$ is de Rham with distinct Hodge–Tate weights $a > b$, then there is a $G$-equivariant isomorphism $B(V_E)_{\mathrm{l.alg}} \xrightarrow{\sim} \mathrm{Sym}^{a-b-1}E^2 \otimes_E \det^{b+1} \otimes_E \pi_p(V_E)$. In particular, $B(V_E)$ contains non-zero locally algebraic vectors.*

*Proof.* In the case when $V_E$ is crystabelline, or is the twist of a semi-stable representation, this has been proved by Colmez [25, Thm. VI.6.50]. In the remaining case (i.e. when $V_E$ is potentially crystalline but not crystabelline), it follows from [25, Thm. VI.6.42], together with a global calculation using the theory developed in this paper. The proof is presented in Subsection 7.4 below. □

3.3.23. **Remark.** In fact Colmez has already shown in [25, Prop. VI.5.1] that if $V_E$ is de Rham with distinct Hodge–Tate weights $a > b$, then the locally algebraic vectors of $B(V_E)$ are necessarily isomorphic (as a $G$-representation) to the tensor product of $\mathrm{Sym}^{a-b-1}E^2 \otimes_E \det^{b+1}$ with a smooth representation of $G$. The point of the preceding theorem is that it gives a precise description of the smooth factor in this tensor product.

The following result is also important in applications. (In [25], Colmez has given a proof of this result using his functor from two-dimensional $G_{\mathbb{Q}_p}$-representations to $G$-representations; see [25, Thm. 0.17 (ii)]. We give a proof here using just the properties of the functor MF.)

3.3.24. **Proposition.** *If $V_E$ is a continuous two-dimensional representation of $G_{\mathbb{Q}_p}$ for which $B(V_E)$ is defined, and if $V_E$ is* (*absolutely*) *irreducible, then $B(V_E)$ is also* (*absolutely*) *topologically irreducible.*

*Proof.* Suppose that $V_E$ is irreducible. Choose a $G$-invariant bounded open lattice $\pi$ in $B(V_E)$ which deforms a representation $\overline{\pi}$ satisfying the conditions of Theorem 3.3.2. Let $W \subset B(V_E)$ be a closed subrepresentation, and write $W_0 = \pi \bigcap W$. Since $W_0$ is saturated in $\pi$, we see that $\mathrm{MF}(W_0)$ is saturated in $\mathrm{MF}(\pi)$ (since MF is exact). Combining this with the fact that $E \otimes_{\mathcal{O}} \mathrm{MF}(\pi) \cong V_E$ is irreducible, we see (again using the exactness of MF) that either $\mathrm{MF}(W_0) = 0$ or $\mathrm{MF}(\pi/W_0) = 0$. Thus either $W_0$ is a finitely generated $\mathcal{O}$-submodule of $\pi$, or else $\pi/W_0$ is a finitely generated $\mathcal{O}$-quotient module of $\pi$. However, since $\overline{\pi}$ admits no finite-dimensional $G$-invariant subrepresentations or quotient representations, we see that one of $W_0$ or $\pi/W_0$ must vanish. Thus either $W = 0$ or else $W = B(V_E)$, and so $B(V_E)$ is topologically irreducible, as claimed. Since the formation of $B(V_E)$ is compatible with extending scalars, we conclude that if $V_E$ is absolutely irreducible, then $B(V_E)$ is absolutely topologically irreducible. □

3.4. **The reducible case.** If $V_E$ is a continuous *reducible* two-dimensional representation of $G_{\mathbb{Q}_p}$ over $E$ satisfying the conditions of Remark 3.3.19 (i.e. containing an $\mathcal{O}$-lattice $V$ in $V_E$ whose reduction $\overline{V}$ satisfies Assumption 3.3.1), then as in Remark 3.3.19, we may associate a Banach space representation $B(V_E)$ of $G$ to $V_E$. In this subsection we will establish some results regarding the structure of $B(V_E)$ in this situation. (By employing the methods of [40, §4], it is possible in fact to verify all the conjectures related to the structure of $B(V_E)$ that we made in [38]. We won't do this here, however, since the somewhat weaker statements that we establish below will suffice for our present purposes.)

Write $V_E$ as an extension

$$0 \to \chi_1 \to V_E \to \chi_2 \to 0, \tag{3.4.1}$$

where each $\chi_i : G_{\mathbb{Q}_p} \to \mathcal{O}^\times \subset E^\times$ is a continuous character.

3.4.2. **Proposition.** *If $\chi_1\chi_2^{-1} \neq \varepsilon$, then $B(V_E)$ is an extension*

$$0 \to \mathrm{Ind}_B^G \chi_1 \otimes \chi_2\varepsilon \to B(V_E) \to \mathrm{Ind}_B^G \chi_2 \otimes \chi_1\varepsilon \to 0,$$

*which is split or not according to whether or not* (3.4.1) *is. Furthermore, the $G$-representations $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\varepsilon$ and $\mathrm{Ind}_B^G \chi_2 \otimes \chi_1\varepsilon$ are topologically irreducible.*

*Proof.* Choose a $G_{\mathbb{Q}_p}$-invariant lattice $V$ in $V_E$ whose reduction $\overline{V}$ satisfies Assumption 3.3.1. Theorem 3.3.13 shows that we may find a lift $\pi$ of $\overline{\pi}$ over $\mathcal{O}$, with central character equal to $\chi_1\chi_2\varepsilon$, and such that $\mathrm{MF}(\pi) \xrightarrow{\sim} V$. Since $\overline{\pi}$ contains no finite-dimensional $G$-invariant subspaces, we see that $\pi$ contains no finitely generated $G$-invariant $\mathcal{O}$-submodules. Proposition 3.2.4 (and the known value of the central character of $\pi$) thus yields an embedding

$$\chi_1 \otimes \chi_2\varepsilon \hookrightarrow \mathrm{Ord}_B(\pi),$$

which in turn, by virtue of the adjointness property of $\mathrm{Ord}_B$, gives rise to a $G$-equivariant map

$$\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\varepsilon \to \pi. \tag{3.4.3}$$

Since $\chi_1 \neq \chi_2\varepsilon$, by assumption, the source of this map becomes topologically irreducible after tensoring with $E$ over $\mathcal{O}$. Consequently, (3.4.3) must be injective. If we let $\pi_1$ denote the saturation in $\pi$ of its image, then $\pi_1$ contains this image with finite index, and consequently

$$\mathrm{MF}(\pi_1) \xrightarrow{\sim} \mathrm{MF}(\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\varepsilon) = \chi_1.$$

Thus, writing $\pi_2 = \pi/\pi_1$, we compute that $\mathrm{MF}(\pi_2) \xrightarrow{\sim} \mathrm{MF}(\pi)/\mathrm{MF}(\pi_1) \xrightarrow{\sim} V/\chi_1 = \chi_2$. Since $\overline{\pi}$ contains at most one finite-dimensional Jordan–Hölder factor, of dimension one if it exists at all, we see that either $\pi_2$ contains no finitely generated $G$-invariant $\mathcal{O}$-submodules, or else that it contains such a submodule of rank one, the quotient of $\pi_2$ by which then contains no such submodules. Letting $\pi_3 = \pi_2$ in the first case, or the quotient of $\pi_2$ by the rank one $G$-invariant submodule in the second case, we see that

$$\mathrm{MF}(\pi_3) \xrightarrow{\sim} \mathrm{MF}(\pi_2) = \chi_2.$$

We thus deduce from Proposition 3.2.4 an isomorphism

$$\mathrm{Ord}_B(\pi_3) \xrightarrow{\sim} \chi_2,$$

and hence a map

$$\mathrm{Ind}_B^G \chi_2 \otimes \chi_1\varepsilon \to \pi_3. \tag{3.4.4}$$

Again by assumption we have that $\chi_2 \neq \chi_1\varepsilon$, and thus this map must be an embedding. If we let $\pi_4$ denote the cokernel of this embedding, then $\mathrm{MF}(\pi_4) = 0$. Thus if we write $\overline{\pi}_4 := \pi_4/\varpi\pi_4$, then $\overline{\pi}_4$ is a $G$-invariant quotient of $\overline{\pi}$ for which $\mathrm{MF}(\overline{\pi}_4) = 0$. On the other hand, $\overline{\pi}$ has no finite-dimensional $G$-invariant quotient. We conclude that $\overline{\pi}_4 = 0$, and thus that $\pi_4 = 0$. Thus we conclude that (3.4.4) is an isomorphism.

Now the Jordan–Hölder factors of $\overline{\pi}$ are precisely the union of those of $\mathrm{Ind}_B^G \overline{\chi}_1 \otimes \overline{\chi}_2\varepsilon$ and those of $\mathrm{Ind}_B^G \overline{\chi}_2 \otimes \overline{\chi}_1\varepsilon$. Also, $\overline{\pi}_1 := \pi_1/\varpi\pi_1$ and $\mathrm{Ind}_B^G \overline{\chi}_1 \otimes \overline{\chi}_2\varepsilon$ have the same Jordan–Hölder factors. Combining this observation with the isomorphism (3.4.4), we conclude that the Jordan–Hölder factors of $\overline{\pi}$ are precisely the union of those of $\overline{\pi}_1$ and those of $\overline{\pi}_3 := \pi_3/\varpi\pi_3$. Consequently it must be the case that $\pi_2 = \pi_3$, and thus that $\pi$ is an extension of $\pi_3(\xrightarrow{\sim} \mathrm{Ind}_B^G \chi_2 \otimes \chi_1\varepsilon)$ by $\pi_1$. Tensoring with $E$ over $\mathcal{O}$ (and recalling that $\pi_1$ contains the image of (3.4.3) with finite index), we find that $B(V_E)$ is an extension of the required form.

Clearly if $B(V_E)$ is a split extension, then the isomorphism $V_E \xrightarrow{\sim} E \otimes_{\mathcal{O}} MF(\pi)$ shows that $V_E$ is itself split. On the other hand, if $V_E$ is split, then we may reverse the roles of $\chi_1$ and $\chi_2$, and, obtaining a corresponding description of $B(V_E)$ as an extension of $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\varepsilon$ by $\mathrm{Ind}_B^G \chi_2 \otimes \chi_1\varepsilon$, conclude in turn that $B(V_E)$ is isomorphic to the direct sum of these two representations. This completes the proof of the proposition. □

3.4.5. **Proposition.** *If $\chi_1\chi_2^{-1} = \varepsilon$, i.e. $\chi_2 = \chi_1\varepsilon^{-1}$, then $B(V_E)$ admits a unique three step Jordan–Hölder filtration (in the category of admissible Banach representations of $G$) $0 = B_0 \subset B_1 \subset B_2 \subset B_3 = B(V_E)$, such that:*

1. *$B_1 \cong (\chi_1 \circ \det) \otimes \widehat{\mathrm{St}}$;*
2. *$B_2/B_1 \cong \chi_1 \circ \det$;*
3. *$B_3/B_1 \cong \mathrm{Ind}_B^G \chi_1\varepsilon^{-1} \otimes \chi_1\varepsilon \ \big(\cong (\chi_1 \circ \det) \otimes (\mathrm{Ind}_B^G \varepsilon^{-1} \otimes \varepsilon)\big)$.*

*Proof.* The proof of this result is similar to that of the preceding proposition. As in that proposition we choose lattices $V$ in $V_E$ and $\pi$ in $B(V_E)$, related by an isomorphism

$$\mathrm{MF}(\pi) \xrightarrow{\sim} V,$$

and we obtain a map

$$\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\varepsilon \to \pi. \tag{3.4.6}$$

Now $\chi_1 = \chi_2\varepsilon$ by assumption, and thus $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\varepsilon = \mathrm{Ind}_B^G \chi_1 \otimes \chi_1$ is not irreducible, but rather is an extension of $(\chi_1 \circ \det) \otimes \widehat{\mathrm{St}}$ by $\chi_1 \circ \det$. Since $\overline{\pi}$ contains no finite-dimensional $G$-invariant subspaces, we see that (3.4.6) must factor to yield an embedding

$$\widehat{\mathrm{St}} \hookrightarrow \pi. \tag{3.4.7}$$

Also, since $\mathrm{St}(= \widehat{\mathrm{St}}/\varpi\widehat{\mathrm{St}})$ is irreducible, the image of this embedding is saturated.

Let $\pi_2$ denote the cokernel of (3.4.7). We see that $\mathrm{MF}(\pi_2) = \chi_1 \otimes \varepsilon$, and (using the known structure of $\overline{\pi}$, namely that is an extension by $(\overline{\chi}_1 \circ \det) \otimes \mathrm{St}$ of a non-split extension of $\mathrm{Ind}_B^G \overline{\chi}_1\overline{\varepsilon} \otimes \overline{\chi}_1\overline{\varepsilon}^{-1}$ by $\overline{\chi}_1 \circ \det$) we see that $\overline{\pi}_2 := \pi_2/\varpi\pi_2$ is a non-split extension

$$0 \to \overline{\chi}_1 \circ \det \to \overline{\pi}_2 \to \mathrm{Ind}_B^G \overline{\chi}_1\overline{\varepsilon} \otimes \overline{\chi}_1\overline{\varepsilon}^{-1} \to 0. \tag{3.4.8}$$

If we let $\pi_3$ denote the quotient of $\pi_2$ by its maximal finitely generated $G$-invariant $\mathcal{O}$-submodule (the latter is of rank at most one), then we conclude as in the proof of the preceding proposition that

$$\mathrm{Ind}_B^G \chi_1\varepsilon^{-1} \otimes \chi_1\varepsilon \xrightarrow{\sim} \pi_3,$$

and thus that

$$\overline{\pi}_3 := \pi_3/\varpi\pi_3 \xrightarrow{\sim} \mathrm{Ind}_B^G \overline{\chi}_1\overline{\varepsilon}^{-1} \otimes \overline{\chi}_1\overline{\varepsilon}.$$

A comparison with (3.4.8) shows that $\pi_2$ must be an extension of $\pi_3$ by a rank one $G$-invariant $\mathcal{O}$-module. Tensoring with $E$ over $\mathcal{O}$, we conclude that $B(V_E)$ has a Jordan–Hölder filtration of the type described in the statement of the proposition. The uniqueness of this filtration follows from the uniqueness of the Jordan–Hölder filtration of $\overline{\pi}$. □

3.4.9. **Remark.** In the situation of Proposition 3.4.5, one can show that the one-dimensional representation $B_2/B_1$ is necessarily isomorphic to the character $\chi_1 \circ \det$. Indeed, as already mentioned, one can confirm the speculations of the discussion of [38, §6.5], namely that the extension $B_2$ of $\chi_1 \circ \det$ by $(\chi_1 \circ \det) \otimes \widehat{\mathrm{St}}$ that embeds as a closed subrepresentation of $B(V_E)$ is isomorphic to $(\chi_1 \circ \det) \otimes B(2, \mathcal{L})$, where $\mathcal{L}$ is the $\mathcal{L}$-invariant of the extension

$$0 \to \chi_1 \to V_E \to \chi_1\varepsilon^{-1} \to 0.$$

3.5. **Weights.** In this subsection we recall the notion of Serre weights and its relation to the mod $p$ local Langlands correspondence.

3.5.1. **Definition.** A Serre weight is an (isomorphism class of) irreducible representation(s) of $\mathrm{GL}_2(\mathbb{F}_p)$ over $k$. The Serre weights are precisely the (isomorphism classes of the) representations $(\mathrm{Sym}^r k^2)^\vee \otimes_k \det^s$, where $0 \le r \le p-1$, and $0 \le s \le p-2$.

3.5.2. **Definition.** If $V$ is a Serre weight, then we write

$$\mathcal{H}(V) := \mathrm{End}_G(c\text{–}\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)}^G V).$$

It follows from [1] that $\mathcal{H}(V) = k[T, Z, Z^{-1}]$, where $T$ is the endomorphism corresponding to the double coset $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, while $Z$ is the element $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ lying in the centre of $G$.

3.5.3. **Remark.** If $\pi$ is any smooth representation of $G$ over $k$, then there is a canonical isomorphism $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \pi) \xrightarrow{\sim} \mathrm{Hom}_G(c\text{–}\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)}^G V, \pi)$, and thus $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \pi)$ is endowed with a canonical action of $\mathcal{H}(V)$.

To any continuous representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ we may attach a set $W(\overline{\rho})$ of Serre weights, as follows. (We essentially follow the recipe of [14, §3], but use a cohomological normalization.)

If $\overline{\rho}$ is absolutely irreducible, then $(\overline{k} \otimes_k \overline{\rho})_{|I_p} \xrightarrow{\sim} \psi_2^{a+pb} \bigoplus \psi_2^{pa+b}$, for some uniquely determined $0 \le b < a \le p-1$, and we set

$$W(\overline{\rho}) := \{(\mathrm{Sym}^{a-b-1} k^2)^\vee \otimes_k \det^a, (\mathrm{Sym}^{p-a+b} k^2)^\vee \otimes_k \det^{b+1}\}.$$

Thus $W(\overline{\rho})$ is a set of two weights.

If $\overline{\rho}$ is absolutely reducible, then the definition of $W(\overline{\rho})$ is given via a slightly more involved rule: If $0 < r \leq p-1$, and $0 \leq s < p-1$, then the representation $(\mathrm{Sym}^r k^2)^\vee \otimes_k \det^s$ lies in $W(\overline{\rho})$ if and only if $\overline{\rho}_{|I_p}$ sits in a short exact sequence of the form

$$0 \to \overline{\varepsilon}^s \to \overline{\rho}_{|I_p} \to \overline{\varepsilon}^{s-1-r} \to 0,$$

while $\det^s$ lies in $W(\overline{\rho})$ if and only if $\overline{\rho}_{|I_p}$ sits in a short exact sequence of the form

$$0 \to \overline{\varepsilon}^s \to \overline{\rho}_{|I_p} \to \overline{\varepsilon}^{s-1} \to 0$$

for which the extension class is furthermore peu ramifiée in the sense of [79].

Thus, if $\overline{\rho}$ is reducible, then $W(\overline{\rho})$ consists of one, two, three or even (in some cases when $p=3$) four weights. (See the table in the proof of [14, Thm. 3.15].)

3.5.4. **Definition.** If $V \in W(\overline{\rho})$, then we define an $\mathcal{H}(V)$ module $\mathrm{m}(V,\overline{\rho})$ as follows:

(1) If $\overline{\rho}$ is absolutely irreducible, then we define $\mathrm{m}(V,\overline{\rho})$ to be one dimensional, with $T$ acting by 0 and $Z$ acting by $\det(\overline{\rho})(p)$. (Here we think of $\det(\overline{\rho})$ as a homomorphism $G_{\mathbb{Q}_p}^{\mathrm{ab}} \to k^\times$, and hence as a homomorphism $\mathbb{Q}_p^\times \to k^\times$, by local class field theory.)

(2) If $\overline{\rho}$ is absolutely reducible, and $V = (\mathrm{Sym}^r k^2)^\vee \otimes_k \det^s$, then we set $\mathrm{m}(V,\overline{\rho}) := \mathrm{Hom}_{I_p}(\overline{\varepsilon}^s,\overline{\rho})$ as a $k$-vector space. Note that $\mathrm{Hom}_{I_p}(\overline{\varepsilon}^s,\overline{\rho})$ is naturally a representation of the quotient group $G_{\mathbb{Q}_p}/I_p$, and so in particular is equipped with an action of the geometric Frobenius at $p$. We equip $\mathrm{m}(V,\overline{\rho})$ with an $\mathcal{H}(V)$-module structure by declaring that $T$ act via geometric Frobenius, and that $Z$ act via the scalar $\det(\overline{\rho})(p)$.

Suppose now that $\overline{\rho}$ satisfies Assumption 3.3.1, and let $\overline{\pi}$ denote the smooth $G$-representation over $k$ attached to $\overline{\rho}$ via Theorem 3.3.2. If $V$ is a Serre weight, then, by Remark 3.5.3, the space $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V,\overline{\pi})$ is naturally an $\mathcal{H}(V)$-module.

3.5.5. **Lemma.** (1) *If* $V \notin W(\overline{\rho})$, *then* $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V,\overline{\pi}) = 0$.

(2) *If* $V \in W(\overline{\rho})$, *then there is an isomorphism of* $\mathcal{H}(V)$*-modules* $\mathrm{m}(V,\overline{\rho}) \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V,\overline{\pi})$.

*Proof.* This is relatively straightforward, based on the explicit description of $\overline{\pi}$ given in Remark 3.3.3, or in [25]. In the case when $\overline{\rho}_{|I_p}$ is not the twist of an extension of 1 by either itself or $\overline{\varepsilon}^{\pm 1}$, it is an immediate consequence of the description of $\overline{\pi}$ given in [13, §20]. ☐

3.6. **Lattices.** This subsection is devoted to the proof of two technical, but important, propositions. To begin with we suppose that $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathcal{O})$ is continuous, that $\rho_E := E \otimes_{\mathcal{O}} \rho$ is irreducible, and that $\overline{\rho}^{\mathrm{ss}} = \chi_1 \oplus \chi_2$, where $\chi_1\chi_2^{-1} \neq \underline{1}, \overline{\varepsilon}^{\pm 1}$. Let $\overline{\pi}$ be the object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ associated to $\overline{\rho}$ via Theorem 3.3.2, and let $\pi$ be the deformation of $\overline{\pi}$ associated to the deformation $\rho$ of $\overline{\rho}$ via Theorem 3.3.13. Following the notation of Remark 3.3.19, we write $B(\rho_E) := E \otimes_{\mathcal{O}} \pi$. Our first goal in this subsection is establish a proposition that allows us to distinguish $\pi$ among the various $\varpi$-adically complete and separated, $G$-invariant $\mathcal{O}$-lattices in $B(\rho_E)$.

Before stating the stating the proposition, we will make a definition. Note that the composite of the isomorphism $S \xrightarrow{\sim} \mathbb{Q}_p^\times$ with the local Artin map defines an injection $S \hookrightarrow G_{\mathbb{Q}_p}^{\mathrm{ab}}$, and hence an anti-diagonal embedding

$$S \hookrightarrow G_{\mathbb{Q}_p}^{\mathrm{ab}} \times S \tag{3.6.1}$$

(defined as the local Artin map on the first factor, and the map $s \mapsto s^{-1}$ on the second factor).

3.6.2. **Definition.** If $V$ is a representation of $G_{\mathbb{Q}_p} \times S$, and if $V^{\mathrm{ab}}$ denote the maximal subobject of $V$ on which $G_{\mathbb{Q}_p}$ acts through its maximal abelian quotient $G_{\mathbb{Q}_p}^{\mathrm{ab}}$ (note that $V^{\mathrm{ab}}$ is a $G_{\mathbb{Q}_p} \times S$-subrepresentation of $V$), then we let $V^{\mathrm{ab},S}$ denote the subspace of $V^{\mathrm{ab}}$ consisting of $S$-fixed vectors, where $S$ acts through the anti-diagonal embedding (3.6.1) and the action of $G_{\mathbb{Q}_p}^{\mathrm{ab}} \times S$-on $V^{\mathrm{ab}}$.

We can now state and prove our first proposition.

3.6.3. **Proposition.** *If $\pi_0$ is a $G$-invariant, $\varpi$-adically complete and separated, $G$-invariant lattice in $B(\rho_E)$, with the property that, for each $n > 0$, the action of $G_{\mathbb{Q}_p}$ on the quotient*

$$\frac{(\rho/\varpi^n\rho) \otimes_{\mathcal{O}/\varpi^n} \mathrm{Ord}_B(\pi_0/\varpi^n\pi_0)}{\big((\rho/\varpi^n\rho) \otimes_{\mathcal{O}/\varpi^n} \mathrm{Ord}_B(\pi_0/\varpi^n\pi_0)\big)^{\mathrm{ab},S}}$$

*factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$, then $\pi_0$ is equal to a scalar multiple of $\pi$.*

*Proof.* According to Proposition 3.2.10, the functor MF induces a bijection between the set of $\varpi$-adically complete and separated, $G$-invariant lattices in $B(\rho_E)$, and the set of $G_{\mathbb{Q}_p}$-invariant lattices in $\rho_E$. Thus it similarly induces a bijection between the set of $\varpi$-adically complete and separated, $G$-invariant lattices in $B(\rho_E)$, modulo scaling, and the set of $G_{\mathbb{Q}_p}$-invariant lattices in $\rho_E$, modulo scaling.

Denote these latter two sets by $\mathcal{L}_\pi$ and $\mathcal{L}_\rho$. Since $\chi_1$ and $\chi_2$ are distinct, and $\rho_E$ is irreducible, the set $\mathcal{L}_\rho$ is a finite (positive) length segment in the tree of all (not necessarily $G$-invariant) lattices in $\rho_E$. We let $\rho_1$ and $\rho_2$ denote the two endpoints of this segment, the labelling chosen so that $\overline{\rho}_1$ is a non-split extension of $\chi_2$ by $\chi_1$, while $\overline{\rho}_2$ is a non-split extension of $\chi_1$ by $\chi_2$. If $\rho_0$ is any (lattice representing an) element of $\mathcal{L}_\rho$, then we write $d_1(\rho_0)$ (resp. $d_2(\rho_0)$) to denote the distance of $\rho_0$ from $\rho_1$ (resp. $\rho_2$). Note that $d_1(\rho_0) + d_2(\rho_0)$ is constant (i.e. independent of $\rho_0$), equal to the length $d$ of the segment $\mathcal{L}_\rho$. For $i = 1, 2$, the natural number $d_i(\rho_0)$ admits the following characterization: for any $n \geq 0$, the quotient $\rho_0/\varpi^n\rho_0$ contains a (necessarily abelian) $G_{\mathbb{Q}_p}$-subrepresentation $\widetilde{\chi}_i(\rho_0)_n$ which is a successive extension of $\min(d_i(\pi_0), n)$ copies of $\chi_i$, and no longer such subrepresentation (i.e. no subrepresentation which is a successive extension of more than this many copies of $\chi_i$). The representations $\widetilde{\chi}_i(\rho_0)_n$ are compatible in an evident way if $n$ varies, namely: if $m \leq n$, then $\widetilde{\chi}_i(\rho_0)_n/\varpi^m\widetilde{\chi}_i(\rho_0)_n \xrightarrow{\sim} \widetilde{\chi}_i(\rho_0)_m$. Note also that $\widetilde{\chi}_i(\rho_0)_n$ is annihilated by $\varpi^{d_i(\rho_0)}$, for any value of $n$, and so $\widetilde{\chi}_i(\rho_0)_n \xrightarrow{\sim} \widetilde{\chi}_i(\rho_0)_{d_i(\rho_0)}$ for any $n \geq d_i(\rho_0)$. If $n \leq d_i(\pi_0)$, then the quotient $(\rho_0/\varpi^n\rho_0)/\widetilde{\chi}_i(\rho_0)_n$ is an abelian representation of $G_{\mathbb{Q}_p}$ (it is a successive extension of $n$ copies of $\chi_{3-i}$), while if $n > d_i(\pi_0)$ then it is not (it contains a non-split extension of $\chi_i$ by $\chi_{3-i}$).

Now choose $\rho_0 = \mathrm{MF}(\pi_0)$. Proposition 3.2.4 gives rise to a $\mathbb{Q}_p^\times$-equivariant embedding

$$\widetilde{\chi}_1(\rho_0)_d \oplus \widetilde{\chi}_2(\rho_0)_d \hookrightarrow \mathrm{Ord}_B(\pi_0/\varpi^d\pi_0),$$

and thus to an embedding

$$\big((\rho/\varpi^{d_1(\rho_0)}\rho) \otimes_{\mathcal{O}/\varpi^{d_1(\rho_0)}\mathcal{O}} \widetilde{\chi}_1(\rho_0)_d\big) \oplus \big((\rho/\varpi^{d_2(\rho_0)}\rho) \otimes_{\mathcal{O}/\varpi^{d_2(\rho_0)}\mathcal{O}} \widetilde{\chi}_2(\rho_0)_d\big) \hookrightarrow (\rho/\varpi^d\rho) \otimes_{\mathcal{O}/\varpi^d} \mathrm{Ord}_B(\pi_0/\varpi^d\pi_0).$$

Taking $n = d$ in the hypothesis of the theorem, we find that for each choice of $i = 1, 2$, the cokernel of the map

$$\big((\rho/\varpi^{d_i(\rho_0)}\rho) \otimes_{\mathcal{O}/\varpi^{d_i(\rho_0)}} \widetilde{\chi}_i(\rho_0)_d\big)^{\mathrm{ab},S} \hookrightarrow (\rho/\varpi^{d_i(\rho_0)}\rho) \otimes_{\mathcal{O}/\varpi^{d_i(\rho_0)}} \widetilde{\chi}_i(\rho_0)_d$$

is again an abelian representation of $G_{\mathbb{Q}_p}$.

Now $(\rho/\varpi^{d_i(\rho_0)}\rho)^{\mathrm{ab}} = \widetilde{\chi}_1(\rho)_{d_i(\rho_0)} \oplus \widetilde{\chi}_2(\rho)_{d_i(\rho_0)}$, and so

$$\big((\rho/\varpi^{d_i(\rho_0)}\rho) \otimes_{\mathcal{O}/\varpi^{d_i(\rho_0)}} \widetilde{\chi}_i(\rho_0)_d\big)^{\mathrm{ab},S} = \widetilde{\chi}_i(\rho)_{d_i(\rho_0)} \otimes_{\mathcal{O}/\varpi^{d_i(\rho_0)}} \widetilde{\chi}_i(\rho_0)_d.$$

Thus we conclude that the action of $G_{\mathbb{Q}_p}$ on the quotient $(\rho/\varpi^{d_i(\rho_0)}\rho)/\widetilde{\chi}_i(\rho)_{d_i(\rho_0)}$ is abelian. This implies that $d_i(\rho_0) \leq d_i(\rho)$ for $i = 1, 2$. However, since $d_1(\rho_0) + d_2(\rho_0) = d_1(\rho) + d_2(\rho) = d$, we conclude that $d_i(\rho_0) = d_i(\rho)$ for $i = 1, 2$, and thus that $\rho_0$ is a scalar multiple of $\rho$. Consequently, $\pi_0$ is a scalar multiple of $\pi$, as claimed. $\square$

Our next result is a variant of the preceding one, in which we work with more general coefficients, but draw a weaker conclusion. To this end we suppose that $A$ is an object of $\mathrm{Comp}(\mathcal{O})$ that is reduced and flat over $\mathcal{O}$, and that $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(A)$ is a continuous representation. If $\mathfrak{p}$ is a point of $\mathrm{Spec}\, A[1/p]$ then we write $\rho(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_A \rho$, where $\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p}$.

We assume that $\rho$ is generically irreducible, i.e. is such that $\rho(\mathfrak{a})$ is irreducible for each minimal prime $\mathfrak{a}$ of $A$. We also suppose given an orthonormalizable admissible representation $\theta$ of $G$ over $A$, with the property that the characters of $E \otimes_{\mathcal{O}} \mathrm{MF}(\theta/\mathfrak{p}\theta)$ and $\rho(\mathfrak{p})$ coincide for any closed point $\mathfrak{p} \in \mathrm{Spec}\, A[1/p]$. This implies that, if $\rho(\mathfrak{p})$ is irreducible, then it is isomorphic to $E \otimes_{\mathcal{O}} \mathrm{MF}(\theta/\mathfrak{p}\theta)$. The point of the following proposition is to provide a criterion for such an isomorphism in certain cases when $\rho(\mathfrak{p})$ is reducible.

3.6.4. **Proposition.** *Suppose that $\mathfrak{p}$ is a closed point of $\mathrm{Spec}\, A[1/p]$ with the property that $\rho(\mathfrak{p})^{\mathrm{ss}} = \chi_1 \oplus \chi_2$, where $\chi_1, \chi_2$ are characters $\chi_i : G_{\mathbb{Q}_p} \to \kappa(\mathfrak{p})^\times$ such that $\chi_1\chi_2^{-1} \neq \underline{1}, \varepsilon^{\pm 1}$, and such that $E \otimes_{\mathcal{O}} (\theta/\mathfrak{p}\theta)$ contains no finite-dimensional $G$-invariant subquotients. If there exists a faithful cofinitely generated $A$-module (in the sense of Definition C.1) such that, for each $n > 0$, the action of $G_{\mathbb{Q}_p}$ on the quotient*

$$\frac{(\rho/\mathfrak{p}^n\rho) \otimes_{A/\mathfrak{p}^n} \mathrm{Ord}_B\big((\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y[\mathfrak{p}^n]\big)}{\Big((\rho/\mathfrak{p}^n\rho) \otimes_{A/\mathfrak{p}^n} \mathrm{Ord}_B\big((\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y[\mathfrak{p}^n]\big)\Big)^{\mathrm{ab},S}}$$

*factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$, then $E \otimes_{\mathcal{O}} \mathrm{MF}(\theta/\mathfrak{p}\theta) \cong \rho(\mathfrak{p})$.*

*Proof.* We wish to reduce ourselves to the consideration of lattices in a representation over a discrete valuation ring, as in the context of the preceding proposition. To this end, we choose a finite morphism $A \to A'$, where $A'$ is an object of $\mathrm{Comp}(\mathcal{O})$ that is normal of dimension two, containing a prime $\mathfrak{p}'$ which maps to $\mathfrak{p}$ under the induced map $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A$. We furthermore can and do choose $A'$ so that $\rho' := A' \otimes_A \rho$ is generically irreducible. We write $\theta' := A' \otimes_A \theta$, and note that it suffices to prove that $E \otimes_{\mathcal{O}} \mathrm{MF}(\theta'/\mathfrak{p}'\theta) \cong \kappa(\mathfrak{p}') \otimes_{A'} \rho'$. Indeed, there are natural isomorphisms $E \otimes_{\mathcal{O}} \mathrm{MF}(\theta'/\mathfrak{p}'\theta) \xrightarrow{\sim} \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} \big(E \otimes_{\mathcal{O}} \mathrm{MF}(\theta/\mathfrak{p}\theta)\big)$ and $\kappa(\mathfrak{p}') \otimes_{A'} \rho' \xrightarrow{\sim} \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} \rho(\mathfrak{p})$. Thus $E \otimes_{\mathcal{O}} \mathrm{MF}(\theta/\mathfrak{p}\theta)$ and $\rho(\mathfrak{p})$ are representations of $G_{\mathbb{Q}_p}$ over $\kappa(\mathfrak{p})$ that become isomorphic over $\kappa(\mathfrak{p}')$, and hence are themselves isomorphic.

We now set $Y' := \mathrm{Hom}_A(A', Y)$; note that $Y'$ is cofinitely generated over $A'$, by Proposition C.13 (1). For any $n > 0$, there are natural isomorphisms

$$(\theta'/\mathfrak{p}'^n\theta') \otimes_{A'/\mathfrak{p}'^n} Y'[\mathfrak{p}'^n] \xrightarrow{\sim} (\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y'[\mathfrak{p}^n]$$

and

$$Y'[\mathfrak{p}^n] \xrightarrow{\sim} \mathrm{Hom}_A(A'/\mathfrak{p}'^n, Y[\mathfrak{p}^n]),$$

and hence (e.g. by Lemma 3.1.17, applied with $A$ taken to be $A/\mathfrak{p}^n$, $\pi$ taken to be $\theta/\mathfrak{p}^n\theta$, which is orthonormalizable over $A/\mathfrak{p}^n$ by Lemma B.6 (4), and $M$ taken to be $A'/\mathfrak{p}'^n$; note that there is no need to pass to completed tensor products, since $A/\mathfrak{p}^n$ is finite over $\mathcal{O}$, so that the usual tensor products are already $\varpi$-adically complete) a natural isomorphism

$$(\theta'/\mathfrak{p}'^n\theta') \otimes_{A'/\mathfrak{p}'^n} Y'[\mathfrak{p}'^n] \xrightarrow{\sim} \mathrm{Hom}_A\big(A'/\mathfrak{p}'^n, (\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y[\mathfrak{p}^n]\big).$$

There are evident isomorphisms

$$(\rho'/\mathfrak{p}'^n\rho') \otimes_{A'/\mathfrak{p}'^n} \mathrm{Ord}_B\Big(\mathrm{Hom}_A\big(A'/\mathfrak{p}'^n, (\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y[\mathfrak{p}^n]\big)\Big)$$
$$\xrightarrow{\sim} \mathrm{Hom}_A\Big(A'/\mathfrak{p}'^n, \mathrm{Ord}_B\big((\rho/\mathfrak{p}^n\rho) \otimes_{A/\mathfrak{p}^n} (\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y[\mathfrak{p}^n]\big)\Big)$$

and

$$\Big((\rho'/\mathfrak{p}'^n\rho') \otimes_{A'/\mathfrak{p}'^n} \mathrm{Ord}_B\Big(\mathrm{Hom}_A\big(A'/\mathfrak{p}'^n, (\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y[\mathfrak{p}^n]\big)\Big)\Big)^{\mathrm{ab},S}$$
$$\xrightarrow{\sim} \mathrm{Hom}_A\Big(A'/\mathfrak{p}'^n, \Big((\rho/\mathfrak{p}^n\rho) \otimes_{A/\mathfrak{p}^n} \mathrm{Ord}_B\big((\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y[\mathfrak{p}^n]\big)\Big)^{\mathrm{ab},S}\Big).$$

Since $\mathrm{Hom}_A(A'/\mathfrak{p}'^n, -)$ is a left-exact functor, we deduce from the hypothesis of the theorem that for each $n > 0$, the action of $G_{\mathbb{Q}_p}$ on the quotient

$$\frac{(\rho'/\mathfrak{p}'^n\rho') \otimes_{A'/\mathfrak{p}'^n} \mathrm{Ord}_B\big((\theta'/\mathfrak{p}'^n\theta') \otimes_{A'/\mathfrak{p}'^n} Y'[\mathfrak{p}'^n]\big)}{\Big((\rho'/\mathfrak{p}'^n\rho') \otimes_{A'/\mathfrak{p}'^n} \mathrm{Ord}_B\big((\theta'/\mathfrak{p}'^n\theta') \otimes_{A'/\mathfrak{p}'^n} Y'[\mathfrak{p}'^n]\big)\Big)^{\mathrm{ab},S}}$$

factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$. Note also that since $A'$ is reduced (being normal), the module $Y'$ is faithful over $A'$, by Proposition C.13 (2). Combining this with the conclusion of the preceding paragraph, we see that we may replace $A$, $\mathfrak{p}$, $\rho$, and $\theta$ by $A'$, $\mathfrak{p}'$, $\rho'$, and $\theta'$, and hence assume that $A$ itself is normal of dimension two.

Let $Y_{\mathrm{ctf}}$ denote the maximal cotorsion free cofinitely generated $A$-submodule of $Y$ (as in Definition C.39, with $\Gamma$ taken to be the trivial group). Proposition C.40 shows that $Y_{\mathrm{ctf}}$ is a faithful $A$-module. For any $n > 0$, the embedding $Y_{\mathrm{ctf}}[\mathfrak{p}^n] \hookrightarrow Y[\mathfrak{p}^n]$ induces a map

$$(\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y_{\mathrm{ctf}}[\mathfrak{p}^n] \hookrightarrow (\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y[\mathfrak{p}^n]$$

which is again an embedding (by Lemma B.6 (1), applied to the orthonormalizable $A/\mathfrak{p}^n$-module $\theta/\mathfrak{p}^n\theta$), and it follows that the action of $G_{\mathbb{Q}_p}$ on the quotient

$$\frac{(\rho/\mathfrak{p}^n\rho) \otimes_{A/\mathfrak{p}^n} \mathrm{Ord}_B\big((\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y_{\mathrm{ctf}}[\mathfrak{p}^n]\big)}{\Big((\rho/\mathfrak{p}^n\rho) \otimes_{A/\mathfrak{p}^n} \mathrm{Ord}_B\big((\theta/\mathfrak{p}^n\theta) \otimes_{A/\mathfrak{p}^n} Y_{\mathrm{ctf}}[\mathfrak{p}^n]\big)\Big)^{\mathrm{ab},S}}$$

factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$. Thus, replacing $Y$ by $Y_{\mathrm{ctf}}$, we may assume that $Y$ is cotorsion free. Then, since $Y$ is cotorsion free and the localization $A_{\mathfrak{p}}$ is a discrete valuation ring (being normal and of dimension one), we find that $E \otimes_{\mathcal{O}} Y[\mathfrak{p}^n]$ is free as an $(A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}})$-module, for any $n > 0$. (Indeed, if we write $N := \mathrm{Hom}_{\mathcal{O}}(Y, \mathcal{O})$, then $N$

is torsion free over $A$, by assumption. Applying Proposition C.11, with $X$ taken to be $Y$ and $M$ taken to be $A/\mathfrak{p}^n$, we find that there is an isomorphism

$$(A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}}) \otimes_A N \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}}(E \otimes_{\mathcal{O}} Y[\mathfrak{p}^n], E).$$

Since $A_{\mathfrak{p}} \otimes_A N$ is finitely generated and torsion free, and hence free, over the discrete valuation ring $A_{\mathfrak{p}}$, we see that the source of this isomorphism is free over $A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}}$, and hence so is the source. Passing to $E$-duals, we find that $E \otimes_{\mathcal{O}} Y[\mathfrak{p}^n]$ is itself free over $(A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}})$, as claimed.) Thus we conclude that the $G_{\mathbb{Q}_p}$-action on

$$\frac{(\rho/\mathfrak{p}^n\rho) \otimes_{A/\mathfrak{p}^n} \operatorname{Ord}_B(\theta/\mathfrak{p}^n\theta)}{\big((\rho/\mathfrak{p}^n\rho) \otimes_{A/\mathfrak{p}^n} \operatorname{Ord}_B(\theta/\mathfrak{p}^n\theta)\big)^{\mathrm{ab},S}}$$

factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$, for any $n > 0$.

Let $\widehat{A}_{\mathfrak{p}}$ denote the completion of the localization $A_{\mathfrak{p}}$, let $\mathcal{K}$ denote the fraction field of $\widehat{A}_{\mathfrak{p}}$, and consider $\widehat{\theta} := \widehat{A}_{\mathfrak{p}} \hat{\otimes}_A \theta$, the $\mathfrak{p}$-adic completion of $A_{\mathfrak{p}} \otimes_A \theta$ (which may also be described as the projective limit $\varprojlim_n E \otimes_{\mathcal{O}} (\theta/\mathfrak{p}^n\theta)$), as well as $\mathcal{K} \otimes_{\widehat{A}_{\mathfrak{p}}} \widehat{\theta}$. We are in the context of the discussion at the end of Subsection 3.2, and we observe that $\operatorname{MF}(\widehat{\theta})$ is an $\widehat{A}_{\mathfrak{p}}$-lattice in $\operatorname{MF}(\mathcal{K} \otimes_{\widehat{A}_{\mathfrak{p}}} \widehat{\theta}) \cong \mathcal{K} \otimes_A \rho$. Since $\widehat{A}_{\mathfrak{p}}$ is a complete discrete valuation ring, and since by assumption $\widehat{\theta}/\mathfrak{p}\widehat{\theta} \cong E \otimes_{\mathcal{O}} \theta/\mathfrak{p}\theta$ contains no finite-dimensional $G$-invariant subquotients, we may apply the same argument as that used to prove Proposition 3.6.3 above to conclude that $\operatorname{MF}(\widehat{\theta})$ is a scalar multiple (in $\mathcal{K} \otimes_{\widehat{A}_{\mathfrak{p}}} \rho$) of $\widehat{A}_{\mathfrak{p}} \otimes_A \rho$, and so in particular that $\operatorname{MF}(\widehat{\theta}) \cong \widehat{A}_{\mathfrak{p}} \otimes_A \rho$. Reducing this isomorphism modulo $\mathfrak{p}$, we find that $E \otimes_{\mathcal{O}} \operatorname{MF}(\theta/\mathfrak{p}\theta) \cong \rho(\mathfrak{p})$, as claimed. □

## 3.7. **Deformations.**

In this subsection we prove some technical results which will allow us to construct certain deformations of $G$-representations. These results will be applied only in the proof of Proposition 6.5.11.

We suppose that $A$ is a finite, flat, reduced, local $\mathcal{O}$-algebra, with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ be a minimal prime ideal of $A$ such that $\mathcal{O} \xrightarrow{\sim} A/\mathfrak{p}$, and let $J$ denote the intersection of the minimal primes of $A$ that are distinct from $\mathfrak{p}$. Thus $J$ is a saturated ideal of $A$ (i.e. $A/J$ is flat over $\mathcal{O}$), and $J \cap \mathfrak{p} = 0$. There is a short exact sequence

$$0 \longrightarrow A \xrightarrow{a \mapsto (a \bmod \mathfrak{p}, a \bmod J)} A/\mathfrak{p} \times A/J \xrightarrow{(a_1,a_2) \mapsto a_1 - a_2} A/(\mathfrak{p} + J) \longrightarrow 0. \tag{3.7.1}$$

Let $s > 0$ be such that $\mathcal{O}/\varpi^s\mathcal{O} \xrightarrow{\sim} A/(J + \mathfrak{p})$. If we let $\imath : A/J \to \mathcal{O}/\varpi^s\mathcal{O}$ denote the composition of the surjection $A/J \to A/(J + \mathfrak{p})$ with the inverse of this isomorphism, then a consideration of the short exact sequence (3.7.1) shows that there is an isomorphism

$$A \xrightarrow{\sim} \{(x, y) \in A/J \times \mathcal{O} \mid \imath(x) \equiv y \bmod \varpi^s\}. \tag{3.7.2}$$

We may use this description of $A$ to construct orthonormalizable $A$-modules by gluing together orthonormalizable modules over $\mathcal{O}$ and $A/J$. Suppose to begin with that $X$ is an orthonormalizable $A$-module. Write $X_J := X/JX$ and $X_{\mathfrak{p}} := X/\mathfrak{p}X$. It follows from Lemma B.6 (4) that $X_J$ is an orthonormalizable $A/J$-module, while $X_{\mathfrak{p}}$ is an orthonormalizable $\mathcal{O}$-module. The transitivity of tensor products shows that there are canonical isomorphisms

$$X_J/\mathfrak{p}X_J \xrightarrow{\sim} X/(\mathfrak{p} + J)X$$

and

$$X_{\mathfrak{p}}/\varpi^s X_{\mathfrak{p}} \xrightarrow{\sim} X/(\mathfrak{p}+J)X,$$

and we define

$$\phi : X_{\mathfrak{p}}/\varpi^s X_{\mathfrak{p}} \xrightarrow{\sim} X_J/\mathfrak{p}X_J$$

to be the composite of the second of these with the inverse of the first.

3.7.3. **Lemma.** *There is a natural isomorphism*

$$X \xrightarrow{\sim} \{(u,v) \in X_{\mathfrak{p}} \times X_J \mid \phi(u) = v\}.$$

*Proof.* This follows by tensoring the short exact sequence (3.7.1) with $X$ over $A$, taking into account the flatness of $X$ over $A$ (Part 1 of Lemma B.6). □

Conversely, we now suppose given an orthonormalizable $\mathcal{O}$-module $X_{\mathfrak{p}}$, an orthonormalizable $A/J$-module $X_J$, and an isomorphism

$$\phi : X_{\mathfrak{p}}/\varpi^s X_{\mathfrak{p}} \xrightarrow{\sim} X_J/\mathfrak{p}X_J.$$

Define

$$X := \{(u,v) \in X_{\mathfrak{p}} \times X_J \mid \phi(u) = v\}. \tag{3.7.4}$$

3.7.5. **Lemma.** *The $A$-module $X$ is orthonormalizable, and there are natural isomorphisms $X/\mathfrak{p}X \xrightarrow{\sim} X_{\mathfrak{p}}$ and $X/JX \xrightarrow{\sim} X_J$.*

*Proof.* Choose a basis $\{\overline{e}_i\}_{i\in I}$ of $X_{\mathfrak{p}}/\varpi^s X_{\mathfrak{p}}$, and lift $\{\overline{e}_i\}_{i\in I}$ (resp. $\{\phi(\overline{e}_i\}_{i\in I}$ to an orthonormal basis $\{e_i^{\mathfrak{p}}\}_{i\in I}$ of $X_{\mathfrak{p}}$ (resp. an orthonormal basis $\{e_i^J\}_{i\in I}$ of $X_J$). One easily verifies, taking into account the isomorphism (3.7.2), that $\{(e_i^{\mathfrak{p}}, e_i^J)\}_{i\in I}$ forms an orthonormal basis for $X$. We leave the (easy) verification of the claimed isomorphisms to the reader. □

We now present a variant of the gluing procedure just discussed. To begin with, write $A' := \mathrm{Hom}_{\mathcal{O}}(A,\mathcal{O})$ (and define $(A/J)'$ analogously). The description (3.7.2) of $A$ gives rise to a corresponding description of $A'$. Before stating it, note that

$$\big(\frac{1}{\varpi^s}(A/J)'/(A/J)'\big)[\mathfrak{p}] \xrightarrow{\sim} \big(A/(\mathfrak{p}+J)\big)^\vee \xrightarrow{\sim} \frac{1}{\varpi^s}\mathcal{O}/\mathcal{O}. \tag{3.7.6}$$

The inverse of this isomorphism gives an embedding

$$\jmath : \frac{1}{\varpi^s}\mathcal{O}/\mathcal{O} \xrightarrow{\sim} \big(\frac{1}{\varpi^s}(A/J)'/(A/J)'\big)[\mathfrak{p}] \hookrightarrow \frac{1}{\varpi^s}(A/J)'/(A/J)',$$

and we have the isomorphism

$$A' \xrightarrow{\sim} \{(u',v') \in \frac{1}{\varpi^s}\mathcal{O} \times \frac{1}{\varpi^s}(A/J)' \mid \jmath(u' \bmod \mathcal{O}) \equiv v' \bmod (A/J)'\};$$

if we regard $A$ as a subring of $\mathcal{O}\times A/J$ via (3.7.2), then the pairing between $A$ and $A'$ is defined via

$$\langle (u,v),(u',v')\rangle := \langle u,u'\rangle - \langle v,v'\rangle.$$

We may re-express the preceding isomorphism via the following exact sequence:

$$0 \longrightarrow A' \longrightarrow \frac{1}{\varpi^s}\mathcal{O} \times \frac{1}{\varpi^s}(A/J)' \xrightarrow{(u',v')\mapsto \jmath(u')-v'} \frac{1}{\varpi^s}(A/J)'/(A/J)'. \tag{3.7.7}$$

Suppose now that we are given an orthonormalizable $A/J$-module $X_J$. If we define $V_J := (A/J)' \otimes_{A/J} X_J$, then the isomorphism (3.7.6) gives rise to a natural isomorphism

$$(\frac{1}{\varpi^s}V_J/V_J)[\mathfrak{p}] \xrightarrow{\sim} (\frac{1}{\varpi^s}X_J)/\mathfrak{p}(\frac{1}{\varpi^s}X_J).$$

Thus if we are given an orthonormalizable $\mathcal{O}$-module $X_{\mathfrak{p}}$ and an isomorphism $\phi : X_{\mathfrak{p}}/\varpi^s X_{\mathfrak{p}} \xrightarrow{\sim} X_J/\mathfrak{p}X_J$, we may equally well regard $\phi$ an an isomorphism

$$\phi : \frac{1}{\varpi^s}X_{\mathfrak{p}}/X_{\mathfrak{p}} \xrightarrow{\sim} (\frac{1}{\varpi^s}V_J/V_J)[\mathfrak{p}] \subset \frac{1}{\varpi^s}V_J/V_J.$$

3.7.8. **Lemma.** *In the context of the preceding discussion, if we write*

$$V := \{(u,v) \in \frac{1}{\varpi^s}X_{\mathfrak{p}} \times \frac{1}{\varpi^s}V_J \,|\, \phi(u \bmod X_{\mathfrak{p}}) = v \bmod V_J\},$$

*then there is an isomorphism $A' \otimes_A X \xrightarrow{\sim} V$, for some orthonormalizable $A$-module $X$.*

*Proof.* If we define the $A$-module $X$ via (3.7.4), then Lemma 3.7.5 shows that $X$ is orthonormalizable. Now tensoring $X$ over $A$ with the exact sequence (3.7.7), and taking into account the flatness of $X$ over $A$ (Lemma B.6 (1), and the isomorphisms $X/\mathfrak{p}X \xrightarrow{\sim} X_{\mathfrak{p}}$ and $X/JX \xrightarrow{\sim} X_J$ of Lemma 3.7.5, we obtain an exact sequence

$$0 \to A' \otimes_A X \to \frac{1}{\varpi^s}X_{\mathfrak{p}} \times \frac{1}{\varpi^s}V_J \to \frac{1}{\varpi^s}V_J/V_J,$$

where the third arrow is defined by $(u,v) \mapsto \phi(u) - v$. The lemma follows. □

The next result gives a characterization of those $A$-modules of the type constructed in the preceding lemma. Before stating it, we note that passing to transposes induces a natural isomorphism

$$A = \mathrm{Hom}_A(A,A) \xrightarrow{\sim} \mathrm{Hom}_A(A',A').$$

Thus if $X$ is any orthonormalizable $A$-module, it follows from Lemma B.6 (3) that there is a natural isomorphism

$$X \xrightarrow{\sim} \mathrm{Hom}_A(A', A' \otimes_A X). \tag{3.7.9}$$

3.7.10. **Lemma.** *If $V$ is an $A$-module, then the following are equivalent:*

(1) *There exists an orthonormalizable $A$-module $X$ and an isomorphism*

$$A' \otimes_A X \xrightarrow{\sim} V.$$

(2) $\mathrm{Hom}_A(A',V)$ *is an orthonormalizable $A$-module, and the natural map*

$$A' \otimes_A \mathrm{Hom}_A(A',V) \to V$$

*is an isomorphism.*

*Furthermore, any isomorphism as in* 1 *induces an isomorphism*

$$X \xrightarrow{\sim} \mathrm{Hom}_A(A',V).$$

*Proof.* Obviously 2 implies 1. Conversely, any isomorphism as in 1 induces the second in the sequence of isomorphisms

$$X \xrightarrow{\sim} \mathrm{Hom}_A(A', A' \otimes_A X) \xrightarrow{\sim} \mathrm{Hom}(A',V),$$

the first being the isomorphism (3.7.9). Thus 1 implies 2, and the final statement of the lemma also holds. □

We are finally ready to explain how the previous results can be used to construct certain deformations of $G$-representations.

3.7.11. **Proposition.** *Let $V$ be a $G$-representation on an $\mathfrak{m}$-adically complete $A$-module, and let $\overline{\pi}$ be a semi-simple object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$. Suppose that:*

(1) *There exists a deformation* $X_J$ *of* $\overline{\pi}$ *over* $A/J$ *such that* $(A/J)' \otimes_{A/J} X_J \xrightarrow{\sim} V[J]$.
(2) $V[\mathfrak{p}]$ *is a deformation of* $\overline{\pi}$ *over* $\mathcal{O}$.
(3) *For each Serre weight* $U$*, the embedding* $\overline{\pi} \xrightarrow{\sim} (V_J/\varpi V_J)[\mathfrak{m}] \hookrightarrow (V/\varpi V)[\mathfrak{m}]$ *induces an isomorphism*

$$\mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(U, \overline{\pi}) \xrightarrow{\sim} \mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}\big(U, (V_J/\varpi V_J)[\mathfrak{m}]\big).$$

(*We refer to Definition* 3.5.1 *for the definition of Serre weight, to Definition* 3.5.2 *for the definition of* $\mathcal{H}(U)$*, and to Remark* 3.5.3 *for the definition of the action of* $\mathcal{H}(U)$ *on the functor* $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(U, -)$*. If* $W$ *is a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation, then we write* $\mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(U, W)$ *to denote the socle of* $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(U, W)$ *as a* $\mathcal{H}(U)$*-representation.*)

*Then there is a deformation* $X$ *of* $\overline{\pi}$ *over* $A$ *and a* $G$*-equivariant* $A$*-linear isomorphism* $A' \otimes_A X \xrightarrow{\sim} V$.

*Proof.* Write $C$ to denote the cokernel of the inclusion $V[J] \hookrightarrow V$, so that we have a short exact sequence

$$0 \to V[J] \to V \to C \to 0 \tag{3.7.12}$$

of $A[G]$-modules. Since

$$\mathcal{O}/\varpi^r \xrightarrow{\sim} A/(\mathfrak{p} + J),$$

it follows that the class in $\mathrm{Ext}^1_{A[G]}(C, V[J])$ represented by (3.7.12) is annihilated by multiplication by $\varpi^s$, and hence that there is a map of $A[G]$-modules $\phi : C \to L[J]/\varpi^s L[J]$ such that (3.7.12) is obtained by pulling back the short exact sequence

$$0 \to V[J] \to \frac{1}{\varpi^s} V[J] \to \frac{1}{\varpi^s} V[J]/V[J] \to 0 \tag{3.7.13}$$

via $\phi$.

Let $K$ denote the kernel of $\phi$. Note that $K$ may also be described as the maximal submodule of $C$ such that pull-back of (3.7.12) along the inclusion $K \hookrightarrow C$ splits; this alternate description of $K$ shows that surjection $V \to C$ restricts to an isomorphism

$$V[\mathfrak{p}] \xrightarrow{\sim} K. \tag{3.7.14}$$

Choose $t$ to be maximal such that $K \subset \varpi^t C$; note that $\varpi^s C \subset K$, and thus that $s \geq t$. There is an embedding $\phi' : \varpi^{-t} K/K \hookrightarrow V[J]/\varpi^t V[J]$, defined as the composite

$$\varpi^{-t} K/K \xrightarrow{\sim} (C/K)[\varpi^t] \overset{\phi}{\hookrightarrow} (V[J]/\varpi^s V[J])[\varpi^t] \\ \xrightarrow{\sim} \varpi^{s-t} V[J]/\varpi^s V[J] \xrightarrow{\sim} V[J]/\varpi^t V[J].$$

Pulling back (3.7.12) via the inclusion $\varpi^{-t} K \hookrightarrow C$ yields a short exact sequence

$$0 \to V[J] \to W \to \varpi^{-t} K \to 0, \tag{3.7.15}$$

obtained by pulling back the short exact sequence (3.7.13) along the embedding $\phi'$.

Suppose now that $s > t$. We claim that $(W/\varpi W)[\mathfrak{m}] \xrightarrow{\sim} \overline{\pi} \oplus \overline{\pi}$, and (consequently) that the exact sequence

$$0 \to (V[J]/\varpi V[J])[\mathfrak{m}] \to (W/\varpi W)[\mathfrak{m}] \to (\varpi^{-t} K/\varpi^{-t+1} K)$$

is then in fact short exact (i.e. exact on the right as well), and split.

To see this, suppose first that $t = 0$. In this case (3.7.15) is split, and the claim is evident. Suppose next that $s > t > 0$. Write

$$B \xrightarrow{\sim} \{(x, y) \in A/J \times \mathcal{O} \mid \imath(x) \equiv y \bmod \varpi^t\}. \tag{3.7.16}$$

It follows from Lemma 3.7.8 that there is an orthonormalizable $B$-module $Y$ such that $B' \otimes_B Y \xrightarrow{\sim} W$. We then compute that

$$(W/\varpi W)[\mathfrak{m}] \xrightarrow{\sim} (B/\mathfrak{m}B)^\vee \otimes_k (Y/\mathfrak{m}Y).$$

Since $s > t$, there is an isomorphism $B/\mathfrak{m}B \xrightarrow{\sim} k \oplus k$. On the other hand,

$$Y/\mathfrak{m}Y \xrightarrow{\sim} (Y/J)/\mathfrak{m}(Y/J) \xrightarrow{\sim} X_J/\mathfrak{m}X_J \xrightarrow{\sim} \overline{\pi}.$$

This proves the claim when $t > 0$.

Continuing to suppose that $s > t$, we see that by virtue of our choice of $t$, the map $(\varpi^{-t}K/\varpi^{-t+1}K) \to C/\varpi C$ is not identically zero, and thus we find that the image of the natural map $(W/\varpi W)[\mathfrak{m}] \to (V/\varpi V)[\mathfrak{m}]$ has a non-zero projection into $C/\varpi C$. Since $\overline{\pi}$ is semi-simple as a $G$-representation, so is $(W/\varpi W)[\mathfrak{m}]$, and thus so is its image in $(V/\varpi V)[\mathfrak{m}]$. Thus this image contains an irreducible $G$-subrepresentation, say $Y$, which embeds into $C/\varpi C$. If $U$ is a Serre weight lying in the $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of $Y$, then the composite

$$\mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(U, Y) \hookrightarrow \mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}\big(U, (V/\varpi V)[\mathfrak{m}]\big) \\ \to \mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(U, C/\varpi C)$$

is injective, and consequently $\mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(U, Y)$ does not lie in the image of the embedding

$$\mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}\big(U, (V_J/\varpi V_J)[\mathfrak{m}]\big) \hookrightarrow \mathrm{soc}_{\mathcal{H}(U)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}\big(U, (V/\varpi V)[\mathfrak{m}]\big).$$

This contradicts the third hypothesis in the statement of the proposition, and hence we conclude that necessarily $s = t$. Thus $K = \varpi^s C$, hence $\varpi^{-s}K = C$ and $\phi' = \phi$, and so in fact $W = V$. The proposition now follows from Lemma 3.7.8. ☐

## 4. Local Langlands at primes $\ell \neq p$

For any finite set $\Sigma_0$ of primes distinct from $p$, write $G_{\Sigma_0} := \prod_{\ell \in \Sigma_0} \mathrm{GL}_2(\mathbb{Q}_\ell)$. This section is devoted to the representation theory of the groups $G_{\Sigma_0}$ with $p$-adic and mod $p$ coefficients, or more generally with coefficients in objects of $\mathrm{Comp}(\mathcal{O})$. More precisely, after recalling the definition of the Kirillov functor, we describe various aspects of the local Langlands correspondence in these contexts. Our main reference is the paper [43].

### 4.1. The Kirillov functor.

In this subsection we recall the definition of the Kirillov functor on smooth $G_{\Sigma_0}$-representations.

**4.1.1. Definition.** Write

$$P_0 := \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \prod_{\ell \in \Sigma_0} \mathbb{Z}_\ell^\times, b \in \prod_{\ell \in \Sigma_0} \mathbb{Z}_\ell\} \subset \prod_{\ell \in \Sigma_0} \mathrm{GL}_2(\mathbb{Z}_\ell) \subset G_{\Sigma_0}.$$

For each $\ell \in \Sigma_0$, define the operator $U_\ell \in \mathcal{O}[\mathrm{GL}_2(\mathbb{Q}_\ell)] \subset \mathcal{O}[G_{\Sigma_0}]$ via the formula

$$U_\ell = \sum_{i=0}^{\ell-1} \begin{pmatrix} \ell & i \\ 0 & 1 \end{pmatrix}.$$

If $X$ is a smooth representation of $G_{\Sigma_0}$ over an object $A$ of $\mathrm{Comp}(\mathcal{O})$, then one easily checks that the operators $U_\ell$ preserve the $A$-submodule of $P_0$-invariant vectors $X^{P_0}$. Define

$$F_{\Sigma_0}(X) = \{x \in X^{P_0} \mid U_\ell x = 0 \text{ for each } \ell \in \Sigma_0\} \subset X^{P_0}.$$

The formation of $F_{\Sigma_0}(X)$ is evidently functorial in $X$ (with values in the category of $A$-modules), and we refer to $F_{\Sigma_0}$ as the Kirillov functor.

The following result summarizes the key properties of $F_{\Sigma_0}$.

4.1.2. **Theorem.** (1) *The functor $F_{\Sigma_0}$ is exact on the category of smooth $G_{\Sigma_0}$-representations over $A$.*
(2) *If $X$ is finitely generated over $A[G]$, then $F_{\Sigma_0}(X)$ is finitely generated over $A$.*
(3) *If $X$ is an absolutely irreducible representation of $G_{\Sigma_0}$ over $k$, then $F_{\Sigma_0}(X)$ is at most one-dimensional over $k$.*

4.1.3. **Definition.** We say that a representation $X$ of $G_{\Sigma_0}$ over $k$ is generic if $X$ contains no non-zero subrepresentations $W$ for which $F_{\Sigma_0}(W) = 0$.

4.1.4. **Lemma.** *Let $f : W \to X$ be a $G_{\Sigma_0}$-equivariant map from one smooth admissible $G_{\Sigma_0}$-representation over $k$ to another. Suppose that:*
(1) *$W$ is generic.*
(2) *The map $f$ induces an injection $F_{\Sigma_0}(W) \hookrightarrow F_{\Sigma_0}(X)$ after applying the Kirillov functor $F_{\Sigma_0}$.*

*Then $F_{\Sigma_0}$ is injective.*

*Proof.* Let $U$ denote the kernel of $f$. Since $F_{\Sigma_0}$ is exact, we see from condition 2 that $F_{\Sigma_0}(U) = 0$. Condition 1 then implies that $U = 0$, as required. □

4.2. **Classical local Langlands for** $\mathrm{GL}_2(\mathbb{Q}_\ell)$**.** Let $A$ be an object of $\mathrm{Comp}(\mathcal{O})$. Suppose that $A$ is furthermore a domain, with field of fractions $\mathcal{K}$, and that $\mathcal{K}$ is of characteristic zero. Let $\ell \neq p$ be prime, and consider a representation $\rho_\ell : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\mathcal{K})$ which is continuous, in the sense that it is obtained from a continuous representation $G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(A)$ (the target being equipped with its $\mathfrak{m}$-adic topology) via extension of scalars. In [43] we define a local Langlands correspondence $\rho_\ell \mapsto \pi(\rho_\ell)$, where $\pi(\rho_\ell)$ is an admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ over $\mathcal{K}$. As usual, this correspondence is defined by first passing from the Galois representation $\rho_\ell$ to a Weil–Deligne representation (see [27, §8] and [43]), then Frobenius semi-simplifying, and then applying a suitably normalized version of the local Langlands correspondence relating Frobenius semi-simple Weil–Deligne representations and admissible smooth $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representations. There is one caveat that we should draw attention to, however: in [43] we apply a generic version of this correspondence. As a consequence, $\pi(\rho_\ell)$ is always generic, but in certain situations is not irreducible.

To be clear, let us recall a more explicit description of the correspondence of [43]:

(1) If $\rho_\ell = \chi_1 \bigoplus \chi_2$, labelled so that $\chi_1\chi_2^{-1} \neq |\ \ |_\ell^{-1}$, then

$$\pi(\rho_\ell) := \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \chi_1|\ \ |_\ell \otimes \chi_2.$$

(This representation is irreducible if $\chi_1\chi_2^{-1} \neq |\ \ |_\ell$, and is a non-trivial extension of $\chi_1 \circ \det$ by $(\chi_1 \circ \det) \otimes \mathrm{St}$ otherwise.)

(2) If $\rho_\ell$ is a non-split extension of $\chi_2$ by $\chi_1$, then either $\chi_1 = \chi_2 = \chi$, in which case
$$\pi(\rho_\ell) := \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \chi| \ \ |_\ell \otimes \chi,$$
or else $\chi_1\chi_2^{-1} = | \ \ |_\ell$, in which case
$$\pi(\rho_\ell) := (\chi_1 \circ \det) \otimes \mathrm{St}.$$

(3) If $\rho_\ell$ is absolutely irreducible, then $\pi(\rho_\ell)$ is a cuspidal representation, with central character equal to $\det(\rho_\ell)| \ \ |_\ell$.

Given $\rho_\ell$ as above, we let $\widetilde{\pi}(\rho_\ell)$ denote the smooth contragredient representation to $\pi(\rho_\ell)$.

4.3. **Mod $p$ local Langlands for $\mathrm{GL}_2(\mathbb{Q}_\ell)$.** In [43], we construct a mod $p$ local Langlands correspondence, whose key properties are summarized in the following theorem:

4.3.1. **Theorem.** *There is a map $\overline{\rho}_\ell \mapsto \overline{\pi}(\overline{\rho}_\ell)$ from the set of isomorphism classes of continuous representations $\overline{\rho}_\ell : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(k)$ to the set of isomorphism classes of finite length admissible smooth $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representations satisfying the following conditions:*

(1) *For any $\overline{\rho}_\ell$, the associated $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representation $\overline{\pi}(\overline{\rho}_\ell)$ is generic, in the sense of Definition 4.1.3 (equivalently, $\overline{\pi}(\overline{\rho}_\ell)$ contains no finite-dimensional subrepresentations).*

(2) *If $K$ is any finite extension of $\mathbb{Q}_p$ with residue field $k$, and if $\rho_\ell : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(K)$ is a continuous representation lifting $\overline{\rho}_\ell : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(k)$, then there is a $\mathrm{GL}_2(\mathbb{Q}_\ell)$-invariant integral lattice $\pi(\rho_\ell)^\circ$ contained in $\pi(\rho_\ell)$ (the admissible smooth $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representation over $K$ attached to $\rho_\ell$ via the local Langlands correspondence of the preceding section), whose mod $\varpi$ reduction $\overline{\pi(\rho_\ell)^\circ}$ admits a $\mathrm{GL}_2(\mathbb{Q}_\ell)$-equivariant embedding $\overline{\pi(\rho_\ell)^\circ} \hookrightarrow \overline{\pi}(\overline{\rho}_\ell)$. Furthermore, the lattice $\pi(\rho_\ell)^\circ$ is uniquely determined up to multiplication by an element of $K^\times$.*

(3) *The representation $\overline{\pi}(\overline{\rho})$ is minimal with respect to satisfying conditions (1) and (2), i.e. given any representation $\overline{\pi}$ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ satisfying these two conditions with respect to $\overline{\rho}$, there is a $\mathrm{GL}_2(\mathbb{Q}_\ell)$-equivariant embedding $\overline{\pi}(\overline{\rho}) \hookrightarrow \overline{\pi}$.*

*Furthermore, the correspondence $\overline{\rho}_\ell \mapsto \overline{\pi}(\overline{\rho}_\ell)$ is characterized by these conditions.*

Given $\overline{\rho}_\ell$ as in the statement of the theorem, we let $\widetilde{\overline{\pi}}(\overline{\rho}_\ell)$ denote the smooth contragredient representation to $\overline{\pi}(\overline{\rho}_\ell)$.

4.4. **Local Langlands in $p$-adic families for the groups $G_{\Sigma_0}$.** Let $A$ be an object of $\mathrm{Comp}(\mathcal{O})$ with maximal ideal $\mathfrak{m}$ and suppose that $A$ is reduced and flat over $\mathcal{O}$ (so that every minimal prime of $A$ has residue characteristic zero). For each point $\mathfrak{p} \in \operatorname{Spec} A$, let $\kappa(\mathfrak{p})$ denote the fraction field of $A/\mathfrak{p}$.

Let $\Sigma_0$ be a finite set of primes not containing $p$, and suppose given a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$. If $\mathfrak{p} \in \operatorname{Spec} A$, then we write $\rho(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_A \rho$. In the case of the maximal ideal $\mathfrak{m}$, we also write $\overline{\rho} := \overline{\rho}(\mathfrak{m}) = k \otimes_A \rho$.

In [43] we prove the following result:

4.4.1. **Theorem.** *There is (up to isomorphism) at most one coadmissible (in the sense of Definition C.23) smooth $G_{\Sigma_0} := \prod_{\ell \in \Sigma_0} \mathrm{GL}_2(\mathbb{Q}_\ell)$-representation $X$ satisfying the following conditions:*

(1) *The* $G_{\Sigma_0}$*-representation* $(X/\varpi X)[\mathfrak{m}]$ *is generic, and furthermore the* $k$*-vector space* $F_{\Sigma_0}\big((X/\varpi X)[\mathfrak{m}]\big)$ *is at most one-dimensional.*

(2) *There is a Zariski dense subset of closed points* $\Pi \subset \operatorname{Spec} A[1/p]$*, such that:*
   (a) *For each* $\mathfrak{p} \in \Pi$*, if for some* $\ell \in \Sigma_0$*, the Galois representation* $\rho(\mathfrak{p})_{G_{\mathbb{Q}_\ell}}$ *is a non-generic principal series representation, then* $\mathfrak{p}$ *is a minimal prime of* $A$*.*
   (b) *For each* $\mathfrak{p} \in \Pi$*, there is a* $\kappa(\mathfrak{p})$*-linear,* $G_{\Sigma_0}$*-equivariant isomorphism*
$$E \otimes_{\mathcal{O}} X[\mathfrak{p}] \xrightarrow{\sim} \bigotimes_{\ell \in \Sigma_0} \pi(\rho(\mathfrak{p})_{\,|G_{\mathbb{Q}_\ell}}).$$
   (c) *The closure in* $X$ *(in the sense of Definition* C.28*) of the saturation in* $X$ *(in the sense of Definition* C.6*) of* $\sum_{\mathfrak{p}\in\Pi} X[\mathfrak{p}]$ *coincides with* $X$ *itself.*

*Any such* $X$ *satisfies the following additional conditions:*

(3) $X$ *is cotorsion free as an* $A$*-module (in the sense of Definition* C.37*).*

(4) *For each minimal prime* $\mathfrak{a}$ *of* $A$*, the tensor product* $\kappa(\mathfrak{a})\otimes_A \widetilde{X}$ *is* $\kappa(\mathfrak{a})$*-linearly and* $\mathrm{GL}_2(\mathbb{Q}_\ell)$*-equivariantly isomorphic to* $\otimes_{\ell\in\Sigma_0}\widetilde{\pi}(\rho_\ell(\mathfrak{a}))$*.*

(5) *For every closed point* $\mathfrak{p}$ *of* $\operatorname{Spec} A[1/p]$*, there is a* $G_{\Sigma_0}$*-equivariant embedding*
$$E \otimes_{\mathcal{O}} X[\mathfrak{p}] \hookrightarrow \bigotimes_{\ell \in \Sigma_0} \pi(\rho(\mathfrak{p})_{\,|G_{\mathbb{Q}_\ell}}).$$

4.4.2. **Definition.** If a coadmissible smooth representation $X$ of $G_{\Sigma_0}$ over $A$ satisfying the conditions of Theorem 4.4.1 exists, then we denote it by $\pi_{\Sigma_0}(\rho)$, and we let $\widetilde{\pi}_{\Sigma_0}(\rho)$ denote the smooth contragredient to $\pi_{\Sigma_0}(\rho)$.

In the case when $S = \{\ell\}$ consists of a single prime, we write $\pi_\ell(\rho)$ and $\widetilde{\pi}_\ell(\rho)$ rather than $\pi_{\{\ell\}}(\rho)$ and $\widetilde{\pi}_{\{\ell\}}(\rho)$ (assuming that they exist). In [43] we prove that $\pi_{\Sigma_0}(\rho)$ exists if and only if $\pi_\ell(\rho)$ exists for each $\ell \in \Sigma_0$, and that $\widetilde{\pi}_{\Sigma_0}(\rho)$ may be identified with the maximal $A$-torsion free quotient of $\otimes_{\ell\in\Sigma_0}\widetilde{\pi}_\ell(\rho)$.

The preceding discussion applies in particular to the case when $A = \mathcal{O}$. In this case, we have the following result.

4.4.3. **Proposition.** *Suppose that* $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ *is continuous, and write* $\rho_E := E\otimes_{\mathcal{O}}\rho$ *for the associated* $E$*-valued representation. In this case the admissible smooth* $G_{\Sigma_0}$*-representations* $\pi_{\Sigma_0}(\rho)$ *does exist, and is an admissible smooth* $G_{\Sigma_0}$*-representation over* $\mathcal{O}$*, which is characterized by the following properties:*

(1) *There is an* $E$*-linear,* $G$*-equivariant isomorphism*
$$E \otimes_{\mathcal{O}} \pi_{\Sigma_0}(\rho) \xrightarrow{\sim} \bigotimes_{\ell \in \Sigma_0} \pi(\rho_{E\,|G_{\mathbb{Q}_\ell}})$$
*(where* $\pi(\rho_{E\,|G_{\mathbb{Q}_\ell}})$ *is the* $\mathrm{GL}_2(\mathbb{Q}_\ell)$*-representation over* $E$ *attached to* $\rho_{E\,|G_{\mathbb{Q}_\ell}}$ *via the local Langlands correspondence discussed in Subsection* 4.2*).*

(2) *The quotient* $\pi_{\Sigma_0}(\rho)/\varpi\pi_{\Sigma_0}(\rho)$ *is generic, in the sense of Definition* 4.1.3*.*

*Proof.* For each $\ell \in \Sigma_0$, let $\pi(\rho_{E|G_{\mathbb{Q}_\ell}})^\circ$ denote the $\mathrm{GL}_2(\mathbb{Q}_\ell)$-invariant $\mathcal{O}$-lattice in $\pi(\rho_{E|G_{\mathbb{Q}_\ell}})$ satisfying condition 2 of Theorem 4.3.1, and set
$$\pi_{\Sigma_0}(\rho) := \otimes_{\ell\in\Sigma_0}\pi(\rho_{E\,|G_\ell})^\circ.$$

The representation $\pi_{\Sigma_0}(\rho)$ satisfies the conditions of Theorem 4.4.1 with respect to $\rho$ (taking $\Pi$ to consist of the zero ideal of $\mathcal{O}$). Thus $\pi_{\Sigma_0}(\rho)$ exists. By construction, it satisfies conditions 1 and 2 of the proposition, and is easily seen to be characterized by those conditions. □

## 5. $\varpi$-adically completed cohomology and Hecke algebras

In this section we recall the basic definitions and results related to the $\varpi$-adically completed cohomology of modular curves, and the associated $\varpi$-adically completed Hecke algebras. For further details the reader may consult [36] and [38].

5.1. **$\widehat{H}^1_{\mathcal{O}}$ and some related spaces.** For any compact open subgroup $K_f$ of the adèlic group $\mathrm{GL}_2(\mathbb{A}_f)$, we let $Y(K_f)$ denote the modular curve

$$Y(K_f) := \mathrm{GL}_2(\mathbb{Q})\backslash\big((\mathbb{C}\setminus\mathbb{R})\times \mathrm{GL}_2(\mathbb{A}_f)\big)/K_f.$$

As is well-known, $Y(K_f)$ admits a canonical model as an algebraic curve over $\mathbb{Q}$, which we again denote by $Y(K_f)$. We write

$$H^1(K_f)_A := H^1_{\text{ét}}\big(Y(K_f)_{/\overline{\mathbb{Q}}}, A\big),$$

where the subscript ét denotes étale cohomology, and where $A$ denotes one of $E$, $\mathcal{O}$, or $\mathcal{O}/\varpi^s\mathcal{O}$ for some $s > 0$. (Mostly we will have $s = 1$, in which case $\mathcal{O}/\varpi\mathcal{O} = k$.) There is a natural action of $G_{\mathbb{Q}}$ on $H^1(K_f)_A$.

If $K^p$ is some fixed compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f^p)$, then we write

$$H^1(K^p)_A := \varinjlim_{K_p} H^1(K_pK^p)_A,$$

where the inductive limit is taken over all compact open subgroups $K_p$ of $G := \mathrm{GL}_2(\mathbb{Q}_p)$, and where again $A$ denotes one of $E$, $\mathcal{O}$, or $\mathcal{O}/\varpi^s\mathcal{O}$. There are natural commuting actions of $G$ and $G_{\mathbb{Q}}$ on $H^1(K^p)_A$.

For each $s > 0$ there is a natural $G\times G_{\mathbb{Q}}$-equivariant isomorphism

$$H^1(K^p)_{\mathcal{O}}/\varpi^s H^1(K^p)_{\mathcal{O}} \xrightarrow{\sim} H^1(K^p)_{\mathcal{O}/\varpi^s\mathcal{O}}. \tag{5.1.1}$$

We write

$$\widehat{H}^1(K^p)_{\mathcal{O}} := \varprojlim_s H^1(K^p)_{\mathcal{O}}/\varpi^s H^1(K^p)_{\mathcal{O}} \xrightarrow{\sim} \varprojlim_s H^1(K^p)_{\mathcal{O}/\varpi^s\mathcal{O}}$$

to denote the $\varpi$-adic completion of $H^1(K^p)_{\mathcal{O}}$. Since the formation of $\varpi$-adic completions is functorial, the $G_{\mathbb{Q}}\times G$-action on $\widehat{H}^1(K^p)_{\mathcal{O}}$ extends to a $\varpi$-adically continuous $G_{\mathbb{Q}}\times G$-action on $\widehat{H}^1(K^p)_{\mathcal{O}}$. The $G$-action on $\widehat{H}^1(K^p)_{\mathcal{O}}$ makes it a $\varpi$-adically admissible $G$-representation over $\mathcal{O}$, in the sense of Definition 3.1.1. (See [36].) The isomorphisms (5.1.1) induce a natural $G_{\mathbb{Q}}\times G$-equivariant isomorphism

$$\widehat{H}^1(K^p)_{\mathcal{O}}/\varpi^s\widehat{H}^1(K^p)_{\mathcal{O}} \xrightarrow{\sim} H^1(K^p)_{\mathcal{O}/\varpi^s\mathcal{O}} \tag{5.1.2}$$

for each $s > 0$.

We write

$$\widehat{H}^1(K^p)_E := E\otimes_{\mathcal{O}}\widehat{H}^1(K^p)_{\mathcal{O}}.$$

This is an $E$-Banach space, having $\widehat{H}^1(K^p)_{\mathcal{O}}$ as unit ball, and equipped with a continuous action of $G_{\mathbb{Q}}\times G$. The $G$-action makes it a unitary admissible Banach space representation of $G$ over $E$.

If $A$ is one of $E$ or $\mathcal{O}$, then we write

$$\widehat{H}^1_A := \varinjlim_{K^p} \widehat{H}^1(K^p)_A,$$

and equip it with the $\mathcal{O}$-linear inductive limit topology.[10] It is also equipped with a continuous action of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$. For any $s > 0$, we also write

$$H^1_{\mathcal{O}/\varpi^s\mathcal{O}} := \varinjlim_{K^p} H^1(K^p)_{\mathcal{O}/\varpi^s\mathcal{O}} = \varinjlim_{K_f} H^1(K_f)_{\mathcal{O}/\varpi^s\mathcal{O}}$$

(the first inductive limit being taken over all compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_f^p)$, and the second over all compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_f)$). It is equipped with a smooth action of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$. (The case when $s = 1$, with coefficients $\mathcal{O}/\varpi\mathcal{O} = k$, will be of most interest.) The isomorphism (5.1.2) induces a natural isomorphism

$$\widehat{H}^1_{\mathcal{O}}/\varpi^s \widehat{H}^1_{\mathcal{O}} \xrightarrow{\sim} H^1_{\mathcal{O}/\varpi^s\mathcal{O}},$$

which is $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant.[11]

Suppose that $\Sigma_0$ is a finite set of primes not containing $p$, write $\Sigma := \Sigma_0 \cup \{p\}$, and write $G_{\Sigma_0} := \prod_{\ell \in \Sigma_0} \mathrm{GL}_2(\mathbb{Q}_\ell)$. Let $K_0^{\Sigma} := \prod_{\ell \notin \Sigma} \mathrm{GL}_2(\mathbb{Z}_\ell)$. If $K_{\Sigma_0}$ is an open subgroup of $G_{\Sigma_0}$, then we write

$$\widehat{H}^1(K_{\Sigma_0})_A := \widehat{H}^1(K_{\Sigma_0} K_0^{\Sigma})_A,$$

for $A = E$ or $\mathcal{O}$, and

$$H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O}} := H^1(K_{\Sigma_0} K_0^{\Sigma})_{\mathcal{O}/\varpi^s\mathcal{O}},$$

for any $s > 0$. In particular, when $s = 1$, we have

$$H^1(K_{\Sigma_0})_k := H^1(K_{\Sigma_0} K_0^{\Sigma})_k.$$

We also write

$$\widehat{H}^1_{A,\Sigma} := \varinjlim_{K_{\Sigma_0}} \widehat{H}^1(K_{\Sigma_0})_A,$$

for $A = E$ or $\mathcal{O}$, and

$$H^1_{\mathcal{O}/\varpi^s\mathcal{O},\Sigma} := \varinjlim_{K_{\Sigma_0}} H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O}},$$

for $s > 0$. The modules $\widehat{H}^1_{A,\Sigma}$ are equipped with a continuous action of $G_{\mathbb{Q}} \times G \times G_{\Sigma_0}$, for $A = E$ or $\mathcal{O}$, while the modules $H^1_{\mathcal{O}/\varpi^s\mathcal{O},\Sigma}$ are equipped with a smooth action of the same group.

There are analogues of all the above constructions with compactly supported cohomology in place of usual cohomology; we denote the resulting spaces by including an extra subscript $c$ (as usual), thus $\widehat{H}^1_{c,\mathcal{O}}$, etc. There are also analogues with cohomology in degree zero rather than degree one, which we denote by $\widehat{H}^0_{\mathcal{O}}$, etc. (See [36, §4] and [38, §7.2].)

We finish this section by noting the following lemma (which can already be found in the proof of [38, Thm. 7.10.7]).

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[10] I.e. we take as a basis of neighbourhoods of the origin in $\widehat{H}^1_A$ those $\mathcal{O}$-submodules $U$ such that $U \bigcap \widehat{H}^1(K^p)_A$ is $\varpi$-adically open in $\widehat{H}^1(K^p)_A$ for each tame level $K^p$. When $A = E$, this is the usual locally convex inductive limit topology.

[11] As remarked in Section 2, the $G$ and $\mathrm{GL}_2(\mathbb{A}_f)$-actions that we consider in this paper are obtained from those considered in [16] by applying the automorphism $g \mapsto (g^{-1})^t$.

5.1.3. **Lemma.** *The space $\widehat{H}^1_E$ contains no one-dimensional $G$-subrepresentations.*

*Proof.* If $U$ is a one-dimensional $G$-invariant subspace of $\widehat{H}^1_E$, then cupping $U$ with an appropriately chosen element of $\widehat{H}^0_E$, if necessary, we may assume that the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on $U$ is trivial, and hence in particular that $U$ is locally algebraic. (Cf. the proof of [38, Lem. 7.3.19].) However, it follows from [38, Thm. 7.4.2], together with the fact that the local factor at $p$ of the $\mathrm{GL}_2(\mathbb{A}_f)$-representation attached to a modular newform is generic [51, p. 354], that $\widehat{H}^1_E$ cannot contain a one-dimensional locally algebraic $G$-subrepresentation. This proves the lemma. □

5.1.4. **Remark.** Although we will not need the result in this paper, we note that the proof of the preceding lemma extends to show that $\widehat{H}^1_E$ contains no non-zero finite-dimensional $G$-subrepresentation. (Use the fact that any finite-dimensional representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ over $E$ is locally algebraic, as follows from [78, LG 5.42, Thm. 2].)

5.2. **Hecke algebras.** Fix a compact open subgroup $K^p$ of $\mathrm{GL}_2(\mathbb{A}_f^p)$. If $K_p$ is a compact open subgroup of $G$, then we let $\mathbb{T}(K_pK^p)$ denote the $\mathcal{O}$-algebra of $G_{\mathbb{Q}}\times G$-equivariant endomorphisms of $H^1(K_pK^p)_E$ generated by the Hecke operators $S_\ell$ and $T_\ell$ for those primes $\ell$ distinct from $p$ and unramified in $K^p$. If $K_p' \subset K_p$ is an inclusion of compact open subgroups of $G$, then there is a natural surjection $\mathbb{T}(K_p'K^p) \to \mathbb{T}(K_pK^p)$.

5.2.1. **Definition.** *We define $\mathbb{T}(K^p) := \varprojlim_{K_p} \mathbb{T}(K_pK^p)$, and we equip $\mathbb{T}(K^p)$ with its projective limit topology, each of the $\mathcal{O}$-algebras $\mathbb{T}(K_pK^p)$ being equipped with its $\varpi$-adic topology.*

The $\mathcal{O}$-algebra $\mathbb{T}(K^p)$ is then topologically generated by the Hecke operators $S_\ell$ and $T_\ell$, and acts faithfully on $\widehat{H}^1(K^p)_{\mathcal{O}}$, and so also on $\widehat{H}^1(K^p)_E$.[12]

The $\mathcal{O}$-algebra $\mathbb{T}(K^p)$ is reduced, commutative, and complete with respect to its natural topology, and in fact decomposes as the product of finitely many complete local $\mathcal{O}$-algebras, which are in bijection with the (finitely many) Galois conjugacy classes of systems of Hecke eigenvalues arising from modular forms of level $K^p$ defined over $k$. Furthermore, its natural topology coincides with the product of the $\mathfrak{m}$-adic topologies on each of its finitely many local factors, and so in particular, it is a compact topological ring. (These claims are easily deduced from Definition 5.2.1.)

For any inclusion $K^{p\prime} \subset K^p$, there is induced a continuous morphism $\mathbb{T}(K^{p\prime}) \to \mathbb{T}(K^p)$ taking $S_\ell$ and $T_\ell$ in the source to $S_\ell$ and $T_\ell$ in the target, for $\ell \neq p$ and unramified in $K^{p\prime}$, and compatible with respect to the natural closed embedding $\widehat{H}^1(K_p)_E \hookrightarrow \widehat{H}^1(K^{p\prime})_E$ and the action of $\mathbb{T}(K^p)$ (resp. $\mathbb{T}(K^{p\prime})$) on its source (resp. target).

We now fix a continuous absolutely irreducible representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$, and a finite set $\Sigma_0$ of primes, *not* containing $p$, chosen so that $\overline{\rho}$ is unramified outside $\Sigma := \Sigma_0 \cup \{p\}$. Assume that $\overline{\rho}$ is modular. (As conjectured by Serre, and proved by Khare, Wintenberger, and Kisin [53, 54, 55, 60], this is implied by the

[12]The topological $\mathcal{O}$-algebra $\mathbb{T}(K^p)$ admits the following alternative definition: namely, when regarded as an algebra of endomorphisms of $\widehat{H}^1(K^p)_E$, it coincides with the weakly closed $\mathcal{O}$-algebra of $G_{\mathbb{Q}} \times G$-equivariant endomorphisms of $\widehat{H}^1(K^p)_E$ topologically generated by the Hecke operators $S_\ell$ and $T_\ell$ for those primes $\ell$ distinct from $p$ and unramified in $K^p$. However, we won't need this interpretation in what follows.

*a priori* weaker assumption that $\overline{\rho}$ is odd.) If $K_{\Sigma_0}$ is a compact open subgroup of $G_{\Sigma_0}$, and if $K_0^{\Sigma}$ has the same meaning as in the preceding subsection, then we write $\mathbb{T}(K_{\Sigma_0}) := \mathbb{T}(K_{\Sigma_0}K_0^{\Sigma})$.

5.2.2. **Definition.** We say that a compact open subgroup $K_{\Sigma_0} \subset G_{\Sigma_0}$ is an allowable level for $\overline{\rho}$ if we may find a maximal ideal $\mathfrak{m}$ of $\mathbb{T}(K_{\Sigma_0})$, having residue field $k$, which is associated to $\overline{\rho}$ in the usual sense, namely

$$T_\ell \bmod \mathfrak{m} = \mathrm{trace}\big(\overline{\rho}(\mathrm{Frob}_\ell)\big), \quad \ell S_\ell \bmod \mathfrak{m} = \det\big(\overline{\rho}(\mathrm{Frob}_\ell)\big)$$

for $\ell \notin \Sigma$ (or equivalently, if there exists a newform of tame level $K_{\Sigma_0}$ whose associated Galois representation lifts $\overline{\rho}$).

If $K_{\Sigma_0}$ is an allowable level for $\overline{\rho}$, then we write $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}} := \mathbb{T}(K_{\Sigma_0})_{\mathfrak{m}}$ to denote the completion of $\mathbb{T}(K_{\Sigma_0})$ at $\mathfrak{m}$; note that $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ is simply a direct factor of $\mathbb{T}(K_{\Sigma_0})$, which by construction has residue field $k$. Since $\overline{\rho}$ is assumed to be modular, any sufficiently small $K_{\Sigma_0} \subset G_{\Sigma_0}$ is an allowable level for $\overline{\rho}$.

5.2.3. **Remark.** The preceding claim merits some explanation. Indeed, if *modular* is simply understood to mean that $\overline{\rho}$ arises from *some* modular form, then *a priori* one could think that it might be necessary to enlarge $\Sigma_0$ first. However, it is known that $\overline{\rho}$ arises from a modular form of level equal to the Artin conductor of $\overline{\rho}$, and so in particular of level divisible only by primes in $\Sigma$. (This follows from the proof of Serre's conjecture in [53, 54, 55, 60]. Of course, for $p > 2$, and for most cases when $p = 2$, it was known earlier, under the rubric "the weak Serre's conjecture implies the strong Serre's conjecture" – see [74] and the references therein.)

As is well-known, for any allowable level $K_{\Sigma_0}$, we may construct a deformation $\rho(K_{\Sigma_0}) : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}})$ of $\overline{\rho}$, unramified outside of $\Sigma$, uniquely determined by the requirement

$$\mathrm{trace}\big(\rho(K_{\Sigma_0})(\mathrm{Frob}_\ell)\big) = T_\ell, \quad \det\big(\rho(K_{\Sigma_0})(\mathrm{Frob}_\ell)\big) = S_\ell$$

for $\ell \notin \Sigma$. (For example, for any compact open subgroup $K_p$ of $G$, if we write $\mathbb{T}(K_pK_{\Sigma_0})_{\overline{\rho}} := \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}} \otimes_{\mathbb{T}(K_{\Sigma_0})} \mathbb{T}(K_pK_{\Sigma_0}K_0^{\Sigma})$, then [18, Thm. 3] constructs a deformation $\rho(K_pK_{\Sigma_0}) : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}(K_pK_{\Sigma_0})_{\overline{\rho}})$, whose characteristic polynomial on $\mathrm{Frob}_\ell$ equals $X^2 - T_\ell X + \ell S_\ell$, for $\ell \notin \Sigma$. Since $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}} := \varprojlim_{K_p} \mathbb{T}(K_pK_{\Sigma_0})_{\overline{\rho}}$ (by Definition 5.2.1), we may construct the deformation $\rho(K_{\Sigma_0})$ as the projective limit of the deformations $\rho(K_pK_{\Sigma_0})$.)

Let $R_{\overline{\rho},\Sigma}$ denote the complete local Noetherian $\mathcal{O}$-algebra with residue field $k$ that parametrizes deformations of $\overline{\rho}$, unramified outside of $\Sigma$, over complete local $\mathcal{O}$-algebras. Let $\rho^{\mathrm{u}}$ denote the universal deformation of $\overline{\rho}$ to a continuous representation $\rho^{\mathrm{u}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(R_{\overline{\rho},\Sigma})$. If we write

$$t_\ell := \mathrm{trace}\big(\rho^{\mathrm{u}}(\mathrm{Frob}_\ell)\big) \in R_{\overline{\rho},\Sigma}, \quad s_\ell := \ell^{-1}\det\big(\rho^{\mathrm{u}}(\mathrm{Frob}_\ell)\big) \in R_{\overline{\rho},\Sigma}$$

($\ell \notin \Sigma$), then $R_{\overline{\rho},\Sigma}$ is topologically generated by the elements $t_\ell$ [18]. For each allowable level $K_{\Sigma_0}$, the deformation $\rho(K_{\Sigma_0})$ of $\overline{\rho}$ determines, and is determined by, a local $\mathcal{O}$-algebra homomorphisms $\phi(K_{\Sigma_0}) : R_{\overline{\rho},\Sigma} \to \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$. This latter map is uniquely determined by the requirement that $\phi(K_{\Sigma_0})(t_\ell) = T_\ell$ for $\ell \notin \Sigma$. One has in addition that $\phi(K_{\Sigma_0})(s_\ell) = S_\ell$, and hence that $\phi(K_{\Sigma_0})$ is surjective, since the elements $S_\ell$ and $T_\ell$ topologically generate $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$. (And one concludes that in fact $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ is topologically generated by the elements $T_\ell$ alone; this is well-known, and also follows directly from the results of [18].)

5.2.4. **Lemma.** (1) *If $K'_{\Sigma_0} \subset K_{\Sigma_0}$ is an inclusion of allowable levels, then the induced map $\mathbb{T}(K'_{\Sigma_0})_{\overline{\rho}} \to \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ is a surjection of complete local Noetherian $\mathcal{O}$-algebras.*

(2) *If furthermore $K_{\Sigma_0}$ is a sufficiently small allowable level, then the induced map $\mathbb{T}(K'_{\Sigma_0})_{\overline{\rho}} \to \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ of part 1 is an isomorphism.*

*Proof.* Claim 1 follows from the fact that the image of $\mathbb{T}(K'_{\Sigma_0})_{\overline{\rho}}$ in $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ is closed (since $\mathbb{T}(K'_{\Sigma_0})_{\overline{\rho}}$ is compact) and contains a set of topological generators of $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ (namely, the elements $T_\ell$, for $\ell \notin \Sigma$). (Alternatively, one may regard each of $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ and $\mathbb{T}(K'_{\Sigma_0})_{\overline{\rho}}$ as quotients of $R_{\overline{\rho},\Sigma}$ — see diagram (5.2.6) below.) As for claim 2, recall that, as is explained in [17] and [63], there is a positive integer $N_{\Sigma_0}$, divisible only by the primes in $\Sigma_0$, with the property that for any modular lift $\rho$ of $\overline{\rho}$, unramified away from $\Sigma$, the prime-to-$p$-part of the conductor of $\rho$ divides $N_{\Sigma_0}$. Thinking of both $\mathbb{T}(K'_{\Sigma_0})_{\overline{\rho}}$ and $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ as quotients of $R_{\overline{\rho},\Sigma}$, we thus see that they coincide as quotients of $R_{\overline{\rho},\Sigma}$, provided (for example) that $K_{\Sigma_0}$ is contained in the congruence subgroup of level $N_{\Sigma_0}$. □

5.2.5. **Definition.** We write $\mathbb{T}_{\overline{\rho},\Sigma} = \varprojlim_{K_{\Sigma_0}} \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$, the projective limit being taken over the allowable levels $K_{\Sigma_0}$ of $G_{\Sigma_0}$. If $\ell \notin \Sigma$, we write $T_\ell$ to denote the element of $\mathbb{T}_{\overline{\rho},\Sigma}$ uniquely determined by the requirement that it map to the element $T_\ell$ in each of the algebras $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$.

Part 1 of Lemma 5.2.4 shows that the transition maps in the projective limit defining $\mathbb{T}_{\overline{\rho},\Sigma}$ are surjective, while part 2 of the lemma shows that they are eventually isomorphisms. In particular, we see that $\mathbb{T}_{\overline{\rho},\Sigma}$ is a complete local Noetherian $\mathcal{O}$-algebra.

If $K'_{\Sigma_0} \subset K_{\Sigma_0}$ is an inclusion of allowable levels, then we obtain a commutative diagram

(5.2.6)
$$\begin{array}{ccc} R_{\overline{\rho},\Sigma} & & \\ \downarrow{\scriptstyle \phi(K'_{\Sigma_0})} & \searrow^{\phi(K_{\Sigma_0})} & \\ \mathbb{T}(K'_{\Sigma_0})_{\overline{\rho}} & \longrightarrow & \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}, \end{array}$$

Passing to the limit over all $K_{\Sigma_0}$, we obtain a local surjection of complete local $\mathcal{O}$-algebras $\phi_\Sigma : R_{\overline{\rho},\Sigma} \to \mathbb{T}_{\overline{\rho},\Sigma}$, uniquely characterized by the requirement that $\phi_\Sigma(t_\ell) = T_\ell$ for $\ell \notin \Sigma$.

5.2.7. **Definition.** We let $\rho^{\mathrm{m}}_\Sigma$ denote the deformation of $\overline{\rho}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$ associated to $\phi_\Sigma$.

Referring again to Lemma 5.2.4, one sees that $\rho^{\mathrm{m}}_\Sigma$ coincides with $\rho(K_{\Sigma_0})$ for sufficiently small $K_{\Sigma_0}$.

As usual, for a point $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$, we will write $\kappa(\mathfrak{p})$ to denote the residue field of $\mathfrak{p}$. We recall in particular that if $\mathfrak{p}$ is a closed point of $\operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$, then $\kappa(\mathfrak{p})$ is a finite extension of $E$.

5.2.8. **Definition.** We say that a closed point $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is a classical closed point if the system of Hecke eigenvalues $\operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p] \to \kappa(\mathfrak{p})$ determined by $\mathfrak{p}$ arises from a classical cuspform of weight $k \geq 2$.

5.2.9. **Definition.** For any $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$, we write $\rho(\mathfrak{p})^\circ := \rho_\Sigma^{\mathrm{m}}/\mathfrak{p}\rho_\Sigma^{\mathrm{m}}$, and $\rho(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \rho_\Sigma^{\mathrm{m}} \cong \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}} \rho(\mathfrak{p})^\circ$.

Thus $\rho(\mathfrak{p})$ is a two-dimensional representation of $G_\mathbb{Q}$ over $\kappa(\mathfrak{p})$, and $\rho(\mathfrak{p})^\circ$ is a $G_\mathbb{Q}$-invariant lattice in $\rho(\mathfrak{p})$, free of rank two over the integral domain $\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}$.

### 5.3. The $\overline{\rho}$-part of $\widehat{H}^1_\mathcal{O}$.

We continue to fix $\overline{\rho}$, $\Sigma_0$, and $\Sigma := \Sigma_0 \cup \{p\}$ as in the preceding subsection.

5.3.1. **Definition.** For each allowable level $K_{\Sigma_0} \subset G_{\Sigma_0}$, and for $A = E$ or $\mathcal{O}$, we write $\widehat{H}^1(K_{\Sigma_0})_{A,\overline{\rho}} := \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}} \otimes_{\mathbb{T}(K_{\Sigma_0})} \widehat{H}^1(K_{\Sigma_0})_A$. Similarly, if $K_p$ is a fixed compact open subgroup of $G$, and if $K_f = K_p K_{\Sigma_0} K_0^\Sigma$, then we write $H^1(K_f)_{A,\overline{\rho}} := \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}} \otimes_{\mathbb{T}(K_{\Sigma_0})} H^1(K_f)_A$.

Since $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ is a direct factor of $\mathbb{T}(K_{\Sigma_0})$, we see that $\widehat{H}^1(K_{\Sigma_0})_{A,\overline{\rho}}$ is naturally a direct summand of $\widehat{H}^1(K_{\Sigma_0})_A$. Since the $\mathbb{T}(K_{\Sigma_0})$-action on $\widehat{H}^1(K_{\Sigma_0})_A$ is $G_\mathbb{Q} \times G$-equivariant, we see that $\widehat{H}^1(K_{\Sigma_0})_{A,\overline{\rho}}$ is $G_\mathbb{Q} \times G$-invariant. (Here, as in Definition 5.3.1, $A = E$ or $\mathcal{O}$.) If $\mathfrak{m}$ denotes the maximal ideal in $\mathbb{T}(K_{\Sigma_0})$ corresponding to $\overline{\rho}$ (so that $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ is the $\mathfrak{m}$-adic completion of $\mathbb{T}(K_{\Sigma_0})$), then $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ may also be described as the $\mathfrak{m}$-adic completion of $\widehat{H}^1(\mathcal{O}, K_{\Sigma_0})$.

If $K'_{\Sigma_0} \subset K_{\Sigma_0}$ is an inclusion of allowable levels of $G_{\Sigma_0}$, then the closed embedding $\widehat{H}^1(K_{\Sigma_0})_A \hookrightarrow \widehat{H}^1(K'_{\Sigma_0})_A$ restricts to a closed embedding $\widehat{H}^1(K_{\Sigma_0})_{A,\overline{\rho}} \hookrightarrow \widehat{H}^1(K'_{\Sigma_0})_{A,\overline{\rho}}$ (again for $A = E$ or $\mathcal{O}$).

5.3.2. **Definition.** For $A = E$ or $\mathcal{O}$, write $\widehat{H}^1_{A,\overline{\rho},\Sigma} := \varinjlim_{K_{\Sigma_0}} \widehat{H}^1(K_{\Sigma_0})_{A,\overline{\rho}}$, where the inductive limit is taken over all allowable levels $K_{\Sigma_0} \subset G_{\Sigma_0}$.

One immediately sees that $\widehat{H}^1_{A,\overline{\rho},\Sigma}$ is a $G_\mathbb{Q} \times G \times G_{\Sigma_0}$-invariant direct summand of $\widehat{H}^1_{A,\Sigma}$. The natural surjection $\mathbb{T}_{\overline{\rho},\Sigma} \to \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ (for each allowable level $K_{\Sigma_0} \subset G_{\Sigma_0}$) allows us to regard $\widehat{H}^1(K_{\Sigma_0})_{A,\overline{\rho}}$ as a $\mathbb{T}_{\overline{\rho},\Sigma}$-module, for any allowable level $K_{\Sigma_0} \subset G_{\Sigma_0}$, and this structure is compatible with the embeddings $\widehat{H}^1(K_{\Sigma_0})_{A,\overline{\rho}} \hookrightarrow \widehat{H}^1(K'_{\Sigma_0})_{A,\overline{\rho}}$. Thus $\widehat{H}^1_{A,\overline{\rho},\Sigma}$ is also naturally a $\mathbb{T}_{\overline{\rho},\Sigma}$-module, and the $G_\mathbb{Q} \times G \times G_{\Sigma_0}$-action on $\widehat{H}^1_{A,\overline{\rho},\Sigma}$ is evidently $\mathbb{T}_{\overline{\rho},\Sigma}$-linear.

We also have a mod $\varpi^s$ analogue of the preceding constructions for any $s > 0$, which we denote by $H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}$ (for any allowable level $K_{\Sigma_0} \subset G_{\Sigma_0}$) and $H^1_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho},\Sigma}$, defined in the evident manner. The isomorphism (5.1.2) gives rise to an isomorphism

$$\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^s \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}} \xrightarrow{\sim} H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}, \tag{5.3.3}$$

and hence also an isomorphism

$$\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^s \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma} \xrightarrow{\sim} H^1_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho},\Sigma}, \tag{5.3.4}$$

for each $s > 0$. (In the particular case when $s = 1$, which will be of the most interest, the preceding isomorphism may be written as $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma} \xrightarrow{\sim} H^1_{k,\overline{\rho},\Sigma}$.)

5.3.5. **Lemma.** *For any allowable level $K_{\Sigma_0} \subset G_{\Sigma_0}$ for $\overline{\rho}$, the module $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ is a $\varpi$-adically admissible representation of $G$ over $\mathbb{T}_{\overline{\rho},\Sigma}$.*

*Proof.* Since $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ is a direct summand of $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}}$, it is a $\varpi$-adically admissible representation of $G$ over $\mathcal{O}$. In order to prove the lemma, it then suffices to observe (taking into account the isomorphism (5.3.3)) that for any $i \geq 0$, each element of $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^i \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$ is annihilated by some power of the maximal ideal of $\mathbb{T}_{\overline{\rho},\Sigma}$. □

5.3.6. **Lemma.** *Let $K_{\Sigma_0} \subset G_{\Sigma_0}$ be an allowable level.*

(1) *The natural map*

$$\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}} \to \big(\widehat{H}^1_{E,\overline{\rho},\Sigma}\big)^{K_{\Sigma_0}}$$

*is an isomorphism.*

(2) *If $K_p$ is a compact open subgroup of $G$, then, writing $K_f := K_p K_{\Sigma_0} K_0^{\Sigma}$, the natural map*

$$H^1(K_f)_{E,\overline{\rho}} \to \big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}\big)^{K_p}$$

*is an isomorphism.*

*Proof.* Claim 1 follows from [36, Prop. 2.2.13]. Claim 2 follows from the isomorphism [36, (4.3.4)], taking the representation $W$ appearing there to be the trivial representation. □

5.3.7. **Definition.** If $\tau \in \mathbb{C} \setminus \mathbb{R}$, then let $\mathrm{GL}_2(\mathbb{Q})_\tau$ denote the stabilizer of $\tau$ in $\mathrm{GL}_2(\mathbb{Q})$. We say that a compact open subgroup $K_f$ of $\mathrm{GL}_2(\mathbb{A}_f)$ is neat if the intersection $\mathrm{GL}_2(\mathbb{Q})_\tau \bigcap gK_f g^{-1} = \{1\}$ for each $\tau \in \mathbb{C} \setminus \mathbb{R}$ and each $g \in \mathrm{GL}_2(\mathbb{A}_f)$. (This condition ensures that $\mathrm{GL}_2(\mathbb{Q})$ acts on $(\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/K_f$ without fixed points.)

5.3.8. **Lemma.** *Let $K_{\Sigma_0} \subset G_{\Sigma_0}$ be an allowable level.*

(1) *The natural map*

$$\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}} \to \big(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}\big)^{K_{\Sigma_0}}$$

*is an isomorphism.*

(2) *If $K_p$ is a compact open subgroup of $G$, and if $K_f := K_p K_{\Sigma_0} K_0^{\Sigma}$ is neat, then the natural map*

$$H^1(K_f)_{\mathcal{O},\overline{\rho}} \to \big(\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)^{K_p}$$

*is an isomorphism, as is the natural map*

$$H^1(K_f)_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}} \to \big(H^1_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho},\Sigma}\big)^{K_p K_{\Sigma_0}},$$

*for any $s > 0$.*

*Proof.* We begin by putting ourselves in the situation of 2. Thus we choose a compact open subgroup $K_p \subset G$, and write $K_f := K_p K_{\Sigma_0} K_0^{\Sigma}$. We furthermore assume that $K_f$ is neat. If $K_p' \subset K_p$ and $K_{\Sigma_0}' \subset K_{\Sigma_0}$ are normal open subgroups, and if we write $K_f' := K_p' K_{\Sigma_0}' K_0^{\Sigma}$, then the Hochschild–Serre spectral sequence gives, for any $s > 0$, a short exact sequence

$$0 \to H^1\big(K_f/K_f', H^0(Y(K_f'), \mathcal{O}/\varpi^s\mathcal{O})\big) \to H^1\big(Y(K_f), \mathcal{O}/\varpi^s\mathcal{O}\big) \\ \to H^1\big(Y(K_f'), \mathcal{O}/\varpi^s\mathcal{O}\big)^{K_f} \to 0.$$

(Note that our assumption that $K_f$ is neat implies that $K_f/K'_f$ acts freely on $Y(K'_f)$, so that we in a context to which the Hochschild–Serre spectral sequence applies.) Since $\overline{\rho}$ is assumed to be irreducible, localizing at $\overline{\rho}$ yields an isomorphism

$$H^1(K_f)_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}} \xrightarrow{\sim} \big(H^1(K'_f)_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}\big)^{K_f}. \tag{5.3.9}$$

Passing to the inductive limit over $K'_p$ and $K'_{\Sigma_0}$ in (5.3.9), one obtains the second of the isomorphisms in 2.

Taking $K'_p = K_p$ in (5.3.9), then passing to the inductive limit over $K_p$, and then to the projective limit over $s$, one obtains an isomorphism

$$\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}} \xrightarrow{\sim} \big(\widehat{H}^1(K_{\Sigma'_0})_{\mathcal{O},\overline{\rho}}\big)^{K_{\Sigma_0}}.$$

Passing to the inductive limit over $K'_{\Sigma_0}$ yields the isomorphism of part 1.

Passing to the inductive limit over $K'_p$ in (5.3.9), then to the projective limit over $s$, and then to the inductive limit over $K'_{\Sigma_0}$, one obtains an isomorphism

$$H^1(K_f)_{\mathcal{O},\overline{\rho}} \xrightarrow{\sim} \big(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}\big)^{K_pK_{\Sigma_0}}.$$

In combination with the isomorphism of part 1, this yields the first of the isomorphisms of part 2. □

5.3.10. **Remark.** A lemma of Carayol [17] shows that even when $K_f$ is not neat, claim 2 of the preceding result holds as long as $\overline{\rho}$ is not induced from a character of $G_{\mathbb{Q}(\sqrt{-1})}$ or $G_{\mathbb{Q}(\sqrt{-3})}$.

5.3.11. **Proposition.** *For any $s > 0$, the space $H^1_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho},\Sigma_0}$ is an admissible smooth representation of $G \times G_{\Sigma_0}$ over $\mathcal{O}/\varpi^s\mathcal{O}$.*

*Proof.* This follows from part 2 of the preceding lemma, and the fact that the space $H^1(K_f)_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}$ is finitely generated over $\mathcal{O}/\varpi^s\mathcal{O}$. □

5.3.12. **Remark.** In fact, the preceding proposition holds generally, without localizing at $\overline{\rho}$; i.e. $H^1_{\mathcal{O}/\varpi^s\mathcal{O},\Sigma_0}$ is an admissible smooth representation of $G \times G_{\Sigma_0}$. However, the analogue of Lemma 5.3.8 does not hold in general.

5.3.13. **Proposition.** *If $K_p$ is any compact open subgroup of $G$, then $(\widehat{H}^1_{E,\overline{\rho},\Sigma})^{K_p}$ is finitely generated as an $E[G_{\Sigma_0}]$-module.*

*Proof.* If $\rho$ is any lift of $\overline{\rho}$, then for any prime $\ell \neq p$, the conductor of $\rho$ at $\ell$ is bounded in terms of the conductor of $\overline{\rho}$ of $\ell$ [17, 63]. Thus if we fix some power $p^n$ of $p$, any classical lift $\rho$ of $\overline{\rho}$ that is unramified outside $\Sigma$, and whose conductor at $p$ is bounded by $p^n$, in fact has bounded conductor. It follows that there are only finitely many such non-isomorphic lifts of any given weight $k \geq 2$.

In particular, there are only finitely many lifts $\rho$ of $\overline{\rho}$ that are classical of weight 2, unramified away from $\Sigma$, and for which $\pi(\rho_{|G_{\mathbb{Q}_p}})$ (the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\rho_{|G_{\mathbb{Q}_p}}$ via the classical local Langlands correspondence at $p$) has non-zero $K_p$-invariants. Replacing $E$ by a finite extension, if necessary, we may assume that all such lifts $\rho$ are defined over $E$. It then follows from [38, Thm. 7.4.2] and the main theorem of [16] that there is an isomorphism

$$(\widehat{H}^1_{E,\overline{\rho},\Sigma})^{K_p} \xrightarrow{\sim} \bigoplus_{\rho} \pi_p(\rho_{|G_{\mathbb{Q}_p}})^{K_p} \otimes \pi_{\Sigma_0}(\rho),$$

where the direct sum runs over all such lifts $\rho$. Since the direct sum is finite, since each of the spaces $\pi_p(\rho_{|G_{\mathbb{Q}_p}})^{K_p}$ is finite-dimensional, and since each of the $G_{\Sigma_0}$-representations $\pi_{\Sigma_0}(\rho)$ is finitely generated, the proposition follows. □

5.3.14. **Corollary.** *If $K_p$ is any compact open subgroup of $G$, then $(\widehat{H}^1_{k,\overline{\rho},\Sigma})^{K_p}$ is finitely generated as a $k[G_{\Sigma_0}]$-module.*

*Proof.* It follows from Lemma 5.3.8 that reduction modulo $\varpi$ induces an isomorphism

$$(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})^{K_p}/\varpi(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})^{K_p} \xrightarrow{\sim} (H^1_{k,\overline{\rho},\Sigma})^{K_p}.$$

Thus Proposition 5.3.13 shows that $(H^1_{k,\overline{\rho},\Sigma})^{K_p}$ is the reduction modulo $\varpi$ of a $G_{\Sigma_0}$-invariant $\mathcal{O}$-lattice in a finitely generated admissible smooth representation of $G_{\Sigma_0}$ over $E$. Since any such lattice is finitely generated over $\mathcal{O}[G_{\Sigma_0}]$, we conclude that $(H^1_{k,\overline{\rho},\Sigma})^{K_p}$ is finitely generated over $k[G_{\Sigma_0}]$, as required. □

5.3.15. **Proposition.** *If $K_p$ is a compact open subgroup of $G$, and if $K_{\Sigma_0} \subset G_{\Sigma_0}$ is an allowable level, chosen so that $K_pK_{\Sigma_0}K_0^{\Sigma}$ is neat, then, for each $s > 0$, $H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}$ is injective as a smooth representation of $K_p$ over $\mathcal{O}/\varpi^s\mathcal{O}$.*

*Proof.* Let $M$ be any finitely generated smooth representation of $K_p$ over $\mathcal{O}/\varpi^s\mathcal{O}$, and write $M^\vee := \mathrm{Hom}_{\mathcal{O}/\varpi^s\mathcal{O}}(M, \mathcal{O}/\varpi^s\mathcal{O})$ to denote the Pontrjagin dual of $M$. Since $K_pK_{\Sigma_0}K_0^{\Sigma}$ is neat, the representation $M^\vee$ induces a local system $\mathcal{M}^\vee$ on each of the modular curves $Y(K'_pK_{\Sigma_0}K_0^{\Sigma})$, as $K'_p$ ranges over the normal open subgroups of $K_p$. Since $M$ is finitely generated and smooth, if $K'_p$ is sufficiently small, then $K'_p$ acts trivially on $M$, and so $\mathcal{M}^\vee$ is the constant local system. Thus, if we write

$$H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O}} \xrightarrow{\sim} \varinjlim_{K'_p} H^1(Y(K'_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{O}/\varpi^s\mathcal{O}),$$

where $K'_p$ runs over all sufficiently small normal open subgroups of $K_p$, then we find that

$$\begin{aligned} \text{(5.3.16)} \quad & \mathrm{Hom}_{K_p}\big(M, H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O}}\big) \\ & \xrightarrow{\sim} \varinjlim_{K'_p} \mathrm{Hom}_{K_p}\big(M, H^1(Y(K'_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{O}/\varpi^s\mathcal{O})\big) \\ & \xrightarrow{\sim} \varinjlim_{K'_p} H^1\big(Y(K'_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{M}^\vee\big)^{K_p}. \end{aligned}$$

On the other hand, the Hochschild-Serre spectral sequence yields, after passing to the limit over all $K'_p$, a short exact sequence

$$\begin{aligned} 0 \to \varinjlim_{K'_p} H^1\big(K_p/K'_p, H^0(Y(K_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{M}^\vee)\big) \to H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{M}^\vee\big) \\ \to \varinjlim_{K'_p} H^1\big(Y(K'_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{M}^\vee\big)^{K_p} \to 0, \end{aligned}$$

which, when combined with (5.3.16), may be rewritten as the short exact sequence

$$\begin{aligned} 0 \to \varinjlim_{K'_p} H^1\big(K_p/K'_p, H^0(Y(K_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{M}^\vee)\big) \to H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{M}^\vee\big) \\ \to \mathrm{Hom}_{K_p}\big(M, H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O}}\big) \to 0. \end{aligned}$$

If we localize at the maximal ideal in the Hecke algebra corresponding to $\overline{\rho}$, the $H^0$ terms in this short exact sequence vanish, since $\overline{\rho}$ is absolutely irreducible by assumption, and so we obtain a natural isomorphism

$$H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}^{\vee}\big)_{\overline{\rho}} \xrightarrow{\sim} \mathrm{Hom}_{K_p}\big(M, H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}\big). \tag{5.3.17}$$

Now if $0 \to M_0 \to M_1 \to M_2 \to 0$ is any short exact sequence of finitely generated smooth $K_p$-representations over $\mathcal{O}/\varpi^s\mathcal{O}$, then we obtain a corresponding short exact sequence of sheaves

$$0 \to \mathcal{M}_2^{\vee} \to \mathcal{M}_1^{\vee} \to \mathcal{M}_0^{\vee} \to 0,$$

and hence a long exact sequence of cohomology

$$\begin{aligned} 0 &\to H^0\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_2^{\vee}\big) \to H^0\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_1^{\vee}\big) \\ &\to H^0\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_0^{\vee}\big) \to H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_2^{\vee}\big) \\ &\to H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_1^{\vee}\big) \to H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_0^{\vee}\big) \to 0. \end{aligned}$$

Again, localizing at the maximal ideal associated to $\overline{\rho}$ annihilates the $H^0$ terms, and hence we obtain a short exact sequence

$$\begin{aligned} 0 &\to H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_2^{\vee}\big)_{\overline{\rho}} \to H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_1^{\vee}\big)_{\overline{\rho}} \\ &\to H^1\big(Y(K_pK_{\Sigma_0}K_0^{\Sigma}),\mathcal{M}_0^{\vee}\big)_{\overline{\rho}} \to 0. \end{aligned}$$

The natural isomorphism (5.3.17) then shows that the corresponding exact sequence of Homs

$$\begin{aligned} 0 &\to \mathrm{Hom}_{K_p}\big(M_2, H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}\big) \to \mathrm{Hom}_{K_p}\big(M_1, H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}\big) \\ &\to \mathrm{Hom}_{K_p}\big(M_0, H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}\big) \end{aligned}$$

is also exact on the right. Thus $\mathrm{Hom}_{K_p}\big(-, H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}\big)$ is an exact functor (on the category of finitely generated smooth $K_p$-representations over $\mathcal{O}/\varpi^s\mathcal{O}$, and hence on the category of all smooth $K_p$-representations over $\mathcal{O}/\varpi^s\mathcal{O}$), and so $H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}$ is indeed injective as a smooth $K_p$-representation over $\mathcal{O}$. This completes the proof of the proposition. □

5.3.18. **Remark.** The method of proof of the preceding proposition is a standard technique in the study of the cohomology of sheaves on modular curves, which can be summarized by the statement that $H^1$ becomes an exact functor after localizing at a non-Eisenstein maximal ideal of the Hecke algebra. The interpretation in terms of injectivity is a useful way to express this technique (and its many consequences) via a simple structural statement.

5.3.19. **Corollary.** *If $K_p$ is a pro-$p$ open subgroup of $G$, and if $K_{\Sigma_0} \subset G_{\Sigma_0}$ is an allowable level, chosen so that $K_pK_{\Sigma_0}K_0^{\Sigma}$ is neat, then, for some $r > 0$, there is an isomorphism $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}} \cong \mathcal{C}(K_p,\mathcal{O})^r$ of $\varpi$-adically admissible $K_p$-representations over $\mathcal{O}$.*

*Proof.* It suffices to show that there is an isomorphism of admissible smooth representations of $K_p$ over $\mathcal{O}$,

$$\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^s\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}} \cong \mathcal{C}(K^p,\mathcal{O}/\varpi^s\mathcal{O})^r, \tag{5.3.20}$$

for some $r > 0$, and each $s > 0$. It will then follow that $r$ is independent of $s$, and we can pass to the projective limit in $s$ to establish the isomorphism in the statement of the corollary.

Since $K_p$ is a pro-$p$ group, the completed group ring $(\mathcal{O}/\varpi^s\mathcal{O})[[K_p]]$ is a (non-commutative) local ring. Thus, if $M$ is any non-zero finitely generated projective $(\mathcal{O}/\varpi^s\mathcal{O})[[K_p]]$-module, then $M$ is isomorphic to $(\mathcal{O}/\varpi^s\mathcal{O})[[K_p]]^r$ for some $r > 0$. Dualizing, we find that if $\pi$ is a smooth admissible $K_p$-module over $\mathcal{O}/\varpi^s\mathcal{O}$ that is also injective, then there is an isomorphism $\pi \cong \mathcal{C}(K^p, \mathcal{O}/\varpi^s\mathcal{O})^r$ of admissible smooth representations of $K_p$ over $\mathcal{O}$, for some $r > 0$. Consequently, to obtain the isomorphism (5.3.20), it suffices to show that $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^s\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ is injective as an admissible smooth representation (or equivalently, by [40, Prop. 2.1.6], simply as a smooth representation) of $K_p$ over $\mathcal{O}/\varpi^s\mathcal{O}$. Taking into account the isomorphism (5.3.3), we must equivalently show that $H^1(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}$ is injective, for each $s > 0$. This follows from the preceding proposition. $\square$

5.4. **A density result.** Recall that for any $G$-representation $V$, we write $V_{\mathrm{l.alg}}$ to denote the $G$-subrepresentation of $V$ consisting of locally algebraic vectors, and that $V_{\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ denotes the subspace of $V$ consisting of vectors which are algebraic under the action of $\mathrm{GL}_2(\mathbb{Z}_p)$.

5.4.1. **Proposition.** *If $K_{\Sigma_0} \subset G_{\Sigma_0}$ is an allowable level, then the space of $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors $\big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}\big)_{\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ is dense in $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$.*

*Proof.* Let $K_p$ be a compact open pro-$p$-subgroup of $G$, chosen to be normal in $\mathrm{GL}_2(\mathbb{Z}_p)$, and also chosen so small that $K_pK_{\Sigma_0}K_0^{\Sigma}$ is neat. Corollary 5.3.19 then shows that $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}} \cong \mathcal{C}(K_p, E)^r$ for some $r > 0$, and hence that the topological dual $\widehat{H}^1(K_{\Sigma_0})'_{E,\overline{\rho}}$ is free as a module over $E \otimes_{\mathcal{O}} \mathcal{O}[[K_p]]$.

This implies in turn that $\widehat{H}^1(K_{\Sigma_0})'_{E,\overline{\rho}}$ is projective as an $E \otimes_{\mathcal{O}} \mathcal{O}[[\mathrm{GL}_2(\mathbb{Z}_p)]]$-module. Indeed, we have a natural isomorphism of functors

$$\mathrm{Hom}_{E\otimes_{\mathcal{O}}\mathcal{O}[[\mathrm{GL}_2(\mathbb{Z}_p)]]}\big(\widehat{H}^1(K_{\Sigma_0})'_{E,\overline{\rho}}, -\big) \xrightarrow{\sim} \mathrm{Hom}_{E\otimes_{\mathcal{O}}\mathcal{O}[[K_p]]}\big(\widehat{H}^1(K_{\Sigma_0})'_{E,\overline{\rho}}, -\big)^{\mathrm{GL}_2(\mathbb{Z}_p)/K_p}$$

(where the superscript indicates passage to invariants under the natural action of $\mathrm{GL}_2(\mathbb{Z}_p)/K_p$ on $\mathrm{Hom}_{E\otimes_{\mathcal{O}}\mathcal{O}[[K_p]]}\big(\widehat{H}^1(K_{\Sigma_0})'_{E,\overline{\rho}}, -\big)$), the target of which is exact (by virtue of the freeness of $\widehat{H}^1(K_{\Sigma_0})'_{E,\overline{\rho}}$ over $E \otimes_{\mathcal{O}} \mathcal{O}[[K_p]]$, together with the fact that passage to invariants under the finite group $\mathrm{GL}_2(\mathbb{Z}_p)/K_p$ is exact).

Since any projective module is a direct summand of a free module, undoing the duality we find that $\widehat{H}^1(K_{\Sigma_0})_{E,\rho}$ may be $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariantly embedded as a topological direct summand of $\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p), E\big)^s$ for some $s > 0$, and thus it suffices to show that $\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p), E)\big)_{\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{l.alg}}$ is dense in $\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p), E\big)_{\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{l.alg}}$. The space $\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p), E\big)_{\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ is evidently equal to the space of polynomial functions (with coefficients in $E$) on $\mathrm{GL}_2(\mathbb{Z}_p)$ (regarded as the $\mathbb{Z}_p$-points of the affine group scheme $\mathrm{GL}_2$). Since the theory of Mahler expansions shows that these are dense in the space $\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p), E\big)$, the proposition follows. $\square$

5.4.2. **Remark.** In fact the proceeding result remains true even without localizing at $\overline{\rho}$. That is, for any compact open subgroup $K^p$ of $\mathrm{GL}_2(\mathbb{A}_f^p)$, the space $\widehat{H}^1(K^p)_{E,\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ is dense in the $E$-Banach space $\widehat{H}^1(K^p)_E$. To see this, note that it suffices to prove the proposition after replacing $K^p$ by an open subgroup $K^{p\prime}$, since averaging over a set of coset representatives of $K^{p\prime}$ in $K^p$ induces a continuous projection from $\widehat{H}^1(K^{p\prime})_E$ onto $\widehat{H}^1(K^p)_E$ which commutes with the $G$-action on these spaces (by [36, Prop. 2.2.13] and its proof). Thus we may shrink $K^p$ if necessary so that $K_f := \mathrm{GL}_2(\mathbb{Z}_p)K^p$ is neat. The proof of [36, Thm. 2.1.5] — together with the fact that each modular curve $Y(K_f)$ is homotopic to a finite simplicial complex of dimension one — shows that $\widehat{H}^1(K^p)_E$ fits into a right exact sequence of $\mathrm{GL}_2(\mathbb{Z}_p)$-representations on $E$-Banach spaces

$$\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p),E\big)^r \to \mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p),E\big)^s \to \widehat{H}^1(K^p)_E \to 0$$

(for some $r, s \geq 0$), and the surjection $\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p),E\big)^s \to \widehat{H}^1(K^p)_E$ induces a map

$$\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p),E\big)^s_{\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}} \to \widehat{H}^1(K^p)_{E,\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}.$$

As we already observed in the preceding proof, the space $\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p),E\big)_{\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ is dense in $\mathcal{C}\big(\mathrm{GL}_2(\mathbb{Z}_p),E\big)$, and so it follows that $\widehat{H}^1(K^p)_{E,\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ is indeed dense in $\widehat{H}^1(K^p)_{E,\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$, as claimed.

5.4.3. **Remark.** The space $\widehat{H}^1(K^p)_E$ may be regarded as a cohomological analogue of the space of $p$-adic modular forms of tame level $K_p$ with coefficients in $E$. The preceding result is then analogous to the result of Katz [52, Thm. 2.1], which shows that the space $\bigoplus_{k\geq 0}\mathcal{M}_k(K^p,E)$ is dense in the space of $p$-adic modular forms. (Here $\mathcal{M}_k(K^p,E)$ denotes the space of weight $k$ modular forms of level $\mathrm{GL}_2(\mathbb{Z}_p)K^p$.)

We fix $\overline{\rho}$, $\Sigma_0$, and $\Sigma := \Sigma_0 \cup \{p\}$ as in the preceding subsections.

5.4.4. **Definition.** If $K_{\Sigma_0} \subset G_{\Sigma_0}$ is any allowable level for $\overline{\rho}$, then we let $\mathcal{C}(K_{\Sigma_0})$ denote the subset of closed points $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}[1/p]$ that are classical and whose associated Galois representations are crystalline locally at $p$. Similarly, we let $\mathcal{C}$ denote the subset of closed points $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ that are classical and whose associated Galois representations are crystalline locally at $p$.

5.4.5. **Corollary.** (1) *If $K_{\Sigma_0} \subset G_{\Sigma_0}$ is any allowable level for $\overline{\rho}$, then the direct sum $\bigoplus_{\mathfrak{p}\in\mathcal{C}(K_{\Sigma_0})}\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}]_{\mathrm{l.alg}}$ is dense in $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$.*

(2) *The direct sum $\bigoplus_{\mathfrak{p}\in\mathcal{C}}\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\mathrm{l.alg}}$ is dense in $\widehat{H}^1_{E,\overline{\rho},\Sigma}$.*

*Proof.* Proposition 5.4.1 shows that $\widehat{H}^1(K_{\Sigma_0})_{E,\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ is dense $\widehat{H}^1(K_{\Sigma_0})_E$. Tensoring with $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ over $\mathbb{T}(K_{\Sigma_0})$, we find that $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ is dense in $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$. Thus $E[G]\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$ (i.e. the $E[G]$-representation generated by $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}}$) is also dense in $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$. From [38, Thm. 7.4.2] we deduce that

$$E[G]\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\mathrm{GL}_2(\mathbb{Z}_p)-\mathrm{alg}} = \bigoplus_{\mathfrak{p}}\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}]_{\mathrm{l.alg}},$$

where the sum is taken over all classical closed points $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}$ corresponding to modular forms of weight $k \geq 2$ and prime-to-$p$ conductor. The main result of [75] shows that the set of such $\mathfrak{p}$ is precisely the set $\mathcal{C}(K_{\Sigma_0})$, and so part 1 of the corollary is proved. Part 2 then follows by passing to the inductive limit in $K_{\Sigma_0}$. $\square$

The next corollary is a well-known result, which follows (for example) from the result of Katz recalled in Remark 5.4.3. It is also easily deduced from the preceding corollary.

5.4.6. **Corollary.** *The set* $\mathcal{C}$ *is Zariski dense in* $\operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}$.

*Proof.* Suppose that $t \in \bigcap_{\mathfrak{p}\in\mathcal{C}} \mathfrak{p}$. Then $t$ annihilates $\bigoplus_{\mathfrak{p}\in\mathcal{C}} \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]$. The preceding corollary shows that this subspace of $\widehat{H}^1_{E,\overline{\rho},\Sigma}$ (in fact, even its subspace of locally algebraic vectors) is dense in all of $\widehat{H}^1_{E,\overline{\rho},\Sigma}$. It follows that $t$ thus annihilates all of $\widehat{H}^1_{E,\overline{\rho},\Sigma}$, and hence (since $\mathbb{T}_{\overline{\rho},\Sigma,\overline{\rho}}$ acts faithfully on $\widehat{H}^1_{E,\overline{\rho},\Sigma}$, by virtue of its very definition) that $t = 0$. □

The final result of this subsection gives a technical strengthening of the preceding result. Before stating it, we define an important class of closed points in $\operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$.

5.4.7. **Definition.** We say that a closed point $\mathfrak{p} \in \operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is allowable if it is a classical closed point for which the local representation $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ is crystabelline and absolutely irreducible, and is not exceptional in the sense of Definition 3.3.18.

5.4.8. **Remark.** It is expected (and is a consequence of Tate's conjecture [21]) that in fact every point $\mathfrak{p}$ of $\mathcal{C}$ for which $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible is allowable.

We now state our result.

5.4.9. **Lemma.**
(1) *If* $K_{\Sigma_0} \subset G_{\Sigma_0}$ *is any allowable level for* $\overline{\rho}$, *then the set of allowable points* $\mathfrak{p} \in \operatorname{Spec}\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}[1/p]$ *is Zariski dense in* $\operatorname{Spec}\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$.
(2) *The set of allowable points* $\mathfrak{p} \in \operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ *is Zariski dense in* $\operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}$.

*Proof.* If $\mathfrak{p} \in \mathcal{C}(K_{\Sigma_0})$ is not allowable, then the theory of the eigencurve allows us to write it as the limit of a sequence of points $\mathfrak{p}_n \in \mathcal{C}(K_{\Sigma_0})$ which are allowable. The first claim of the lemma follows from this observation together with the preceding corollary. The second claim follows from the first and the fact that $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}} \xrightarrow{\sim} \mathbb{T}_{\overline{\rho},\Sigma}$ for a sufficiently small choice of $K_{\Sigma_0}$. □

5.5. **The Galois action on** $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$. We continue to fix $\overline{\rho}$, $\Sigma_0$, and $\Sigma := \Sigma_0 \cup \{p\}$ as in the preceding subsections. In this subsection we show that it is possible to "factor out" the Galois action from $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma_0}$. To this end, we define

$$U_\Sigma := \operatorname{Hom}_{\mathbb{T}_{S,\overline{\rho}}[G_{\mathbb{Q}}]}(\rho^{\mathrm{m}}_\Sigma, \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}).$$

There is a natural evaluation map

$$\rho^{\mathrm{m}}_\Sigma \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} U_\Sigma \to \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}. \tag{5.5.1}$$

Reducing each side of this map modulo $\varpi$, and passing to $\mathfrak{m}$-torsion parts, we obtain a map

$$\overline{\rho} \otimes_k (U_\Sigma/\varpi U_\Sigma)[\mathfrak{m}] \to H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]. \tag{5.5.2}$$

5.5.3. **Proposition.** *The evaluation map* (5.5.1) *is an isomorphism.*

*Proof.* If $K_{\Sigma_0}$ is an allowable level for $\overline{\rho}$, then Lemma 5.3.8 provides a natural isomorphism

$$\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}) \xrightarrow{\sim} (\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})^{K_{\Sigma_0}}, \tag{5.5.4}$$

from which we in turn deduce a natural isomorphism

$$U_\Sigma^{K_{\Sigma_0}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}]}\big(\rho_\Sigma^{\mathrm{m}}, \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big). \tag{5.5.5}$$

Passing to $K_{\Sigma_0}$-invariants in the source and target of (5.5.1) thus yields a map

$$\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} U_\Sigma^{K_{\Sigma_0}} \to \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}. \tag{5.5.6}$$

To prove the proposition, it suffices to show that this map is an isomorphism for each allowable level $K_{\Sigma_0}$.

We begin by considering the map

$$E \otimes_{\mathcal{O}} \rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} U_\Sigma^{K_{\Sigma_0}} \to \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}. \tag{5.5.7}$$

obtained by tensoring (5.5.6) with $E$ over $\mathcal{O}$. If $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is any classical closed point, then $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}]_{\mathrm{l.alg}}$ (which the isomorphism (5.5.4) shows is naturally isomorphic to $(\widehat{H}^1_{E,\overline{\rho},\Sigma})^{K_{\Sigma_0}}[\mathfrak{p}]_{\mathrm{l.alg}}$) is isomorphic (as a $G_{\mathbb{Q}}$-representation) to a direct sum of copies of $\rho(\mathfrak{p})$ [38, Thm. 7.4.2]. Thus the map

$$\rho(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} (E \otimes_{\mathcal{O}} U_\Sigma^{K_{\Sigma_0}}[\mathfrak{p}]) \to \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}],$$

obtained by passing to $\mathfrak{p}$-torsion parts in the source and target of (5.5.7), contains $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}]_{\mathrm{l.alg}}$ in its image.

Now $K_{\Sigma_0}$ acts smoothly on $\widehat{H}^1_{E,\overline{\rho},\Sigma}$, and so averaging over $K_{\Sigma_0}$ induces a continuous projection onto the space of $K_{\Sigma_0}$-invariants, which by (5.5.4) we identify with $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$. It thus follows from Corollary 5.4.5 that $\bigoplus_{\mathfrak{p}} \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}]_{\mathrm{l.alg}}$ (where $\mathfrak{p}$ ranges over the classical closed points of $\mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$) is dense in $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}]$, and hence we see that the map (5.5.7) has dense image.

There are natural isomorphisms

$$(U_\Sigma)^{K_{\Sigma_0}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}]}\big(\rho_\Sigma^{\mathrm{m}}, \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big) \xrightarrow{\sim} \big((\rho_\Sigma^{\mathrm{m}})^\vee \otimes_{\mathbb{T}_{S,\overline{\rho}}} \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)^{G_{\mathbb{Q}}}.$$

Since $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ is a $\varpi$-adically admissible representation of $G$ over $\mathcal{O}$, the same is true of $(\rho_\Sigma^{\mathrm{m}})^\vee \otimes_{\mathbb{T}_{S,\overline{\rho}}} \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ (which is isomorphic as a $G$-representation to a direct sum of two copies of $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$), and thus of its closed and saturated subrepresentation $\big((\rho_\Sigma^{\mathrm{m}})^\vee \otimes_{\mathbb{T}_{S,\overline{\rho}}} \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)^{G_{\mathbb{Q}}}$, by [39, Prop. 2.4.13]. Hence $(U_\Sigma)^{K_{\Sigma_0}}$ is an admissible $G$-representation over $\mathcal{O}$, and thus so is $\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} U_\Sigma^{K_{\Sigma_0}}$ (which is isomorphic as a $G$-representation to a direct sum of two copies of $U_\Sigma^{K_{\Sigma_0}}$). Thus (5.5.7) is a continuous $G$-equivariant $E$-linear map between admissible continuous representations of $G$ over $E$, and so necessarily has closed image (as follows from the results of [77]; see [34, Prop. 6.2.9] or Proposition 3.1.3 above for explicit statements). Since we have already seen that the map (5.5.1) has dense image, we conclude that it is in fact surjective.

Thus, in order to show that (5.5.6) is an isomorphism, it suffices, by Lemma 3.1.6, to show that the map

$$\overline{\rho} \otimes_k (U_\Sigma/\varpi U_\Sigma)[\mathfrak{m}] \to H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}], \tag{5.5.8}$$

obtained by reducing (5.5.6) modulo $\varpi$ and passing to $\mathfrak{m}$-torsion parts of the source and target, is injective. A consideration of the isomorphism (5.5.5) shows that there

is a natural embedding
$$(U_{\Sigma}^{K_{\Sigma_0}}/\varpi U_{\Sigma}^{K_{\Sigma_0}})[\mathfrak{m}] \hookrightarrow \mathrm{Hom}_{k[G_{\mathbb{Q}}]}(\overline{\rho}, H^1(K_{\Sigma_0})_{k,\overline{\rho}}),$$
such that the natural evaluation map
$$\overline{\rho} \otimes_k \mathrm{Hom}_{k[G_{\mathbb{Q}}]}(\overline{\rho}, H^1(K_{\Sigma_0})_{k,\overline{\rho}}) \to H^1(K_{\Sigma_0})_{k,\overline{\rho}}$$
extends (5.5.8). The injectivity of (5.5.8) is thus proved once we note that this evaluation map is injective, since $\overline{\rho}$ is an absolutely irreducible $G_{\mathbb{Q}}$-representation. □

5.5.9. **Corollary.** *The evaluation map* (5.5.2) *is an isomorphism.*

*Proof.* This follows by reducing the isomorphism (5.5.1) mod $\varpi$ and passing to $\mathfrak{m}$-torsion parts. □

5.5.10. **Remark.** The analogue of Proposition 5.5.3 for cohomology at finite levels has been proved by Carayol [18, Thm. 4].

5.5.11. **Remark.** In Section 6, we will use a similar method to furthermore "factor out" the $G$-action from $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$ (under some appropriate additional hypotheses on $\overline{\rho}$).

5.6. **Ordinary forms.** We present an interpretation of one of the main results of the theory of $p$-adically ordinary modular forms, namely the reducibility of the associated Galois representations locally at $p$, from the representation-theoretic view-point of this note. (See Remark 5.6.10 below for an indication of the connection between the statements we give and the more traditional formulation in terms of $p$-adically ordinary eigenforms.)

To begin with, suppose that $\pi$ is a smooth representation of $G$ over $E$, which contains a $\varpi$-adically separable $G$-invariant $\mathcal{O}$-submodule $\pi^0$ (more succinctly, a $G$-invariant $\mathcal{O}$-lattice) which is admissible, in the sense that $\pi^0/\varpi\pi^0$ (and hence $\pi^0/\varpi^n\pi^0$ for any $n \geq 0$) is an admissible smooth representation of $G$. Note that $\pi$ itself is then necessarily admissible. Although the $G$-representations $\pi$ or $\pi^0$ do not quite fit into the context considered in [39] (since the coefficients of $\pi$ are $E$, while the $\mathcal{O}$-module $\pi^0$ will typically not be $p$-adically complete), we will begin by observing that the basic definitions of that paper apply to them.

Indeed, if we write $N_0 := \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{Z}_p)$, and then set $T^+ := \{t \in T \, tN_0t^{-1} \subset N_0\}$, then [39, Def. 3.1.3] applies perfectly well to $\pi$ and $\pi_0$, and so we may define the Hecke $T^+$-action on $\pi^{N_0}$ and $\pi_0^{N_0}$ (and, of course, the Hecke $T^+$-action on the latter is the restriction of the Hecke $T^+$-action on the former).

5.6.1. **Remark.** Since $\pi$ is an admissible smooth $G$-representation over $E$, it is in particular an admissible locally analytic $G$-representation over $G$, and so the discussion of [35, §3.4] applies to define an action of $T^+$ on $\pi^{N_0}$. This action is closely related to the Hecke $T^+$-action on $\pi^{N_0}$ defined in the preceding paragraph. Indeed, it differs from it simply by a twist by the modulus character $| \ |_p \otimes | \ |_p^{-1}$. (Since the definition of the $T^+$-action in [35, §3.4] involves an averaging which does not appear in the definition of the Hecke $T^+$-action given by [39, Def. 3.1.3].)

For each integer $r \geq 0$, write $I_r := \left\{ g \in \mathrm{GL}_2(\mathbb{Z}_p) \,|\, g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$. Note that if $t \in T^+$, then $tI_rt^{-1} \subset I_r$ for each $r \geq 0$, and so the Hecke $T^+$-action on $\pi$ (resp. $\pi^{N_0}$)

preserves $\pi^{I_r}$ (resp. $\pi_0^{I_r}$) for each $r \geq 0$. We define $A_r$ to be the image of $\mathcal{O}[T^+]$ in $\mathrm{End}_{\mathcal{O}}(\pi^{I_r})$ under the mapping induced by the Hecke $T^+$-action. Since $A_r$ preserves the separated sublattice $\pi_0^{I_r}$ of the finite-dimensional $E$-vector space $\pi^{I_r}$, we see that $A_r$ is a finite $\mathcal{O}$-algebra, and so there is an isomorphism $A_r \xrightarrow{\sim} A_{r,\mathfrak{m}}$, where $\mathfrak{m}$ ranges over the finitely many maximal ideals of $A_r$. We say that a maximal ideal $\mathfrak{m}$ is ordinary if the image of $T^+$ is disjoint from $\mathfrak{m}$, write $A_{r,\mathrm{ord}} := \prod_{\mathrm{ordinary}\ \mathfrak{m}} A_{r,\mathfrak{m}}$ (a direct factor of $A_r$), and let $e_{r,\mathrm{ord}}$ denote the idempotent in $A$ which projects onto $A_{\mathrm{ord}}$.

5.6.2. **Definition.** Define $\mathrm{Ord}_B(\pi) := \bigcup_r e_{r,\mathrm{ord}}\pi^{I_r}$ and $\mathrm{Ord}_B(\pi_0) := \bigcup_r e_{r,\mathrm{ord}}\pi_0^{I_r}$. Note that $\mathrm{Ord}_B(\pi)$ (resp. $\mathrm{Ord}_B(\pi_0)$) is naturally an $E[T]$-module (resp. an $\mathcal{O}[T]$-module).

Evidently $E \otimes_{\mathcal{O}} \mathrm{Ord}_B(\pi_0) \xrightarrow{\sim} \mathrm{Ord}_B(\pi)$. The following lemma shows that the formation of $\mathrm{Ord}_B$ is similarly compatible with reduction modulo powers of $\varpi$.

5.6.3. **Lemma.** *For any integer $n \geq 0$, the surjection $\pi_0 \to \pi_0/\varpi^n\pi_0$ induces an embedding $\mathrm{Ord}_B(\pi_0)/\varpi^n\mathrm{Ord}_B(\pi_0) \xrightarrow{\sim} \mathrm{Ord}_B(\pi/\varpi^n\pi)$ (where $\mathrm{Ord}_B(\pi_0)$ is defined according to Definition* 5.6.2, *while $\mathrm{Ord}_B(\pi_0/\varpi^n\pi_0)$ is defined according to* [39, Def. 3.1.9]). *If the natural map $\pi_0^{N_0} \to (\pi_0/\varpi^r\pi_0)^{N_0}$ is furthermore a surjection, then this embedding is in fact an isomorphism.*

*Proof.* Clearly

$$\mathrm{Ord}_B(\pi_0)/\varpi^n\mathrm{Ord}_B(\pi_0) \xrightarrow{\sim} \bigcup_r e_{r,\mathrm{ord}}(\pi_0^{I_r}/\varpi^n\pi_0^{I_r}) \tag{5.6.4}$$

(the union taking place in $\pi_0/\varpi^r\pi_0$). The proof of [39, Lem. 3.1.5] provides a natural isomorphism

$$\begin{aligned} e_{r,\mathrm{ord}}(\pi_0^{I_r}/\varpi^r\pi_0^{I_r}) &\xrightarrow{\sim} \mathrm{Hom}_{(\mathcal{O}/\varpi^n\mathcal{O})[T^+]}\big((\mathcal{O}/\varpi^n\mathcal{O})[T], \pi_0^{I_r}/\varpi^r\pi_0^{I_r}\big) \\ &= \mathrm{Hom}_{(\mathcal{O}/\varpi^n\mathcal{O})[T^+]}\big((\mathcal{O}/\varpi^n\mathcal{O})[T], \pi_0^{I_r}/\varpi^r\pi_0^{I_r}\big)_{T-\mathrm{fin}} \end{aligned}$$

for each $r \geq 0$ (where the equality holds since $\pi_0^{I_r}/\varpi^r\pi_0^{I_r}$ is a finite $\mathcal{O}/\varpi^n\mathcal{O}$-module), and thus a natural isomorphism

$$\bigcup_r e_{r,\mathrm{ord}}(\pi_0^{I_r}/\varpi^n\pi_0^{I_r}) \xrightarrow{\sim} \varinjlim_r \mathrm{Hom}_{(\mathcal{O}/\varpi^n\mathcal{O})[T^+]}\big((\mathcal{O}/\varpi^n\mathcal{O})[T], \pi_0^{I_r}/\varpi^r\pi_0^{I_r}\big)_{T-\mathrm{fin}}. \tag{5.6.5}$$

Since $\pi_0^{N_0} = \bigcup_r \pi_0^{I_r}$, and hence $\pi_0^{N_0}/\varpi^n\pi_0^{N_0} = \bigcup_r(\pi_0/\varpi^n\pi_0)^{I_r}$ (the union taking place in $\pi_0/\varpi^r\pi_0$), it follows from [39, Lem. 3.2.2] that

$$\begin{aligned} &\varinjlim_r \mathrm{Hom}_{(\mathcal{O}/\varpi^n\mathcal{O})[T^+]}\big((\mathcal{O}/\varpi^n\mathcal{O})[T], \pi_0^{I_r}/\varpi^r\pi_0^{I_r}\big)_{T-\mathrm{fin}} \\ &\quad\xrightarrow{\sim} \mathrm{Hom}_{(\mathcal{O}/\varpi^n\mathcal{O})[T^+]}\big((\mathcal{O}/\varpi^n\mathcal{O})[T], \pi_0^{N_0}/\varpi^r\pi_0^{N_0}\big)_{T-\mathrm{fin}} \\ &\quad\hookrightarrow \mathrm{Hom}_{(\mathcal{O}/\varpi^n\mathcal{O})[T^+]}\big((\mathcal{O}/\varpi^n\mathcal{O})[T], (\pi_0/\varpi^n\pi_0)^{N_0}\big) =: \mathrm{Ord}_B(\pi_0/\varpi^n\pi_0). \end{aligned} \tag{5.6.6}$$

The isomorphisms (5.6.4) and (5.6.5), together with the embedding (5.6.6), provide the required embedding, which is clearly an isomorphism if $\pi_0^{N_0}/\varpi\pi_0^{N_0} = (\pi_0/\varpi\pi_0)^{N_0}$. □

5.6.7. **Lemma.** *If $\pi$ is absolutely irreducible as a $G$-representation, then* $\mathrm{Ord}_B(\pi)$ *is either zero or one-dimensional. In the latter case, if $\alpha \otimes \beta$ denotes the character through which $T$ acts on* $\mathrm{Ord}_B(\pi)$*, and* $\mathrm{WD}(\pi)$ *denotes the representation of the Weil–Deligne group attached to $\pi$ via the local Langlands correspondence* (*normalized as in Section* 2)*, then there is a Weil–Deligne-equivariant embedding* $\alpha \hookrightarrow \mathrm{WD}(\pi)$ (*where we regard $\alpha$ as a representation of the Weil–Deligne group via the isomorphism* $\mathrm{W}_p^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{Q}_p^\times$ *of local class field theory*)*.*

*Proof.* Write $\pi_{\mathrm{fs}}^{N_0} := E[T] \otimes_{E[T^+]} \pi^{N_0}$. (Here the subscript fs stands for "finite slope".) It follows from the discussion of Remark 5.6.1 together with [35, Prop. 4.3.4] and its proof that the natural projection $\pi \to \pi_N$ induces an isomorphism of $T$-representations $\pi_{\mathrm{fs}}^{N_0} \xrightarrow{\sim} \pi_N \otimes (|\ \ |_p^{-1} \otimes |\ \ |_p)$. (This is a reformulation of Casselman's theory of the canonical lifting of the Jacquet module [19, §4].)

Since $\pi$ is absolutely irreducible, its Jacquet module $\pi_N$ is either zero, one, or two-dimensional, and thus the same is true of $\pi_{\mathrm{fs}}^{N_0}$. Clearly $\mathrm{Ord}_B(\pi)$ is identified with that subrepresentation of $\pi_{\mathrm{fs}}^{N_0}$ on which $T$ acts (possibly after an extension of scalars) via unitary characters (i.e. characters with values in $\mathcal{O}^\times$, rather than merely in $E^\times$). In the case when $\pi_N$ is two-dimensional, if $\chi_1 \otimes \chi_2$ is one character appearing in $\pi_N$, the other is equal to $\chi_2|\ \ |_p \otimes \chi_1|\ \ |_p^{-1}$, and thus the representation $\pi_N \otimes (|\ \ |_p^{-1} \otimes |\ \ |_p)$ can contain at most a one-dimensional subspace on which $T^+$ acts via a unitary character. Thus in all cases $\mathrm{Ord}_B(\pi)$ is at most one-dimensional.

If $\mathrm{Ord}_B(\pi) = \alpha \otimes \beta$ is one-dimensional, then we obtain an embedding $\alpha|\ \ |_p \otimes \beta|\ \ |_p^{-1} \hookrightarrow \pi_N$, and so indeed an embedding $\alpha \hookrightarrow \mathrm{WD}(\pi)$, as claimed. □

In the following lemma, we change our notation. Namely, we will let $\pi$ denote a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ attached to a weight two cuspidal newform defined over $E$, and let $\pi_f$ denote the finite part of $\pi$ (i.e. the restricted tensor product of the local components $\pi_\ell$ for all finite primes $\ell$), which is an admissible smooth representation of $\mathrm{GL}_2(\mathbb{A}_f)$ which we can take to be defined over $E$ [20, Prop. 3.2]. (Here we are implicitly using our chosen identification $\mathcal{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$, as well as our embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$.) If $K^p$ is any compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f^p)$, then $\pi_f^{K^p}$ is an admissible smooth representation of $G$ to which the previous results apply. (Note that $\pi_f^{K^p}$ embeds into $H^1(K_p)_{\mathcal{E}}$, which contains the admissible $G$-invariant $\mathcal{O}$-lattice $H^1(K_p)_{\mathcal{O}}$, and thus $\pi_f^{K^p}$ also contains an admissible $G$-invariant $\mathcal{O}$-lattice.)

5.6.8. **Lemma.** *As in the preceding paragraph, let $\pi$ be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A})$ *attached to a weight two cuspidal newform defined over $E$, so that $\pi_f$ is an admissible smooth representation of* $\mathrm{GL}_2(\mathbb{A}_f)$ *over $E$. If $\rho_\pi : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ denotes the $\varpi$-adic Galois representation attached to $\pi$, then for any compact open subgroup $K^p$ of* $\mathrm{GL}_2(\mathbb{A}_f^p)$*, the action of $G_{\mathbb{Q}_p}$ on the cokernel of the natural embedding*

$$\big(\rho_\pi \otimes_E \mathrm{Ord}_B(\pi_f^{K^p})\big)^{\mathrm{ab},S} \hookrightarrow \rho_\pi \otimes_E \mathrm{Ord}_B(\pi_f^{K^p})$$

*factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$.* (*We refer to Definition* 3.6.2 *for an explanation of the notation* $(-)^{\mathrm{ab},S}$*.*)

*Proof.* Let $\pi_f^p$ denote the restricted tensor product of the local components $\pi_\ell$ (with $\ell \neq p$), so that $\pi_f = \pi_p \otimes_E \pi_f^p$, and hence $\pi_f^{K^p} = \pi_p \otimes (\pi_f^p)^{K_p}$. Evidently, then, we have that $\mathrm{Ord}_B(\pi_f^{K^p}) \xrightarrow{\sim} \mathrm{Ord}_B(\pi_p) \otimes_E (\pi_f^p)^{K^p}$, and so

$$\big(\rho_\pi \otimes_E \mathrm{Ord}_B(\pi_f^{K^p})\big)^{\mathrm{ab},S} \xrightarrow{\sim} \big(\rho_\pi \otimes_E \mathrm{Ord}_B(\pi_p)\big)^{\mathrm{ab},S} \otimes_E (\pi_f^p)^{K^p}.$$

Thus, to prove the lemma, it suffices to prove that the action of $G_{\mathbb{Q}_p}$ on the cokernel of the embedding

$$\big(\rho_\pi \otimes_E \mathrm{Ord}_B(\pi_p)\big)^{\mathrm{ab},S} \hookrightarrow \rho_\pi \otimes_E \mathrm{Ord}_B(\pi_p) \tag{5.6.9}$$

factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$.

Lemma 5.6.7 shows that $\mathrm{Ord}_B(\pi_p)$ is either zero or one-dimensional. If $\mathrm{Ord}_B(\pi_p)$ vanishes, then there is nothing to prove, and so we may suppose that $\mathrm{Ord}_B(\pi_p)$ is one-dimensional, with the $T$-action begin given by the character $\alpha \otimes \beta$. Note that $S$ then acts on $\mathrm{Ord}_B(\pi_p)$ via $\alpha$. It follows from the main result of [75] that, if $\mathrm{WD}(\rho_{\pi\,|G_{\mathbb{Q}_p}})$ denotes the Weil–Deligne representation attached to the potentially semi-stable Dieudonné module of $\rho_{\pi\,|G_{\mathbb{Q}_p}}$, then $\mathrm{WD}(\rho_{\pi\,|G_{\mathbb{Q}_p}})$ and $\mathrm{WD}(\pi_p)$ coincide up to Frobenius semi-simplification, and Lemma 5.6.7 then implies that $\alpha$ is a subrepresentation of $\mathrm{WD}(\rho_{\pi\,|G_{\mathbb{Q}_p}})$. Since $\alpha$ is a unitary (i.e. $\mathcal{O}^\times$-valued) character of $\mathbb{Q}_p^\times$, we see that in fact $\rho_{\pi\,|G_{\mathbb{Q}_p}}$ is reducible, and contains $\alpha$ as a subrepresentation (now thinking of $\alpha$ as a character $G_{\mathbb{Q}_p} \to \mathcal{O}^\times$). Thus $\alpha \otimes_E \mathrm{Ord}_B(\pi_p)$ is a subspace of $\big(\rho_\pi \otimes_E \mathrm{Ord}_B(\pi_p)\big)^{\mathrm{ab},S}$, and hence the cokernel of (5.6.9) is a quotient of $(\rho_\pi/\alpha) \otimes_E \mathrm{Ord}_B(\pi_p)$. Since $\rho_\pi/\alpha$ is a one-dimensional representation of $G_{\mathbb{Q}_p}$, the $G_{\mathbb{Q}_p}$-action on this space certainly factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$. □

5.6.10. **Remark.** One needn't appeal to [75] to deduce the reducibility of $\rho_\pi$ in the preceding argument. Indeed, if we write $J_r := \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix} I_r$, then when restricted to $\pi_p^{J_r}$, the Hecke action of the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ corresponds precisely to the action of the $U_p$-operator on modular forms of level $\Gamma_1(p^r)$ (together with some auxiliary tame level); see e.g. [36, Prop. 4.4.2]. Thus $\mathrm{Ord}_B(\pi_p)$ is non-zero if and only if the newform attached to a twist of $\pi$ is a ordinary at $p$, in the sense that it (or one of its $p$-stabilizations, if its conductor is prime-to-$p$) has a $p$-adic unit $U_p$-eigenvalue.[13] The fact that $\rho_\pi$ is reducible locally at $p$ is then well-known, and is due to Deligne in the case when the conductor of $\pi$ is prime-to-$p$ [28], and to Wiles in general [85, Thm. 2] (building on earlier results of Mazur–Wiles [66, §8, Prop. 1]). Indeed, Lemma 5.6.8 is essentially a reformulation of [85, Thm. 2] (taking into account the fact that we use cohomological conventions for $\rho_\pi$ and geometric conventions in our normalization of the reciprocity maps of class field theory, while in [85] the opposite conventions are used).

5.6.11. **Theorem.** *For any $n \geq 0$, the action of $G_{\mathbb{Q}_p}$ on the cokernel of the embedding*

$$\mathrm{Ord}_B(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^n \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})^{\mathrm{ab},S} \hookrightarrow \mathrm{Ord}_B(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^n \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})$$

*factors through $G_{\mathbb{Q}_p}^{\mathrm{ab}}$. (We refer to Definition 3.6.2 for an explanation of the notation $(-)^{\mathrm{ab},S}$.)*

[13] Thus the theory of ordinary parts [39] is not quite an abstraction of the theory of ordinary modular forms, but rather of what are usually called nearly ordinary modular forms.

*Proof.* Since $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^n\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma} := \varinjlim_{K_{\Sigma_0}} \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^n\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$, where $K_{\Sigma_0}$ runs over all allowable levels for $\overline{\rho}$ in $G_{\Sigma_0}$, it follows from [39, Lem. 3.2.2] that there is a natural isomorphism

$$\varinjlim_{K_{\Sigma_0}} \mathrm{Ord}_B\big(\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^n\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big) \xrightarrow{\sim} \mathrm{Ord}_B(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^n\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}).$$

Thus, to prove the theorem, it suffices to prove that the action of $G_{\mathbb{Q}_p}$ on the cokernel of the embedding

$$\mathrm{Ord}_B\big(\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^n\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)^{\mathrm{ab},S} \hookrightarrow \mathrm{Ord}_B\big(\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^n\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)$$

factors through $K_{\Sigma_0}$, for each allowable level $K_{\Sigma_0}$.

The smooth $G$-representation $H^1(K_{\Sigma_0})_{E,\overline{\rho}}$ contains $H^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ as an admissible $G$-invariant $\mathcal{O}$-lattice, and so Definition 5.6.2 applies to each of $H^1(K_{\Sigma_0})_{E,\overline{\rho}}$ and $H^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$.[14] Lemma 5.3.8 implies that the natural map

$$H^1(K_{\Sigma_0})^{I_r}_{\mathcal{O},\overline{\rho}}/\varpi^n H^1(K_{\Sigma_0})^{I_r}_{\mathcal{O},\overline{\rho}} \to \big(\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^n\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)^{I_r}$$

is an isomorphism for each $r \geq 0$ (indeed, both spaces are naturally identified with $H^1(I_r K_{\Sigma_0} K_0^{\Sigma})_{\mathcal{O}/\varpi^n\mathcal{O},\overline{\rho}}$). Passing to the inductive limit over $r$, we find that the natural map

$$H^1(K_{\Sigma_0})^{N_0}_{\mathcal{O},\overline{\rho}}/\varpi^n H^1(K_{\Sigma_0})^{N_0}_{\mathcal{O},\overline{\rho}} \to \big(\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^n\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)^{N_0}$$

is surjective, and hence, by Lemma 5.6.3, that

$$\mathrm{Ord}_B\big(H^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)/\varpi^n\mathrm{Ord}_B\big(H^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big) \xrightarrow{\sim} \mathrm{Ord}_B\big(\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^n\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big).$$

Consequently, it suffices to prove that $G_{\mathbb{Q}_p}$ acts on the cokernel of the embedding

$$\mathrm{Ord}_B\big(H^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)^{\mathrm{ab},S} \hookrightarrow \mathrm{Ord}_B\big(H^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big)$$

through $G^{\mathrm{ab}}_{\mathbb{Q}_p}$, or equivalently, that $G_{\mathbb{Q}_p}$ acts on the cokernel of the embedding

$$\mathrm{Ord}_B\big(H^1(K_{\Sigma_0})_{E,\overline{\rho}}\big)^{\mathrm{ab},S} \hookrightarrow \mathrm{Ord}_B\big(H^1(K_{\Sigma_0})_{E,\overline{\rho}}\big) \tag{5.6.12}$$

through $G^{\mathrm{ab}}_{\mathbb{Q}_p}$.

We may write $H^1(K_{\Sigma_0})_{E,\overline{\rho}}$ as the inductive limit of its finitely generated $G$-subrepresentations $V$, and so it suffices for each such $V$ to show that the $G_{\mathbb{Q}_p}$-action on the cokernel of $\mathrm{Ord}_B(V)^{\mathrm{ab},S} \hookrightarrow \mathrm{Ord}_B(V)$ factors through $G^{\mathrm{ab}}_{\mathbb{Q}_p}$. Now if we extend scalars sufficiently, we may write

$$V \cong \oplus_\pi \rho_\pi \otimes \pi_f^{K_{\Sigma_0}K_0^{\Sigma_0}},$$

[14] The only reason to cut down to a fixed auxiliary level $K_{\Sigma_0}$ is to be be able to apply the preceding results, which, for the sake of simplicity, were developed in the context of an admissible representation of $G$, rather than in the context of an admissible representation of $\mathrm{GL}_2(\mathbb{A}_f)$.

where $\pi$ ranges over a finite collection of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ corresponding to a cuspidal newforms of weight two.[15] The required property of the cokernel of (5.6.12) is now seen to follow from Lemma 5.6.8. $\square$

5.6.13. **Corollary.** *The action of $G_{\mathbb{Q}_p}$ on the cokernel of the embedding*

$$\mathrm{Ord}_B(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})^{\mathrm{ab},S} \hookrightarrow \mathrm{Ord}_B(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})$$

*factors through $G^{\mathrm{ab}}_{\mathbb{Q}_p}$.*

*Proof.* This follows directly from the preceding theorem, by passing to the projective limit over $n$. $\square$

5.7. **Multiplicities of weights in $H^1_{k,\overline{\rho},\Sigma}$.** We give a representation-theoretic reformulation of the classical "mod $p$ multiplicity one" results related to the appearance of Galois representations in the mod $p$ (or more precisely, in our setting, mod $\varpi$) cohomology of modular curves.

5.7.1. **Definition.** Let $V$ be a Serre weight, as in Definition 3.5.1. We say that $V$ is a global weight of $\overline{\rho}$ if $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma})$ is non-zero. We let $W^{\mathrm{gl}}(\overline{\rho})$ denote the set of global Serre weights of $\overline{\rho}$.

5.7.2. **Remark.** Since every element of the $\mathbb{T}_{\overline{\rho},\Sigma}$-module $H^1_{k,\overline{\rho},\Sigma}$ is annihilated by some power of $\mathfrak{m}$, the same is true of the $\mathbb{T}_{\overline{\rho},\Sigma}$-module $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma})$. Thus $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma})$ is non-zero if and only if $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])\big(= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma})[\mathfrak{m}]\big)$ is non-zero.

We will frequently impose the following hypothesis on $\overline{\rho}$:

5.7.3. **Assumption.** The representation $\overline{\rho}$ is non-scalar locally at $p$, i.e. the restriction $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is not the direct sum of two copies of the same character.

5.7.4. **Remark.** It is known that $W^{\mathrm{gl}}(\overline{\rho}) = W(\overline{\rho})$ (here, to ease notation, we are writing $W(\overline{\rho}) := W(\overline{\rho}_{|G_{\mathbb{Q}_p}})$ to denote the set of Serre weights attached to $\overline{\rho}_{|G_{\mathbb{Q}_p}}$, as in Subsection 3.5); indeed, this is the weight part of Serre's conjecture. (See [14, Thm. 3.15].) However, we will not need this result in this paper (other than in the proof of Theorem 3.3.22); we will we only need the simpler result that $W^{\mathrm{gl}}(\overline{\rho}) \subset W(\overline{\rho})$. For those $\overline{\rho}$ whose restriction to $G_{\mathbb{Q}_p}$ satisfies Assumptions 3.3.1, the results of this paper will then give a new proof that $W^{\mathrm{gl}}(\overline{\rho}) = W(\overline{\rho})$. (See Remark 6.2.15 below.)

If $V \in W^{\mathrm{gl}}(\overline{\rho})$, then Remark 5.7.2 shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ is a non-zero $k$-vector space. It is equipped with commuting actions of $\mathcal{H}(V)$ (see Remark 3.5.3), $G_{\mathbb{Q}}$, and $G_{\Sigma_0}$, and so is an $\mathcal{H}(V)[G_{\mathbb{Q}} \times G_{\Sigma_0}]$-module. We write $\mathrm{soc}_{\mathcal{H}(V)} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ to denote the socle of $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ as an $\mathcal{H}(V)$-module. This is again an $\mathcal{H}(V)[G_{\mathbb{Q}} \times G_{\Sigma_0}]$-module.

[15] The reason for restricting attention from $H^1(K_{\Sigma_0})_{E,\overline{\rho}}$ to its finitely generated subrepresentations is just to be sure that only finitely many $\pi$ contribute to $V \otimes_E \overline{\mathbb{Q}}_p$, so that we may indeed find a common finite extension of $E$ over which they are all defined. This is a purely technical issue, related to the fact that all our definitions have been made in the context of a finite extension of $\mathbb{Q}_p$, rather than (say) in the context of $\overline{\mathbb{Q}}_p$.

5.7.5. **Definition.** We say that a weight $(\mathrm{Sym}^r k^2)^\vee \otimes_k \det^s$ in $W^{\mathrm{gl}}(\overline{\rho})$ (where $0 \leq r \leq p-1$ and $0 \leq s \leq p-2$) is good if either $r < p-1$, or $r = p-1$ and the weight $\det^s$ does not lie in $W^{\mathrm{gl}}(\overline{\rho})$.

5.7.6. **Remark.** Note that $W^{\mathrm{gl}}(\overline{\rho})$ always contains a good weight. Indeed, if $(\mathrm{Sym}^r k^2)^\vee \otimes_k \det^s$ is a weight in $W^{\mathrm{gl}}(\overline{\rho})$ that is not good, then it must be the case that $r = p-1$ and that $\det^s$ is also a weight in $W^{\mathrm{gl}}(\overline{\rho})$; the weight $\det^s$ is then good.

5.7.7. **Theorem.** (1) *$W^{\mathrm{gl}}(\overline{\rho}) \subset W(\overline{\rho})$.*

(2) *If $\overline{\rho}$ satisfies Assumption 5.7.3, if $V \in W^{\mathrm{gl}}(\overline{\rho})$ is a good weight (in the sense of Definition 5.7.5), and if we write $\mathrm{soc}_{\mathcal{H}(V)}\,\mathrm{m}(V,\overline{\rho})$ to denote the $\mathcal{H}(V)$-socle of the $\mathcal{H}(V)$ module $\mathrm{m}(V,\overline{\rho})$ defined in Definition 3.5.4 (to ease notation, we write $\mathrm{m}(V,\overline{\rho})$ rather then $\mathrm{m}(V,\overline{\rho}_{|G_{\mathbb{Q}_p}})$), then there is an isomorphism of $\mathcal{H}(V)[G_{\mathbb{Q}}]$-modules*

$$F_{\Sigma_0}\Big(\mathrm{soc}_{\mathcal{H}(V)}\big(\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])\big)\Big) \cong \overline{\rho} \otimes_k \mathrm{soc}_{\mathcal{H}(V)}\,\mathrm{m}(V,\overline{\rho}).$$

(3) *If $\overline{\rho}$ satisfies Assumption 5.7.3, then for any weight $V \in W^{\mathrm{gl}}(\overline{\rho})$, the $G_{\Sigma_0}$-representation $\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ is generic (in the sense of Definition 4.1.3).*

*Proof.* The proof of part 1 is standard, but for completeness, we recall it here (in a form adapted to the representation-theoretic view-point of this paper). At the same time, we will prove that any simple $\mathcal{H}(V)$-submodule of

$$\mathrm{soc}_{\mathcal{H}(V)}\,\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$$

also appears in $\mathrm{soc}_{\mathcal{H}(V)}\,\mathrm{m}(V,\overline{\rho})$.

First of all, since the formation of $W^{\mathrm{gl}}(\overline{\rho})$ and $W(\overline{\rho})$ are compatible with twisting $\overline{\rho}$, we need only consider the case when $V \in W^{\mathrm{gl}}(\overline{\rho})$ is of the form $(\mathrm{Sym}^r k^2)^\vee$. For any allowable level $K_{\Sigma_0}$ for $\overline{\rho}$, reduction modulo $\varpi$ induces a map of $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}]$-modules

$$\mathrm{Hom}_{\mathcal{O}[\mathrm{GL}_2(\mathbb{Z}_p)]}\big((\mathrm{Sym}^r \mathcal{O}^2)^\vee, \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big) \to \mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}\big(V, H^1(K_{\Sigma_0})_{k,\overline{\rho}}\big), \tag{5.7.8}$$

which, if $K_{\Sigma_0}$ is chosen to be sufficiently small, is surjective, by Proposition 5.3.15. If we write $K_f := \mathrm{GL}_2(\mathbb{Z}_p)K_{\Sigma_0}$, and let $\mathcal{V}_r$ denote the locally constant sheaf on the curve $Y(K_f)$ corresponding to the representation $\mathrm{Sym}^r E^2$ of $\mathrm{GL}_2(\mathbb{Q})$, then there are isomorphisms

$$E \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}[\mathrm{GL}_2(\mathbb{Z}_p)]}\big((\mathrm{Sym}^r \mathcal{O}^2)^\vee, \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}}\big) \xrightarrow{\sim} \mathrm{Hom}_{E[\mathrm{GL}_2(\mathbb{Z}_p)]}\big((\mathrm{Sym}^r E^2)^\vee, \widehat{H}^1(K_{\Sigma_0})_E\big) \xrightarrow{\sim} H^1\big(Y(K_f), \mathcal{V}_r\big)_E,$$

where the last isomorphism follows from [36, (4.3.4)]. Recall from Subsection 3.5 that $\mathcal{H}(V) = k[T, Z, Z^{-1}]$, which we regard as a quotient of the ring $\mathcal{O}[T, Z, Z^{-1}]$ in the evident way. We make $H^1\big(Y(K_f), \mathcal{V}_r\big)$ a module over $\mathcal{O}[T, Z, Z^{-1}]$ by having $T$ act via the classical Hecke operator $T_p$, and $Z$ act via the matrix $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$. One then verifies that this action of $\mathcal{O}[T,Z,Z^{-1}]$ preserves

$$\mathrm{Hom}_{\mathcal{O}[\mathrm{GL}_2(\mathbb{Z}_p)]}\big((\mathrm{Sym}^r\mathcal{O}^2)^\vee, \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}\big),$$

and that the map (5.7.8) is a map of $\mathcal{O}[T,Z,Z^{-1}]$-modules.

Since, by part 2 of Lemma 5.3.8, the space $\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}\big(V, H^1(K_{\Sigma_0})_{k,\overline{\rho}}\big)$ is finite-dimensional, the same is true of $\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}\big(V, H^1(K_{\Sigma_0})_{k,\overline{\rho}}[\mathfrak{m}]\big)$. Thus, by extending scalars sufficiently, we may assume that the $\mathcal{H}(V)$-socle of this latter space is a direct sum of one-dimensional $\mathcal{H}(V)$-submodules. Fix a non-zero $\mathcal{H}(V)$-eigenvector in $\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}\big(V, H^1(K_{\Sigma_0})_{k,\overline{\rho}}[\mathfrak{m}]\big)$, on which $\mathcal{H}(V)$ acts via the system of eigenvalues $\lambda : \mathcal{H}(V) \to k$. Taking into account the surjectivity of (5.7.8), the Deligne–Serre lemma shows that, extending scalars further if necessary, we may lift this eigenvector to an eigenvector for $\mathbb{T}(K_{\Sigma_0})[T,Z]$ in $H^1\big(Y(K_f), \mathcal{V}_r\big)_E$. This latter eigenvector corresponds to a weight $r+2$ cuspidal eigenform $f$ of level $K_{\Sigma_0}$ defined over $E$, with $T_p$-eigenvalue $a_p$ that is congruent to $\lambda(T)$ mod $\varpi$, and whose associated Galois representation $\rho_f$ lifts $\overline{\rho}$. Since $2 \leq r+2 \leq p+1$, the known results relating the weight $r+2$ and the eigenvalue $a_p$ to the structure of $\overline{\rho} := \rho_f \bmod \varpi$ (see e.g. [33]) then imply that $V \in W(\overline{\rho})$ (proving 1), and that $\lambda$ appears in $\mathrm{m}(V,\overline{\rho})$.

We turn to proving part 2 of the theorem. Let $V \in W^{\mathrm{gl}}(\overline{\rho})$, and hence, by what we have already proved, in $W(\overline{\rho})$. Again, by twisting, we may assume that $V = (\mathrm{Sym}^r k^2)^\vee$ for some $0 \leq r \leq p-1$. Extending scalars, if necessary, we may also assume that $\mathrm{soc}_{\mathcal{H}(V)}\,\mathrm{m}(V,\overline{\rho})$ is spanned by $\mathcal{H}(V)$-eigenvectors. Let $\lambda : \mathcal{H}(V) \to k$ be a system of eigenvalues appearing in $\mathrm{soc}_{\mathcal{H}(V)}\,\mathrm{m}(V,\overline{\rho})$. Assumption 5.7.3 implies that the $\lambda$-eigenspace of $\mathrm{m}(V,\overline{\rho})$ is one-dimensional. We will show that there is an isomorphism

$$F_{\Sigma_0}\big(\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])^{\mathcal{H}(V)=\lambda}\big) \cong \overline{\rho}. \tag{5.7.9}$$

Since we have shown that any irreducible constituent of

$$\mathrm{soc}_{\mathcal{H}(V)}\,\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$$

is an irreducible constituent of $\mathrm{soc}_{\mathcal{H}(V)}\,\mathrm{m}(V,\overline{\rho})$, this will serve to establish part 2 of the theorem.

As in Definition 4.1.1, write

$$P_0 := \{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix} \mid a \in \prod_{\ell\in\Sigma_0} \mathbb{Z}_\ell^\times,\, b \in \prod_{\ell\in\Sigma_0}\mathbb{Z}_\ell\} \subset \prod_{\ell\in\Sigma_0}\mathrm{GL}_2(\mathbb{Z}_\ell) \subset G_{\Sigma_0}.$$

To see that the isomorphism (5.7.9) holds, fix an admissible level $K_{\Sigma_0}$ for $\overline{\rho}$ containing $P_0$, and chosen small enough for $K_f := GL_2(\mathbb{Z}_p)K_{\Sigma_0}$ to be neat. Let $\overline{\mathcal{V}}_k$ denote the locally constant sheaf on $Y(K_f)$ corresponding to the representation $\mathrm{Sym}^r k^2$ of $GL_2(\mathbb{F}_p)$, and then note that

$$\mathrm{Hom}_{k[GL_2(\mathbb{Z}_p)]}(V, H^1(K_{\Sigma_0})_{k,\overline{\rho}}) \xrightarrow{\sim} H^1\big(Y(K_f), \overline{\mathcal{V}}_r\big)_{\mathfrak{m}} \tag{5.7.10}$$

(using the Hochschild–Serre spectral sequence as in the proof of Lemma 5.3.8 (2), and taking into account the fact that $\overline{\rho}$ is irreducible). As we already noted above, under this isomorphism the action of $\mathcal{H}(V)$ becomes identified with the classical Hecke action at $p$. It then follows from the classical mod $p$ multiplicity one results for modular curves (see [33] for example) together with the isomorphism (5.7.10) that

$$\mathrm{Hom}_{k[GL_2(\mathbb{Z}_p)]}\big(V, H^1(K_{\Sigma_0})_{k,\overline{\rho}}[\mathfrak{m}]\big)^{\mathcal{H}(V)=\lambda, U_\ell=0\,\forall\ell\in\Sigma_0} \cong \rho.$$

Passing to an inductive limit over all $K_{\Sigma_0}$ containing $P_0$, and recalling the definition of $F_{\Sigma_0}$ (see Definition 4.1.1), we obtain the isomorphism (5.7.9).

Part 3 of the theorem follows from Ihara's lemma. Indeed,

$$\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbb{Z}_p)]}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \hookrightarrow V^\vee \otimes_k H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}],$$

and so it suffices to show that this latter representation is generic as a $G_{\Sigma_0}$-representation. If it is not, then it contains a finite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$, for some $\ell \in \Sigma_0$, contradicting Ihara's lemma. This completes the proof of the theorem. □

## 6. Local-global compatibility and the structure of $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$

In this section we first introduce a local-global compatibility conjecture, namely Conjecture 6.1.6, which refines Conjecture 1.1.1. We also recall from [42] the (somewhat more technical) Conjecture 6.1.4. The remainder of the section is then devoted to stating and proving our main results in the direction of these two conjectures. The statements of the main results are presented in Subsection 6.2, and we explain there how they follow from the more technically involved results that are the subject of the following subsections. Subsection 6.2 also contains the derivations of Theorem 1.2.1 and Theorem 1.2.6 from these results.

Throughout this section we fix an absolutely irreducible modular Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$, *whose restriction to $G_{\mathbb{Q}_p}$ we furthermore assume satisfies Assumption 3.3.1.* We also fix a finite set of primes $\Sigma_0$, not containing $p$, such that $\Sigma := \Sigma_0 \cup \{p\}$ contains all primes at which $\overline{\rho}$ is ramified. We freely employ the notation related to $\overline{\rho}$, $\Sigma_0$, and $\Sigma$ that was introduced in the preceding section.

### 6.1. **A refined local-global compatibility conjecture.**

Our goal in this subsection is to state a refinement Conjecture 1.1.1, and to establish some of its consequences. In order to state the conjecture, we will need to apply the local Langlands correspondences, both at $p$ and away from $p$, to $\rho^{\mathrm{m}}_{\Sigma}$. We begin with the local correspondence at $p$.

By construction, $\rho^{\mathrm{m}}_{\Sigma}$ is a deformation of $\overline{\rho}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$. Furthermore, it follows from Corollary 5.4.6 that the restriction $\rho^{\mathrm{m}}_{\Sigma\,|G_{\mathbb{Q}_p}}$ lies in the subgroupoid $\mathrm{Def}^{\mathrm{crys}}(\overline{\rho}_{|G_{\mathbb{Q}_p}})$ of $\mathrm{Def}(\overline{\rho}_{|G_{\mathbb{Q}_p}})$. Thus, if we let $\overline{\pi}$ denote the admissible smooth $G$-representation over $k$ attached to $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ by Theorem 3.3.2, then Theorem 3.3.13 shows that there is a deformation $\pi^{\mathrm{m}}_{\Sigma}$ of $\overline{\pi}$ to an admissible $G$-representation on an orthonormalizable $\mathbb{T}_{\overline{\rho},\Sigma}$-module, uniquely determined, up to isomorphism, by the requirements that $\mathrm{MF}(\pi^{\mathrm{m}}_{\Sigma}) \xrightarrow{\sim} \rho^{\mathrm{m}}_{\Sigma\,|G_{\mathbb{Q}_p}}$ and that $\pi^{\mathrm{m}}_{\Sigma}$ admits $\det(\rho^{\mathrm{m}}_{\Sigma\,|G_{\mathbb{Q}_p}})\varepsilon$ as a central character.

We introduce the following notation, in analogy with the notation related to $\rho^{\mathrm{m}}_{\Sigma}$ that was introduced in Definition 5.2.9.

6.1.1. **Definition.** For any $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}$, we write $\pi(\mathfrak{p})^\circ := \pi^{\mathrm{m}}_{\Sigma}/\mathfrak{p}\pi^{\mathrm{m}}_{\Sigma}$, and $\pi(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi^{\mathrm{m}}_{\Sigma} \cong \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}} \pi(\mathfrak{p})^\circ$. (Note that if $\mathfrak{p}$ is a closed point of $\mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}$, then by construction $\pi(\mathfrak{p}) = B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big)$; see Remark 3.3.19 above.)

We now consider the local correspondence away from $p$, related to which we have the following result.

6.1.2. **Theorem.** *The coadmissible smooth representation $\pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})$ of $G_{\Sigma_0}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$ (as defined in* Definition 4.4.2 *above) exists. Furthermore, if $p > 2$ then there is an isomorphism $\big(\pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})/\varpi\pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})\big)[\mathfrak{m}] \xrightarrow{\sim} \otimes_{\ell\in\Sigma_0}\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}})$.*

*Proof.* This is proved in [42]. In the case when $\overline{\rho}$ satisfies Assumption 5.7.3, the existence of $\pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})$ also follows from Corollary 6.4.13, together with Theorems 6.4.16 and 6.4.20, below. □

6.1.3. **Remark.** I expect that the second statement of Theorem 6.1.2 holds also when $p = 2$, but this is currently not known in general; we have only the weaker statement that, for each $\ell \in \Sigma_0$, there is a $G_{\Sigma_0}$-equivariant embedding

$$\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}}) \hookrightarrow \big(\pi_\ell(\rho^{\mathrm{m}}_{\Sigma})/\varpi\pi_\ell(\rho^{\mathrm{m}}_{\Sigma})\big)[\mathfrak{m}],$$

which is an isomorphism when $\overline{\rho}_{|G_{\mathbb{Q}_\ell}}$ is non-scalar, and has cokernel of dimension at most one otherwise. Lassina Dembélé [29] has provided computational evidence that this map is indeed an isomorphism even in those cases when $\overline{\rho}_{|G_{\mathbb{Q}_\ell}}$ is scalar.

If $\mathfrak{p} \in \operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is a closed point, then by Theorem 4.4.1 (5) there is an embedding $E \otimes_{\mathcal{O}} \pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})[\mathfrak{p}] \hookrightarrow \otimes_{\ell\in\Sigma_0}\pi_\ell(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_\ell}})$. In [42] we made the following conjecture:

6.1.4. **Conjecture.** *For any closed point $\mathfrak{p} \in \operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$, there is an isomorphism*

$$\pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})[\mathfrak{p}] \cong \pi_{\Sigma_0}\big(\rho(\mathfrak{p})^\circ\big),$$

*and hence also an isomorphism*

$$E \otimes_{\mathcal{O}} \pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})[\mathfrak{p}] \cong \pi_{\Sigma_0}(\rho(\mathfrak{p}))\ \big(:= \otimes_{\ell\in\Sigma_0}\pi_\ell(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_\ell}})\big).$$

6.1.5. **Remark.** In order to prove Conjecture 6.1.4, it suffices to show that, if $\Sigma_0^{\mathrm{ng}} \subset \Sigma_0$ denotes the set of primes at which $\rho(\mathfrak{p})$ is non-generic principal series, and if $\chi : G_{\mathbb{Q}} \to E^\times$ is a character chosen so that $\chi \otimes \rho(\mathfrak{p})$ is unramified at each $\ell \in \Sigma_0^{\mathrm{ng}}$, then $\chi \otimes \rho(\mathfrak{p})$ is promodular of level divisible only by primes in $\Sigma_0 \setminus \Sigma_0^{\mathrm{ng}}$.

We now state the main conjecture of this section.

6.1.6. **Conjecture.** *There is a $G_{\mathbb{Q}} \times G \times G_{\Sigma_0}$-equivariant, $\mathbb{T}_{\overline{\rho},\Sigma}$-linear isomorphism*

$$\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi^{\mathrm{m}}_{\Sigma} \mathbin{\overset{\curlywedge}{\otimes}}_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma}) \xrightarrow{\sim} \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}. \tag{6.1.7}$$

(*Here $\overset{\curlywedge}{\otimes}$ denotes the completed tensor product of Definition* C.43.)

We will also state a slightly weaker variation of this conjecture, but before doing so, we introduce some further notation.

6.1.8. **Definition.** If $\theta$ is a deformation of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$, then for any $\mathfrak{p} \in \operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}$, we write (in analogy with the notation related to $\pi^{\mathrm{m}}_{\Sigma}$ that was introduced in Definition 6.1.1):

$$\theta(\mathfrak{p})^\circ := \theta/\mathfrak{p}\theta,$$

and

$$\theta(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \cong \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}} \theta(\mathfrak{p})^\circ.$$

6.1.9. **Conjecture.** *There is a deformation $\theta$ of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$, such that for every closed point $\mathfrak{p} \in \operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ there is an isomorphism $\theta(\mathfrak{p}) \cong \pi(\mathfrak{p})$, and such that there is a $G_{\mathbb{Q}} \times G \times G_{\Sigma_0}$-equivariant, $\mathbb{T}_{\overline{\rho},\Sigma}$-linear isomorphism*

$$\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \mathbin{\overset{\curlywedge}{\otimes}}_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma}) \xrightarrow{\sim} \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}. \tag{6.1.10}$$

(*Here $\overset{\curlywedge}{\otimes}$ denotes the completed tensor product of Definition* C.43.)

6.1.11. **Remark.** If $\overline{\rho}$, and thus $\overline{\pi}$ (by [61, Lem. 2.1.2]), has only trivial endomorphisms, so that $\mathrm{Def}(\overline{\pi})$ is representable, then the condition that $\theta(\mathfrak{p}) \cong \pi(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ implies that $\theta \cong \pi_\Sigma^{\mathrm{m}}$ (since the closed points of $\operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ are Zariski dense in $\operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$). Thus in this case Conjecture 6.1.9 is equivalent to Conjecture 6.1.6.

In general, Conjecture 6.1.6 implies Conjecture 6.1.9 (as one sees by taking $\theta = \pi_\Sigma^{\mathrm{m}}$). The reason for considering Conjecture 6.1.9 is that, on the one hand, we will be able to establish this conjecture in some situations in which we cannot establish the stronger Conjecture 6.1.6, and on the other, it is already strong enough to imply the main results described in the introduction.

In order to relate Conjectures 6.1.4 and 6.1.9 to Conjecture 1.1.1, we establish a slight strengthening of [12, Prop. 3.2.3] and [38, Prop. 7.7.7]. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be a continuous, absolutely irreducible representation, unramified outside the finite set of primes $\Sigma$. Let $K_{\Sigma_0}$ be a compact open subgroup of $G_{\Sigma_0}$, and let $\mathfrak{p}$ denote the ideal in $\mathbb{T}(K_{\Sigma_0})[1/p]$ generated by the elements $T_\ell - \mathrm{trace}\big(\rho(\mathrm{Frob}_\ell)\big)$ for $\ell \not\in \Sigma$ (so $\mathfrak{p}$ is either a closed point of $\operatorname{Spec} \mathbb{T}(K_{\Sigma_0})$, if $\rho$ is promodular of tame level $K_{\Sigma_0}$, or else is the unit ideal).

6.1.12. **Proposition.** *The inclusion*

$$\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \widehat{H}^1(K_{\Sigma_0})_E[\mathfrak{p}]) \subset \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \widehat{H}^1(K_{\Sigma_0})_E)$$

*is an equality.*

*Proof.* The map $\mathrm{Frob}_\ell \mapsto \ell S_\ell$ (for $\ell \not\in \Sigma$) extends to a continuous character $G_{\mathbb{Q},\Sigma} \to \mathbb{T}(K_{\Sigma_0})^\times$, where $G_{\mathbb{Q},\Sigma}$ denotes the Galois group of the maximal extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ unramified outside $\Sigma$.

Define $U := \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \widehat{H}^1(K_{\Sigma_0})_E)$. The $\mathbb{T}(K_{\Sigma_0})$-action on $\widehat{H}^1(K_{\Sigma_0})_E$ induces a $\mathbb{T}(K_{\Sigma_0})$-action on $U$. Evaluation gives a map

$$\rho \otimes_E U \to \widehat{H}^1(K_{\Sigma_0})_E, \tag{6.1.13}$$

which is injective, since $\rho$ is absolutely irreducible, and is $\mathbb{T}(K_{\Sigma_0})[G_{\mathbb{Q}}]$-linear, if $\mathbb{T}(K_{\Sigma_0})[G_{\mathbb{Q}}]$ acts on the tensor product through the action of $G_{\mathbb{Q}}$ on the first factor and $\mathbb{T}(K_{\Sigma_0})$ on the second factor. Applying the Eichler–Shimura relations as in the proof of [12, Prop. 3.2.3], we find that

$$\rho(\mathrm{Frob}_\ell)(v) \otimes \big(T_\ell - \mathrm{trace}(\rho(\mathrm{Frob}_\ell))\big)u + v \otimes (\ell S_\ell - \det(\rho(\mathrm{Frob}_\ell))u = 0 \tag{6.1.14}$$

for every $v \in \rho$, $u \in U$, and $\ell \not\in \Sigma$. If $\ell$ is such that $\rho(\mathrm{Frob}_\ell)$ does not act by a scalar, then by choosing $v$ so that $\rho(\mathrm{Frob}_\ell)(v)$ and $v$ are linearly independent, we conclude that

$$T_\ell - \mathrm{trace}(\rho(\mathrm{Frob}_\ell)) = \ell S_\ell - \det(\rho(\mathrm{Frob}_\ell)) = 0$$

on $U$. Since the second of these quantities is the difference of two continuous characters on $G_{\mathbb{Q},\Sigma}$, and since the $\mathrm{Frob}_\ell$ for which $\rho(\mathrm{Frob}_\ell)$ is not scalar generate a dense subgroup of $G_{\mathbb{Q},\Sigma}$ (by Cebotarev density, together with the fact that $\rho$ is absolutely irreducible, and so in particular not identically scalar), we conclude that these two characters coincide identically on $G_{\mathbb{Q},\Sigma}$, and thus in particular that $\ell S_\ell = \det(\rho(\mathrm{Frob}_\ell))$ on $U$ for every $\ell \not\in \Sigma$. From (6.1.14) we conclude that $T_\ell = \mathrm{trace}(\rho(\mathrm{Frob}_\ell))$ on $U$ for every $\ell \not\in \Sigma$, and thus that (6.1.13) has image lying in $\widehat{H}^1(K_{\Sigma_0})_E[\mathfrak{p}]$. This proves the proposition. □

6.1.15. **Remark.** Although we will not need it in what follows, we note that the preceding proposition shows that if $\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \widehat{H}^1_{E,\Sigma}) \neq 0$, then necessarily $\mathfrak{p}$ is a proper ideal of $\mathbb{T}(K_{\Sigma_0})[1/p]$, and thus $\rho$ is promodular. (This is a strengthening of [38, Cor. 7.7.9].) In particular, Conjecture 1.1.1 implies that every odd irreducible continuous two-dimensional representation of $G_{\mathbb{Q}}$ over $E$ is promodular. (Theorem 1.2.3 provides strong evidence in this direction.)

6.1.16. **Corollary.** *For each closed point $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$, each of the inclusions*

$$\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]) \hookrightarrow \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma}) \hookrightarrow \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho(\mathfrak{p}), \widehat{H}^1_{E,\Sigma})$$

*is an equality.*

*Proof.* We may verify the claim of the equality after making an extension of scalars, if necessary, and thus we may assume that $\kappa(\mathfrak{p}) = E$. The claim of the corollary is then seen to follow from Proposition 6.1.12, by passing to the limit over all compact open subgroups $K_{\Sigma_0}$ of $G_{\Sigma_0}$, taking into account the fact that the inclusion $\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}] \subset \widehat{H}^1_{E,\Sigma}[\mathfrak{p}]$ is an equality, since the ideal $\mathfrak{p}$ is associated to the deformation $\rho(\mathfrak{p})$ of $\overline{\rho}$. □

We now prove the following proposition, which relates Conjectures 6.1.4 and 6.1.9 to Conjecture 1.1.1.

6.1.17. **Proposition.** *If Conjectures* 6.1.4 *and* 6.1.9 *hold for our fixed choice of $\overline{\rho}$, and for every possible choice of $\Sigma_0$, then Conjecture* 1.1.1 *holds for every promodular lift $V$ of $\overline{\rho}$.*

*Proof.* Let $V$ be a promodular lift of $\overline{\rho}$ over $E$, and let $\mathfrak{p}$ denote the closed point of $\mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}$ corresponding to $V$. Corollary 6.1.16 yields the equality

$$\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]) = \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_{E,\Sigma}). \tag{6.1.18}$$

Tensoring with $E$ over $\mathcal{O}$, then passing to $\mathfrak{p}$-torsion parts, and taking into account Lemma 6.1.19 below, together with Conjecture 6.1.4, the isomorphism (6.1.10) gives rise to an isomorphism

$$\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}] \xrightarrow{\sim} V \otimes_E B(V_{|G_{\mathbb{Q}_p}}) \otimes_E \bigotimes_{\ell \in \Sigma_0} \pi_\ell(V),$$

from which we deduce an isomorphism

$$\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]) \xrightarrow{\sim} B(V_{|G_{\mathbb{Q}_p}}) \otimes_E \bigotimes_{\ell \in \Sigma_0} \pi_\ell(V).$$

Combined with the equality (6.1.18) and passing to the inductive limit over all $\Sigma_0$, this yields the isomorphism of Conjecture 1.1.1. □

6.1.19. **Lemma.** *If Conjecture* 6.1.4 *holds, if $\theta$ is an orthonormalizable admissible representation of $G$ over $\mathbb{T}_{\overline{\rho},\Sigma}$, and if $\mathfrak{p}$ is a point of* $\mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$*, then there is an isomorphism*

$$\big(\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})\big)[\mathfrak{p}] \xrightarrow{\sim} \rho(\mathfrak{p})^{\circ} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}} \theta(\mathfrak{p})^{\circ} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}} \pi_{\Sigma_0}(\rho(\mathfrak{p})^{\circ}).$$

*Proof.* By definition,

$$\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma}) = \varinjlim_{H} \rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \,\hat{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})^{H},$$

where $H$ runs over the open subgroups of $G_{\Sigma_0}$, and so it suffices to prove that

$$\big(\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \,\hat{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma_0}(\rho_\Sigma^{\mathrm{m}})^H\big)[\mathfrak{p}] \xrightarrow{\sim} \rho(\mathfrak{p})^\circ \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}} \theta(\mathfrak{p})^\circ \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}} \pi_{\Sigma_0}(\rho(\mathfrak{p})^\circ)^H,$$

for each open subgroup $H$. Since $\rho_\Sigma^{\mathrm{m}}$ is free of rank two over $\mathbb{T}_{\overline{\rho},\Sigma}$, while $\theta$ is an orthonormalizable admissible $G$-representation over $\mathbb{T}_{\overline{\rho},\Sigma}$, the tensor product $\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta$ is again an orthonormalizable admissible $G$-representation over $\mathbb{T}_{\overline{\rho},\Sigma}$, and thus (taking into account our assumption that Conjecture 6.1.4 holds) the required isomorphism is provided by Lemma 3.1.17, applied with $M := \mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}$, $X := \pi_{\Sigma_0}(\rho_\Sigma^{\mathrm{m}})^H$, and $\pi := \rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta$. (Note that we may and do omit the $\varpi$-adic completion on the right hand side of this isomorphism, since $\pi_{\Sigma_0}(\rho(\mathfrak{p})^\circ)^H$ is a free $\mathcal{O}$-module of finite rank.) □

We write $H^1_{k,\overline{\rho}}[\mathfrak{m}] := \varinjlim_{\Sigma_0} H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]$. Conjecture 6.1.9 has the following implication for the structure of the mod $\varpi$ cohomology space $H^1_{k,\overline{\rho}}[\mathfrak{m}]$.

6.1.20. **Proposition.** *If Conjecture* 6.1.9 *holds for our fixed choice of* $\overline{\rho}$, *and for every possible choice of* $\Sigma_0$, *and if* $p > 2$, *then there is a* $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$*-equivariant,* $k$*-linear, isomorphism*

$$\overline{\rho} \otimes_k \overline{\pi} \otimes_k \bigotimes_{\ell \neq p}{}' \,\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}}) \xrightarrow{\sim} H^1_{k,\overline{\rho}}[\mathfrak{m}],$$

*where, for each* $\ell \neq p$, *the representation* $\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}})$ *of* $\mathrm{GL}_2(\mathbb{Q}_\ell)$ *is attached to* $\overline{\rho}_{|G_{\mathbb{Q}_\ell}}$ *via the mod* $p$ *local Langlands correspondence of Theorem* 4.3.1.

*Proof.* Reducing the isomorphism (6.1.10) modulo $\varpi$ and then passing to $\mathfrak{m}$-torsion parts, and taking into account Lemma C.45 and the second statement of Theorem 6.1.2, we obtain an isomorphism

$$\overline{\rho} \otimes_k \overline{\pi} \otimes_k \overline{\pi}_{\Sigma_0}(\overline{\rho}) \xrightarrow{\sim} H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}].$$

Passing to the inductive limit over all finite sets $\Sigma_0$ then induces the required isomorphism. □

6.1.21. **Remark.** I expect that the preceding proposition also holds when $p = 2$; see Remark 6.1.3. From the weaker statement made in that remark relating $\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}})$ and $\big(\pi_\ell(\rho_\Sigma^{\mathrm{m}})/\varpi\pi_\ell(\rho_\Sigma^{\mathrm{m}})\big)[\mathfrak{m}]$, for each $\ell \in \Sigma_0$, we may deduce a correspondingly weaker statement relating $\overline{\rho} \otimes_k \overline{\pi} \otimes_k \bigotimes_{\ell \neq p}{}' \,\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}})$ and $H^1_{k,\overline{\rho}}[\mathfrak{m}]$. In particular, we see that the former always embeds into the latter.

6.1.22. **Remark.** In the context of Proposition 6.1.20, in the case when $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is not scalar, we see that Conjecture 6.1.9 has as a consequence various classical "mod $p$ multiplicity one results"; see Remark 1.2.8 above. On the other hand, in the case when $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ *is* scalar, the representation $\overline{\pi}$ is the direct sum of two copies of the same parabolically induced representation (see Remark 3.3.3 (1)), and so we find that Conjecture 6.1.9 implies a *mod* $p$ *multiplicity two* result for such $\overline{\rho}$. This is consistent with [84, Cor. 4.4 and 4.5], which establish unconditionally a *mod* $p$ *multiplicity* $\geq 2$ result for such $\overline{\rho}$.

6.1.23. **Remark.** Suppose (for the duration of this remark) that the restriction to $G_{\mathbb{Q}_p}$ of the absolutely irreducible modular representation $\overline{\rho}$ *does not* satisfy Assumption 3.3.1. We nevertheless expect that there will be a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant,

$k$-linear, isomorphism

$$\overline{\rho} \otimes_k \overline{\pi} \otimes_k \bigotimes_{\ell \neq p}{}' \,\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}}) \xrightarrow{\sim} H^1_{k,\overline{\rho}},$$

where, just as in the statement of Proposition 6.1.20, for each $\ell \neq p$, the representation $\overline{\pi}(\overline{\rho}_{|G_{\mathbb{Q}_\ell}})$ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ is attached to $\overline{\rho}_{|G_{\mathbb{Q}_\ell}}$ via the mod $p$ local Langlands correspondence of Theorem 4.3.1, and where $\overline{\pi}$ is *some* representation of $G$ over $k$ which depends only on $\overline{\rho}_{|G_{\mathbb{Q}_p}}$. However, in this case, we do not expect remark 3.3.4 to hold, i.e., we do not expect that the semi-simplification of $\overline{\pi}$ will match with $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ via the semi-simple mod $p$ local Langlands correspondence of [8].

As an example, we will describe the form that we expect $\overline{\pi}$ to take in the case when $p \geq 5$,

$$\overline{\rho}_{|G_{\mathbb{Q}_p}} \sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$$

for some $k$-valued character $\chi$ of $G_{\mathbb{Q}_p}$, and $\overline{\rho}$ is indecomposable: Let $\underline{1}$ denote the trivial character of $G$, and let St denote the Steinberg representation of $G$. The space $\mathrm{Ext}^1(\underline{1}, \mathrm{St})$ is two-dimensional, and so we may construct the *universal extension*

$$0 \to \mathrm{St} \to U \to W \to 0,$$

where $W = \mathrm{Ext}^1(\underline{1}, \mathrm{St})^\vee$ is a two-dimensional $k$-vector space equipped with trivial $G$-action. We then expect $\overline{\pi}$ to be a non-split extension

$$0 \to (\chi \circ \det) \otimes (\mathrm{Ind}_B^G \overline{\varepsilon}^{-1} \otimes \overline{\varepsilon}) \to \overline{\pi} \to (\chi \circ \det) \otimes U \to 0.$$

(Any two such non-split extensions are isomorphic.)

We expect a similar phenomenon to occur in the context of Conjecture 1.1.1, when $V_{|G_{\mathbb{Q}_p}} \sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & \varepsilon \end{pmatrix}$. Namely, while in this case we expect a formula of the type stated in the conjecture to hold, we don't expect $B(V_{|G_{\mathbb{Q}_p}})$ to be the representation associated to $V_{|G_{\mathbb{Q}_p}}$ via the $p$-adic local Langlands correspondence as described for the reducible representation $V_{|G_{\mathbb{Q}_P}}$ in [38], but rather to be a slightly longer representation (longer in the sense that it will have an additional one-dimensional topological Jordan–Hölder factor). Thus, in such a case, we don't expect Conjecture 1.1.1 to be true in the precise form that it was stated in [38].

6.2. **Main results.** We begin with the following result, which gives some information about the multiplicity spaces $\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \widehat{H}^1_{E,\overline{\rho},\Sigma})$, for promodular liftings $\rho$ of $\overline{\rho}$.

6.2.1. **Theorem.** *If* $\mathfrak{p}$ *is a closed point of* $\mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$*, then there is a non-zero continuous* $G$*-equivariant map* $B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big) \to \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma})$.

*Proof.* Remark 6.4.3 below shows that there is a non-zero $G_{\mathbb{Q}} \times G$-equivariant homomorphism $\rho(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \pi(\mathfrak{p}) \to \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]$. Since $B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big) = \pi(\mathfrak{p})$, the present theorem follows. □

Related to the preceding theorem, we have the following result, which extends [12, Thm. 5.7.2]. The latter result treats the case when $\mathfrak{p}$ is a classical point for which $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ is reducible, and potentially crystalline with distinct Hodge–Tate weights, but in fact its proof applies directly to the more general situation considered in Proposition 6.2.2. We present a slight variant of that proof here.

6.2.2. **Proposition.** *If* $\mathfrak{p}$ *is a closed point of* $\operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ *for which* $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ *is neither the direct sum of two characters, nor an extension of a character by itself, then any non-zero continuous* $\kappa(\mathfrak{p})$*-linear,* $G$*-equivariant map*

$$B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big) \to \operatorname{Hom}_{E[G_{\mathbb{Q}}]}(\rho(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma})$$

*is necessarily a closed embedding.*

*Proof.* If $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ is irreducible, then the same is true of $B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big)$, by Proposition 3.3.24, and the proposition is immediate. Thus we may assume that $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ is the non-split extension of two distinct characters, say an extension of $\chi_2$ by $\chi_1$.

If $\chi_1\chi_2^{-1} \neq \varepsilon$, then Proposition 3.4.2 shows that $B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big)$ is a non-split extension of $\operatorname{Ind}_{\overline{B}}^{G}\chi_2\otimes\chi_1\varepsilon$ by $\operatorname{Ind}_{\overline{B}}^{G}\chi_1\otimes\chi_2\varepsilon$, these two representations being topologically irreducible, while if $\chi_1\chi_2^{-1} = \varepsilon$, then Proposition 3.4.5 shows that $B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big)$ is a non-split extension of the topologically irreducible representation $\operatorname{Ind}_{\overline{B}}^{G}\chi_2 \otimes \chi_1\varepsilon$ by a representation which is itself a non-split extension of a one-dimensional representation by the topologically irreducible representation $(\chi_1 \circ \det) \otimes_E \widehat{\mathrm{St}}$.

We claim that any $\kappa(\mathfrak{p})$-linear, $G$-equivariant map

$$B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big) \to \operatorname{Hom}_{E[G_{\mathbb{Q}}]}(\rho(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma})$$

which is not an embedding must factor through the quotient $\operatorname{Ind}_{\overline{B}}^{G}\chi_2 \otimes \chi_1\varepsilon$ of $B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big)$. This is obvious in the first case. In the second case, it follows from the fact that $\widehat{H}^1_E$ contains no one-dimensional $G$-invariant subspaces, by Lemma 5.1.3.

The adjointness between induction and the functor of ordinary parts provides an isomorphism

$$\begin{aligned}(6.2.3)\quad &\operatorname{Hom}_{\kappa(\mathfrak{p})[G]}(\operatorname{Ind}_{\overline{B}}^{G}\chi_2 \otimes \chi_1\varepsilon, \operatorname{Hom}_{E[G_{\mathbb{Q}}]}(\rho(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma})\\ &\xrightarrow{\sim} \operatorname{Hom}_{\kappa(\mathfrak{p})[T]}\big(\chi_2 \otimes \chi_1\varepsilon, \operatorname{Ord}_B(\operatorname{Hom}_{E[G_{\mathbb{Q}}]}(\rho(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma}))\big)\\ &\xrightarrow{\sim} \operatorname{Hom}_{\kappa(\mathfrak{p})[G_{\mathbb{Q}}\times T]}\big(\rho(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} (\chi_2 \otimes \chi_1\varepsilon), \operatorname{Ord}_B(\widehat{H}^1_{E,\overline{\rho},\Sigma})\big).\end{aligned}$$

We claim that the target of this isomorphism vanishes. Granting this, we find that the source also vanishes, and the proposition is proved.

To see the claimed vanishing, first recall that, by Corollary 5.6.13, the $G_{\mathbb{Q}_p}$-action on $\operatorname{Ord}_B(\widehat{H}^1_{E,\overline{\rho},\Sigma})/\operatorname{Ord}_B(\widehat{H}^1_{E,\overline{\rho},\Sigma})^{\mathrm{ab},S}$ factors through $G^{\mathrm{ab}}_{\mathbb{Q}_p}$. On the other hand, since $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ is non-split (by assumption), the $G_{\mathbb{Q}_p}$-action on $\rho(\mathfrak{p})$ does not factor through $G^{\mathrm{ab}}_{\mathbb{Q}_p}$. It follows that the image of any non-zero $\kappa(\mathfrak{p})[G_{\mathbb{Q}} \times T]$-linear map

$$\phi : \rho(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} (\chi_2 \otimes \chi_1\varepsilon) \to \operatorname{Ord}_B(\widehat{H}^1_{E,\overline{\rho},\Sigma})$$

(which is necessarily injective, since $\rho(\mathfrak{p})$ is injective as a $G_{\mathbb{Q}}$-representation) must have non-trivial intersection with $\operatorname{Ord}_B(\widehat{H}^1_{E,\overline{\rho},\Sigma})^{\mathrm{ab},S}$. On the other hand, since $\chi_1 \neq \chi_2$ (again by assumption), we see that $\big(\rho(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} (\chi_2 \otimes \chi_1\varepsilon)\big)^{\mathrm{ab},S} = 0$. Thus the image of $\phi$ necessarily has trivial intersection with $\operatorname{Ord}_B(\widehat{H}^1_{E,\overline{\rho},\Sigma})^{\mathrm{ab},S}$, and hence any such $\phi$ in fact vanishes, as claimed. □

6.2.4. **Theorem.** *Conjecture* 6.1.4 *holds.*

*Proof.* As noted in Remark 6.1.5, in proving Conjecture 6.1.4, we may replace the closed point $\mathfrak{p}$ by the closed point $\mathfrak{p}_\chi$ for which $\rho(\mathfrak{p}_\chi) \cong \chi \otimes \rho(\mathfrak{p})$, chosen so that

$\rho(\mathfrak{p}_\chi)$ is unramified at each prime $\ell \in \Sigma_0^{\mathrm{ng}}$, and then (now writing $\mathfrak{p}$ rather than $\mathfrak{p}_\chi$) we must prove that $\rho(\mathfrak{p})$ is promodular of level divisible only by primes in $\Sigma_0 \setminus \Sigma_0^{\mathrm{ng}}$.

We will remove the primes $\ell \in \Sigma_0^{\mathrm{ng}}$ from the level of $\rho(\mathfrak{p})$ one at a time. Thus we fix such an $\ell$, write $\Sigma_0^* := \Sigma_0 \setminus \{\ell\}$, and similarly write $\Sigma^* := \Sigma \setminus \{\ell\} = \Sigma_0^* \cup \{p\}$. Let $J_\ell$ denote an Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_\ell)$. (We use the letter $J$ to avoid confusion with $I_\ell$, the inertia subgroup of $G_{\mathbb{Q}_\ell}$.) Since $\rho(\mathfrak{p})$ is unramified non-generic principal series at $\ell$, we see that $\rho(\mathfrak{p})$ is in fact promodular of tame level $K_{\Sigma_0^*} J_\ell K_0^\Sigma$, for some compact open subgroup $K_{\Sigma_0^*}$ of $G_{\Sigma_0^*}$.

Write $K_{\Sigma_0} := K_{\Sigma_0^*} J_\ell$, an open subgroup of $G_{\Sigma_0}$. Remark 6.4.3 below shows that there is a non-zero $G_{\mathbb{Q}}$-equivariant map (which is necessarily an embedding, since $\rho(\mathfrak{p})$ is irreducible)

$$\rho(\mathfrak{p}) \hookrightarrow \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}]. \tag{6.2.5}$$

Our goal is to show that $\mathfrak{p}$ is promodular of level $K_{\Sigma_0^*}$, and we will do this by showing that $\widehat{H}^1(K_{\Sigma_0^*})_{E,\overline{\rho}}[\mathfrak{p}]$ is non-zero.

Of course, we have to use the fact that $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_\ell}}$ is unramified non-generic principal series. More precisely, we will use the following facts: $\rho(\mathfrak{p})$ is unramified locally at $\ell$, so that $\rho(\mathfrak{p})(\mathrm{Frob}_\ell)$ is defined, and the ratio of the eigenvalues of $\rho(\mathfrak{p})(\mathrm{Frob}_\ell)$ is equal to $\ell^{\pm 1}$, so that in particular $\rho(\mathfrak{p})(\mathrm{Frob}_\ell)$ is not a scalar.

To begin with, we define an extension $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}^*$ of $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$, by adjoining the operator $U_\ell$ (whose definition was given in Definition 4.1.1). If $f_\ell \in G_{\mathbb{Q}_\ell}$ denotes a choice of lift of $\mathrm{Frob}_\ell \in G_{\mathbb{Q}_\ell}/I_\ell$, and if $x^2 + ax + b \in \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}[x]$ denotes the characteristic polynomial of $\rho(K_{\Sigma_0})(f_\ell)$, then $U_\ell^2 + aU_\ell + b = 0$, and so $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}^*$ is finite over $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$. Since also $\rho(\mathfrak{p})$ is an irreducible $G_{\mathbb{Q}}$-representation, we see that we may find a prime $\mathfrak{p}^*$ of $\mathbb{T}_{\overline{\rho},\Sigma}^*$ lying over $\mathfrak{p}$, such that (6.2.5) may be refined to an embedding

$$\rho(\mathfrak{p}) \hookrightarrow \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}^*].$$

Since $\rho(\mathfrak{p})$ is unramified locally at $\ell$, in fact this embedding factors through an embedding

$$\rho(\mathfrak{p}) \hookrightarrow \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}[\mathfrak{p}^*]. \tag{6.2.6}$$

Below we will construct an exact sequence

$$0 \to \big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}\big)_{\ell-\mathrm{new}} \to \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell} \to \big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}\big)_{\ell-\mathrm{old}} \to 0 \tag{6.2.7}$$

of $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}^*[G_{\mathbb{Q}_\ell} \times G]$-modules, with the following properties: the action of $\mathrm{Frob}_\ell$ on $\big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}\big)_{\ell-\mathrm{new}}$ is scalar, given by the element $U_\ell \in \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}^*$, and there is an isomorphism of $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}[G_{\mathbb{Q}_\ell} \times G]$-modules

$$\big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}\big)_{\ell-\mathrm{old}} \xrightarrow{\sim} \big(\widehat{H}^1(K_{\Sigma_0^*})_{E,\overline{\rho}}\big)^{\oplus 2}. \tag{6.2.8}$$

If (6.2.6) were to factor through $\big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}\big)_{\ell-\mathrm{new}}$, then we would find that $\rho(\mathfrak{p})(\mathrm{Frob}_\ell)$ is equal to a scalar, namely the image of $U_\ell$ modulo $\mathfrak{p}^*$. On the other hand, we have already observed that $\rho(\mathfrak{p})(\mathrm{Frob}_\ell)$ is not a scalar. Thus (6.2.6) must not factor through $\big(\widehat{H}^1\big)_{\ell-\mathrm{new}}^{I_\ell}[\mathfrak{p}^*]$, consequently the composite of (6.2.6) with the map $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}[\mathfrak{p}^*] \to \big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}\big)_{\ell-\mathrm{old}}[\mathfrak{p}*]$ must be non-zero, and so in particular $\big(\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}^{I_\ell}\big)_{\ell-\mathrm{old}}[\mathfrak{p}*] \neq 0$. A consideration of the isomorphism (6.2.8) then shows that $\widehat{H}^1(K_{\Sigma_0^*})_{E,\overline{\rho}}[\mathfrak{p}] \neq 0$, and thus we conclude that $\rho(\mathfrak{p})$ is promodular of tame level $K_{\Sigma_0^*}$, as required.

It remains to construct the exact sequence (6.2.7). This essentially arises from a consideration of the nearby cycles spectral sequence for computing the $p$-adic étale cohomology of the modular curves $X(K_pK_{\Sigma_0}K_0^{\Sigma})$ over $\operatorname{Spec}\mathbb{Z}_\ell$ (for compact open subgroups $K_p$ of $G$). However, since we have to work with finite and integral coefficients, we will find it simpler, and more concrete, to instead consider the $p$-power torsion in the Néron models of the Jacobians of these curves.

Fix, for the moment, a compact open subgroup $K_p$ of $G$ chosen small enough to ensure that $K_pK_{\Sigma_0^*}K_0^{\Sigma^*}$ is neat, in the sense of Definition 5.3.7. For any compact open subgroup $K_f$ of $\mathrm{GL}_2(\mathbb{A}_f)$, we let $X(K_f)$ denote the completion of the modular curve $Y(K_f)$; it is a projective smooth curve over $\mathbb{Q}$, we write $J(K_f) := \operatorname{Pic}^0\big(X(K_f)\big)$, an abelian variety over $\mathbb{Q}$, and we let $\mathcal{J}(K_f)$ denote the Néron model of $J(K_f)$ over $\mathbb{Z}_\ell$. The modular curve $X(K_pK_{\Sigma_0^*}K_0^{\Sigma^*})$ has good reduction at $\ell$, while the modular curve $X(K_pK_{\Sigma_0}K_0^{\Sigma})$ admits a semi-stable model over $\operatorname{Spec}\mathbb{Z}_\ell$, with special fibre equal to the union of two copies of the special fibre of $X(K_pK_{\Sigma_0^*}K_0^{\Sigma^*})$. Thus $\mathcal{J}(K_pK_{\Sigma_0^*}K_0^{\Sigma^*})$ is an abelian scheme, while the special fibre of $\mathcal{J}(K_pK_{\Sigma_0}K_0^{\Sigma})$ is an extension of a finite étale group scheme by a semi-abelian variety. More precisely, we have an exact sequence

$$0 \to \mathcal{J}(K_pK_{\Sigma_0}K_0^{\Sigma})^0_{/\mathbb{F}_\ell} \to \mathcal{J}(K_pK_{\Sigma_0}K_0^{\Sigma})_{/\mathbb{F}_\ell} \to \Phi \to 0, \tag{6.2.9}$$

where the superscript 0 denotes the connected component of the identity, and $\Phi$, the group scheme of connected components of $\mathcal{J}(K_pK_{\Sigma_0}K_0^{\Sigma})_{/\mathbb{F}_\ell}$, is a finite étale group scheme over $\mathbb{F}_\ell$, as well as an exact sequence

$$\begin{aligned}0 \to \mathcal{T} \to \mathcal{J}(K_pK_{\Sigma_0}K_0^{\Sigma})^0_{/\mathbb{F}_\ell} \\ \to \mathcal{J}(K_pK_{\Sigma_0^*}K_0^{\Sigma^*})_{/\mathbb{F}_\ell} \times \mathcal{J}(K_pK_{\Sigma_0^*}K_0^{\Sigma^*})_{/\mathbb{F}_\ell} \to 0,\end{aligned} \tag{6.2.10}$$

where $\mathcal{T}$ is a torus.

We now tensor with $\mathcal{O}$ over $\mathbb{Z}_p$, and then localize at the maximal ideal $\mathfrak{m}$ of $\mathbb{T}(K_{\Sigma_0})$ corresponding to $\overline{\rho}$. Since the Hecke action on $\Phi$ is Eisenstein [73, Cor., p. 140] (note that our assumption that $K_pK_{\Sigma_0^*}K_0^{\Sigma^*}$ is neat ensures that condition (i) of the corollary is satisfied), while $\mathfrak{m}$ is associated to the absolutely irreducible Galois representation $\overline{\rho}$, we see that (6.2.9) gives rise to an isomorphism

$$\big(\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{J}(K_{\Sigma_0}K_0^{\Sigma})^0_{/\mathbb{F}_\ell}[p^r]\big)_{\mathfrak{m}} \xrightarrow{\sim} \big(\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{J}(K_{\Sigma_0}K_0^{\Sigma})_{/\mathbb{F}_\ell}[p^r]\big)_{\mathfrak{m}},$$

and hence that (6.2.10) gives rise to a short exact sequence of $\mathbb{T}(K_{\Sigma_0})^*_{\overline{\rho}}[G_{\mathbb{Q}_\ell/I_\ell}]$-modules

$$\begin{aligned}0 \to \big(\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{T}[p^r]\big)_{\mathfrak{m}} \to \big(\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{J}(K_pK_{\Sigma_0}K_0^{\Sigma})_{/\mathbb{F}_\ell}[p^r]\big)_{\mathfrak{m}} \\ \to \big(\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{J}(K_pK_{\Sigma_0^*}K_0^{\Sigma^*})_{/\mathbb{F}_\ell}[p^r]\big)_{\mathfrak{m}} \times \big(\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{J}(K_pK_{\Sigma_0^*}K_0^{\Sigma^*})_{/\mathbb{F}_\ell}[p^r]\big)_{\mathfrak{m}} \to 0.\end{aligned} \tag{6.2.11}$$

It is known (see e.g. [72, Prop. 3.8]) that $\mathrm{Frob}_\ell$ (the *geometric* Frobenius) acts on $\big(\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{T}[p^r]\big)_{\mathfrak{m}}$ via $\ell^{-1}U_\ell$.

Now for any smooth projective curve $X$ over a field $k$, with separable closure $k^s$, there is a natural isomorphism $H^1_{\text{ét}}(X_{/k^s}, \mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{\sim} \operatorname{Pic}^0(X)[p^r](-1)$, where $(-1)$ denotes a twist by the inverse cyclotomic character. In particular, there is an isomorphism $H^1\big(X(K_{\Sigma_0}K_0^{\Sigma}), \mathbb{Z}/p^r\mathbb{Z}\big) \xrightarrow{\sim} J(K_{\Sigma_0}K_0^{\Sigma})[p^r](-1)$. Furthermore, under this isomorphism, the $I_\ell$-invariants on the left hand side are identified with the subgroup $\mathcal{J}(K_{\Sigma_0}K_0^{\Sigma})_{/\mathbb{F}_\ell}[p^r]$ of the right-hand side. Thus we may rewrite the inverse cyclotomic twist of (6.2.11) as a short exact sequence of $\mathbb{T}(K_{\Sigma_0})^*_{\overline{\rho}}[G_{\mathbb{Q}_\ell/I_\ell}]$-modules

$$0 \to \Big(H^1\big(X(K_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{O}/p^r\mathcal{O}\big)^{I_\ell}_{\mathfrak{m}}\Big)_{\ell-\mathrm{new}} \to H^1\big(X(K_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{O}/p^r\mathcal{O}\big)^{I_\ell}_{\mathfrak{m}} \to \Big(H^1\big(X(K_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{O}/p^r\mathcal{O}\big)^{I_\ell}_{\mathfrak{m}}\Big)_{\ell-\mathrm{old}} \to 0,$$

with the properties that $\mathrm{Frob}_\ell$ acts on the first non-trivial term via $U_\ell$, and that there is a $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}[G_{\mathbb{Q}_\ell/I_\ell}]$-equivariant isomorphism

$$H^1\big(X(K_pK_{\Sigma_0}K_0^{\Sigma}), \mathcal{O}/p^r\mathcal{O}\big)^{I_\ell}_{\mathfrak{m}}\Big)_{\ell-\mathrm{old}} \xrightarrow{\sim} \Big(H^1\big(X(K_pK_{\Sigma_0^*}K_0^{\Sigma *}), \mathcal{O}/p^r\mathcal{O}\big)_{\mathfrak{m}}\Big)^{\oplus 2}.$$

Passing to the inductive limit over $K_p$, and then to the projective limit over $r$, and noting that, since we have localized at a non-Eisenstein maximal ideal, we may replace the closed modular curves by the corresponding open ones without changing the spaces we are computing, we obtain the desired short exact sequence (6.2.7). ☐

6.2.12. **Remark.** The preceding argument is a $p$-adic analogue of the argument used to prove [72, Thm. 6.1], an argument sometimes known as "Mazur's principle". Another approach to proving Conjecture 6.1.4 is via promodularity theorems. For example, if $p > 2$, $\overline{\rho}_{|G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible, and $\overline{\rho}$ is $p$-distinguished, then Conjecture 6.1.4 follows from Theorem 1.2.3. Indeed, that theorem shows that any lift of $\overline{\rho}$ that is ramified at only finitely many primes, and is unramified at a prime $\ell \neq p$, is promodular of level prime-to-$\ell$, and Conjecture 6.1.4 follows from this, as was noted in Remark 6.1.5.

6.2.13. **Theorem.** *If $\overline{\rho}$ is $p$-distinguished (i.e. if the semi-simplification of $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is not the direct sum of two copies of the same character), then Conjecture* 6.1.6 *holds.*

*Proof.* This follows from Corollary 6.4.13 and Theorems 6.4.9, 6.4.16, and 6.4.20 below. ☐

*Proof of Theorem 1.2.1.* Let $V$ be a promodular two-dimensional representation of $G_{\mathbb{Q}}$ unramified away from a finite set of primes, as in the statement of the theorem, and let $\overline{\rho} := \overline{V}$ denote the reduction mod $\varpi$ of $V$. By hypothesis, $\overline{\rho}$ satisfies the running assumptions of this section, and $V$ corresponds to a closed point $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ for some $\Sigma$ containing the ramified primes for $V$ as well as $p$. Theorem 6.2.1 shows that there is a non-zero map

$$B(V_{|G_{\mathbb{Q}_p}}) \to \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]). \tag{6.2.14}$$

If $V_{|G_{\mathbb{Q}_p}}$ is neither the direct sum of two characters, nor an extension of a character by itself, then Proposition 6.2.2 shows that (6.2.14) is in fact an embedding. Arguing as in the proof of [38, Prop. 7.8.7], we then see that (6.2.14) gives rise to a map

$$B(V_{|G_{\mathbb{Q}_p}}) \otimes_E \bigotimes_{\ell \notin \Sigma}{}' \pi_\ell(V) \to \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(V, \widehat{H}^1_E),$$

which is an embedding if (6.2.14) is. This proves claim 1 of the theorem. Claim 2 of the theorem follows directly from Theorems 6.2.4 and 6.2.13 and Proposition 6.1.17. □

*Proof of Theorem 1.2.6.* This theorem follows from Theorem 6.2.13 and Proposition 6.1.20. □

6.2.15. **Remark.** In the proof of Theorem 1.2.6 we used various mod $p$ multiplicity one theorems, but we did not use level-lowering at the prime $p$ (i.e. [72, Thm. 6.1] in the case when $\ell = p$, and its generalizations), the existence of companion forms [48], or the more difficult part of [86, Thm. 2.1 (ii)] in which the mod $p$ nebentypus character has conductor prime to $p$. Furthermore, each of these results is easily deduced from Theorem 1.2.6 (for those $\overline{\rho}$ satisfying the hypotheses of the theorem). In particular, Theorem 1.2.6 gives a new proof that $W^{\mathrm{gl}}(\overline{\rho}) = W(\overline{\rho})$ for those $\overline{\rho}$ to which it applies.

Of course this last-mentioned result is now well-understood, both in the context of proving that "the weak Serre's conjecture implies the strong Serre's conjecture", and as a part of the much stronger result that any odd $\overline{\rho}$ is in fact modular [53, 54, 55, 60]. Nevertheless, it seems interesting to note that it can also be proved by the methods of the $p$-adic Langlands program.

6.3. **The $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-modules $X(\theta)$.** In this subsection we begin our investigation of Conjectures 6.1.6 and 6.1.9. Throughout the subsection we fix a deformation $\theta$ of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$, and use the notation of Definition 6.1.8. The deformation $\theta$ will always be assumed to satisfy the following key hypothesis:

6.3.1. **Assumption.** *If $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is an allowable closed point , then there is a $\kappa(\mathfrak{p})$-linear $G$-equivariant isomorphism $\theta(\mathfrak{p}) \cong \pi(\mathfrak{p})$.*

6.3.2. **Remark.** If $\operatorname{End}_G(\overline{\pi}) = k$, then $\operatorname{Def}(\overline{\pi})$ is representable, and so, since the allowable closed points $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ are Zariski dense in $\operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$ (by Lemma 5.4.9), we see that Assumption 6.3.1 implies that $\theta \cong \pi_{\Sigma}^{\mathrm{m}}$. However, this need not be true in general (i.e. if $\operatorname{Def}(\overline{\pi})$ is not representable).

6.3.3. **Lemma.** *For any closed point $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$, there is an isomorphism $\theta(\mathfrak{p})^{\mathrm{ss}} \cong \pi(\mathfrak{p})^{\mathrm{ss}}$.*

*Proof.* Applying $MF$ to $\theta$, we obtain a deformation $\psi$ of $\overline{\rho}$ over $\operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$. For each point $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$, we write $\psi(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \psi$. Assumption 6.3.1 implies that $\psi(\mathfrak{p}) \cong \rho(\mathfrak{p})$ for each allowable closed point $\mathfrak{p}$. Lemma 5.4.9 shows that these points are Zariski dense in $\operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$, and hence we conclude that the characters of $\psi$ and $\rho_{\Sigma}^{\mathrm{m}}$ coincide on $\operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$. The lemma follows from this. □

Our goal is now to introduce, and to begin the study of, a certain $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-module, denoted $X(\theta)$, the investigation of which is intimately connected to the investigation of Conjectures 1.1.1 and 6.1.6.

6.3.4. **Definition.** Write $X(\theta) := \operatorname{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}} \times G]-\mathrm{cont}}(\rho_{\Sigma}^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta, \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})$, the $\mathbb{T}_{\overline{\rho},\Sigma}$-module of $\mathbb{T}_{\overline{\rho},\Sigma}$-linear, $G_{\mathbb{Q}} \times G$-equivariant homomorphisms from $\rho_{\Sigma}^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma}^{\mathrm{m}}$ to $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$ that are continuous, when the source is given its $\mathfrak{m}$-adic topology and the target is given its $\mathcal{O}$-linear inductive limit topology. The module $X(\theta)$ is equipped in a natural way with a $\mathbb{T}_{\overline{\rho},\Sigma}$-linear action of $G_{\Sigma_0}$ (induced by the action of $G_{\Sigma_0}$ on $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$).

6.3.5. **Remark.** Ideally, for the purposes of this paper we would like to be able to restrict our attention to the case when $\theta = \pi_\Sigma^{\mathrm{m}}$. However, due to limitations in our methods, we have to develop our results for more general $\theta$ (satisfying Assumption 6.3.1).

6.3.6. **Remark.** For any allowable level $K_{\Sigma_0}$ for $\overline{\rho}$, Lemma 5.3.8 yields an identification

$$X(\theta)^{K_{\Sigma_0}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_\mathbb{Q}\times G]-\mathrm{cont}}(\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta, \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}). \tag{6.3.7}$$

6.3.8. **Remark.** If $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is a closed point, then from the definition of $X(\theta)$ we obtain natural isomorphisms

$$X(\theta)[\mathfrak{p}] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}[G_\mathbb{Q}\times G]}\big(\rho(\mathfrak{p})^\circ \otimes_\mathcal{O} \theta(\mathfrak{p})^\circ, \widehat{H}^1_{\mathcal{O},S,\overline{\rho}}[\mathfrak{p}]\big), \tag{6.3.9}$$

and

$$E \otimes_\mathcal{O} X(\theta)[\mathfrak{p}] \xrightarrow{\sim} \mathrm{Hom}_{\kappa(\mathfrak{p})[G_\mathbb{Q}\times G]-\mathrm{cont}}(\rho(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \theta(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]). \tag{6.3.10}$$

(We do not need to include a continuity condition in the space of homomorphisms considered in (6.3.9) only because the source and target are both naturally endowed with the $\varpi$-adic topology, with respect to which the indicated homomorphisms, being in particular $\mathcal{O}$-linear, are automatically continuous.)

Also, there is an induced embedding

$$\begin{aligned}(X(\theta)/\varpi X(\theta))[\mathfrak{m}] &\hookrightarrow \mathrm{Hom}_{k[G_\mathbb{Q}\times G]}(\overline{\rho} \otimes_k \overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \\ &\xrightarrow{\sim} \mathrm{Hom}_{k[G_\mathbb{Q}\times G]}(\overline{\rho} \otimes_k \overline{\pi}, H^1_{k,\overline{\rho},\Sigma}).\end{aligned} \tag{6.3.11}$$

6.3.12. **Theorem.** *The $\mathbb{T}_{\overline{\rho},\Sigma}$-module $X(\theta)$ is a cofinitely generated and coadmissible smooth representation of $G_{\Sigma_0}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$ (in the sense of Definitions* C.23 *and* C.32).

*Proof.* We begin by showing that $X(\theta)$ is a smooth representation of $G_{\Sigma_0}$, that is, that $X(\theta) = \bigcup_{K_{\Sigma_0}} X(\theta)^{K_{\Sigma_0}}$, where $K_{\Sigma_0}$ runs over all allowable levels for $\overline{\rho}$ in $G_{\Sigma_0}$. Equivalently, taking into account the isomorphism (6.3.7), we have to show that any continuous $\mathbb{T}_{\overline{\rho},\Sigma}$-linear, $G_\mathbb{Q} \times G$-equivariant map $\phi : \rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho}}} \pi_\Sigma^{\mathrm{m}} \to \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$ factors through $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$, for some allowable level $K_{\Sigma_0}$.

It follows from Lemmas 5.3.6 and 5.3.8 that

$$\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}} = \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}} \bigcap \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$$

(the intersection taking place in $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$), and thus it suffices to show that any continuous $\mathbb{T}_{\overline{\rho},\Sigma}$-linear, $G_\mathbb{Q} \times G$-equivariant map $\phi : \rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho}}} \pi_\Sigma^{\mathrm{m}} \to \widehat{H}^1_{E,\overline{\rho},\Sigma}$ (where the source is given its $\mathfrak{m}$-adic topology and the target is given its locally convex inductive limit topology) factors through $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$, for some $K_{\Sigma_0}$. In fact we will show that this is true of any $\mathcal{O}$-linear continuous map. Since $\widehat{H}^1_{E,\overline{\rho},\Sigma}$ is the inductive limit of the Banach spaces $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$, with closed transition maps, any bounded subset of $\widehat{H}^1_{E,\overline{\rho},\Sigma}$ is contained in some $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$ [76, Prop. 5.6]. Thus it suffices to show that the image of $\phi$ is bounded. For this, let $U$ be an open neighbourhood of 0 in $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$. Then $\phi^{-1}(U)$ is a neighbourhood of 0 in $\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho}}} \pi_\Sigma^{\mathrm{m}}$, since $\phi$ is continuous, and so contains $\mathfrak{m}^s\big(\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho}}} \pi_\Sigma^{\mathrm{m}}\big)$ for some $s \geq 0$. In particular it contains $\varpi^s\big(\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho}}} \pi_\Sigma^{\mathrm{m}}\big)$, and hence $\varpi^s\phi(\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho}}} \pi_\Sigma^{\mathrm{m}}) \subset U$. Thus the image of $\phi$ is indeed bounded, as claimed.

We now turn to showing that $X(\theta)$ is cofinitely generated and coadmissible as a smooth representation of $G_{\Sigma_0}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$. Again taking into account the isomorphism (6.3.7), we see that for each allowable level $K_{\Sigma_0}$ for $\overline{\rho}$, the $\mathbb{T}_{\overline{\rho},\Sigma}$-module $X(\theta)^{K_{\Sigma_0}}$ inherits the properties of being $\varpi$-adically separated and complete, and $\mathcal{O}$-torsion free, directly from the corresponding properties of $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$. Furthermore, the continuity property that the elements of $X(\theta)^{K_{\Sigma_0}}$ are assumed to satisfy implies that the action map $\mathbb{T}_{\overline{\rho},\Sigma}\times X(\theta)^{K_{\Sigma_0}}\to X(\theta)^{K_{\Sigma_0}}$ is continuous, when $\mathbb{T}_{\overline{\rho},\Sigma}$ is given is $\mathfrak{m}$-adic topology and $X(\theta)$ is given its $\varpi$-adic topology. Thus, in order to prove that the smooth $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-module $X(\theta)$ is cofinitely generated and coadmissible, it suffices, by Lemma C.25 and Proposition C.33, to prove that the smooth $k[G_{\Sigma_0}]$-module $(X(\theta)/\varpi X(\theta))[\mathfrak{m}]$ is finitely generated and admissible. Taking into account the embedding (6.3.11), it suffices in turn to show that $\mathrm{Hom}_{k[G_{\mathbb{Q}}\times G]}(\overline{\rho}\otimes_k\overline{\pi},H^1_{k,\overline{\rho},\Sigma})$ is a finitely generated admissible smooth $G_{\Sigma_0}$-representation over $k$. If we let $\overline{W}\subset\overline{\pi}$ denote a finite-dimensional $k$-vector subspace of $\overline{\pi}$ that generates $\overline{\pi}$ over $G$, and if we let $K_p\subset G$ denote a compact open subgroup that fixes $W$ pointwise, then restriction to $\overline{\rho}\otimes_k W$ induces an embedding

$$\mathrm{Hom}_{k[G_{\mathbb{Q}}\times G]}(\overline{\rho}\otimes_k\overline{\pi},H^1_{k,\overline{\rho},\Sigma})\hookrightarrow\overline{\rho}^{\vee}\otimes_k W^{\vee}\otimes_k(H^1_{k,\overline{\rho},\Sigma})^{K_p}. \tag{6.3.13}$$

It follows from Proposition 5.3.11 and Corollary 5.3.14 that the target of (6.3.13) is a finitely generated admissible smooth representation of $G_{\Sigma_0}$ over $k$, and thus so is its source. □

6.3.14. **Definition.** If $\mathfrak{p}\in\mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is a classical closed point, then we let $M(\mathfrak{p})_E$ denote the closure in $\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]$ (or equivalently, in $\widehat{H}^1_{E,\overline{\rho},\Sigma}$) of $\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\mathrm{l.alg}}$. Write $M(\mathfrak{p})_{\mathcal{O}}:=M(\mathfrak{p})_E\bigcap\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$ (the intersection taking place in $\widehat{H}^1_{E,\overline{\rho},\Sigma}$).

Note that $M(\mathfrak{p})_E$ and $M(\mathfrak{p})_{\mathcal{O}}$ are each invariant under $G_{\mathbb{Q}}\times G\times G_{\Sigma_0}$, and that, by construction, $M(\mathfrak{p})_{\mathcal{O}}$ is saturated in $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$.

6.3.15. **Proposition.** *Let $\mathfrak{p}\in\mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ be a closed point which is allowable in the sense of Definition* 5.4.7.

(1) *There is a $G_{\mathbb{Q}}\times G\times G_{\Sigma_0}$-equivariant isomorphism*

$$M(\mathfrak{p})_E\xrightarrow{\sim}\rho(\mathfrak{p})\otimes_E\pi(\mathfrak{p})\otimes_E\pi_{\Sigma_0}(\rho(\mathfrak{p})).$$

(2) *The inclusion $M(\mathfrak{p})_E\hookrightarrow\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]$ induces an isomorphism*

$$\mathrm{Hom}_{E[G_{\mathbb{Q}}\times G]}(\rho(\mathfrak{p})\otimes_E\pi(\mathfrak{p}),M(\mathfrak{p})_E)\xrightarrow{\sim}\mathrm{Hom}_{E[G_{\mathbb{Q}}\times G]}(\rho(\mathfrak{p})\otimes_E\pi(\mathfrak{p}),\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]).$$

*Proof.* Let $\widetilde{\pi}_p(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}})$ be the locally algebraic representation attached to $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$, as in the statement of [38, Conj. 3.3.1 (7)]. Since $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ is crystabelline, absolutely irreducible, and not exceptional, the $E$-Banach representation $\pi(\mathfrak{p})$ of $G$ associated to $\mathfrak{p}$ is isomorphic to the universal unitary completion (in the sense of [37]) of $\widetilde{\pi}_p(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}})$ [3]. It then follows from [38, Thm. 7.4.2], together with the main results of [16] and [75], that there is an isomorphism

$$M(\mathfrak{p})_E\xrightarrow{\sim}\rho(\mathfrak{p})\otimes_E\pi(\mathfrak{p})\otimes_E\pi_{\Sigma_0}(\rho(\mathfrak{p})),$$

proving 1.

Again using the fact that $\pi(\mathfrak{p})$ is isomorphic to the universal unitary completion of the locally algebraic representation $\widetilde{\pi}_p(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}})$, we find that

$$\begin{aligned}\mathrm{Hom}_{E[G_{\mathbb{Q}}\times G]}\big(\rho(\mathfrak{p})\otimes_E \pi(\mathfrak{p}), \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]\big) \\ = \mathrm{Hom}_{E[G_{\mathbb{Q}}\times G]}\big(\rho(\mathfrak{p})\otimes_E \widetilde{\pi}_p(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}), \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]\big) \\ = \mathrm{Hom}_{E[G_{\mathbb{Q}}\times G]}\big(\rho(\mathfrak{p})\otimes_E \widetilde{\pi}_p(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}), \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\mathrm{l.alg}}\big) \\ = \mathrm{Hom}_{E[G_{\mathbb{Q}}\times G]}\big(\rho(\mathfrak{p})\otimes_E \widetilde{\pi}_p(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}), M(\mathfrak{p})\big) \\ = \mathrm{Hom}_{E[G_{\mathbb{Q}}\times G]}\big(\rho(\mathfrak{p})\otimes_E \pi(\mathfrak{p}), M(\mathfrak{p})_E\big),\end{aligned}$$

proving 2. □

The preceding proposition, together with Assumption 6.3.1, allows us to rewrite the isomorphism (6.3.10), for those $\mathfrak{p}$ satisfying the hypotheses of the proposition, as an isomorphism

$$(6.3.16)\qquad \big(E\otimes_{\mathcal{O}} X(\theta)\big)[\mathfrak{p}] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}\times G]}\big(\rho(\mathfrak{p})\otimes_E \pi(\mathfrak{p}), M(\mathfrak{p})_E\big).$$

6.3.17. **Corollary.** *If $\mathfrak{p}\in \operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is a closed point which is allowable in the sense of Definition* 5.4.7*, then there is a $G_{\Sigma_0}$-equivariant $\kappa(\mathfrak{p})$-linear isomorphism*

$$(6.3.18)\qquad E\otimes_{\mathcal{O}} X(\theta)[\mathfrak{p}] \xrightarrow{\sim} \pi_{\Sigma_0}(\rho(\mathfrak{p})),$$

*and evaluation induces an isomorphism*

$$(6.3.19)\qquad \rho(\mathfrak{p})\otimes_E \pi(\mathfrak{p})\otimes_E (E\otimes_{\mathcal{O}} X(\theta)[\mathfrak{p}]) \xrightarrow{\sim} M(\mathfrak{p})_E.$$

*Proof.* Both claims follow from the isomorphism (6.3.16), together with the isomorphism of part 1 of Proposition 6.3.15. □

The following proposition provides a refinement of Corollary 6.3.17, for certain points $\mathfrak{p}$.

6.3.20. **Proposition.** *Suppose that $\mathfrak{p}\in\operatorname{Spec}\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is a closed point which is allowable in the sense of Definition* 5.4.7*. Furthermore, suppose that $E\xrightarrow{\sim}\kappa(\mathfrak{p})$, and that $\overline{\rho}$ satisfies Assumption* 5.7.3*. Then the isomorphism* (6.3.18) *induces a $G_{\Sigma_0}$-equivariant $\mathcal{O}$-linear isomorphism*

$$(6.3.21)\qquad X(\theta)[\mathfrak{p}] \xrightarrow{\sim} \pi_{\Sigma_0}(\rho(\mathfrak{p})^\circ)$$

(*where $\pi_{\Sigma_0}(\rho(\mathfrak{p})^\circ)$ is the admissible smooth representation of $G_{\Sigma_0}$ over $\mathcal{O}$ attached to $\rho(\mathfrak{p})^\circ$ by the local Langlands correspondence of Section* 4).

*Proof.* The assumption that $E\xrightarrow{\sim}\kappa(\mathfrak{p})$ implies that $\mathcal{O}\xrightarrow{\sim}\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}$. Thus $X(\theta)[\mathfrak{p}]$ is an $\mathcal{O}$-lattice in the admissible smooth $G_{\Sigma_0}$-representation $E\otimes_{\mathcal{O}} X(\theta)[\mathfrak{p}]$. The inclusion $X(\theta)[\mathfrak{p}]\subset X(\theta)$ induces an embedding

$$X(\theta)[\mathfrak{p}]/\varpi X(\theta)[\mathfrak{p}] \hookrightarrow (X(\theta)/\varpi X(\theta))[\mathfrak{m}],$$

and it follows from Corollary 6.3.24 below that $X(\theta)[\mathfrak{p}]/\varpi X(\theta)[\mathfrak{p}]$ is a generic $G_{\Sigma_0}$-representation, in the sense of Definition 4.1.3. The proposition follows from this and Corollary 6.3.17, together with the characterization of $\pi_{\Sigma_0}(\rho(\mathfrak{p})^\circ)$ provided by Proposition 4.4.3. □

6.3.22. **Proposition.** *If $\overline{\rho}$ satisfies Assumption* 5.7.3*, then* $\mathrm{Hom}_G(\overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ *is a generic $G_{\Sigma_0}$-representation.*

*Proof.* Suppose first that $\overline{\pi}$ is irreducible, and let $V$ be a weight in $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}\,\overline{\pi}$ (the $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of $\overline{\pi}$). In the case when $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}$ contains a one-dimensional representation, we furthermore choose $V$ to be one-dimensional. Since $\overline{\pi}$ is irreducible, $V$ generates $\overline{\pi}$ over $G$. Thus the restriction map $\mathrm{Hom}_G(\overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ is injective. If the target is non-zero, then we find that $V$ belongs to the set of global weights $W^{\mathrm{gl}}(\overline{\rho})$. It is furthermore good, and so the proposition follows in this case from part 3 of Theorem 5.7.7.

Suppose next that $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is an extension $0 \to \chi_1 \to \overline{\rho} \to \chi_2 \to 0$, and that $\chi_1\chi_2^{-1} \neq \overline{\varepsilon}$. The representation $\overline{\pi}$ then sits in a short exact sequence $0 \to \overline{\pi}_1 \to \overline{\pi} \to \overline{\pi}_2 \to 0$, where $\overline{\pi}_1 = \mathrm{Ind}_B^G\,\chi_1 \otimes \chi_2\overline{\varepsilon}$ and $\overline{\pi}_2 = \mathrm{Ind}_B^G\,\chi_2 \otimes \chi_1\overline{\varepsilon}$ are irreducible principle series. In this case we choose $V_i$ to be a weight in $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}\,\overline{\pi}_i$ (in fact, $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}\,\overline{\pi}_i$ is irreducible, so there is a unique choice for $V_i$). There is an exact sequence

$$(6.3.23)\quad 0 \to \mathrm{Hom}_G(\overline{\pi}_2, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \to \mathrm{Hom}_G(\overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \to \mathrm{Hom}_G(\overline{\pi}_1, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]),$$

and for each $i = 1, 2$, an embedding

$$\mathrm{Hom}_G(\overline{\pi}_i, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]), \hookrightarrow \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V_i, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]).$$

If either of the spaces $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V_i, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ is non-zero then $V_i$ is a weight in $W^{\mathrm{gl}}(\overline{\rho})$, which is automatically good. Part 3 of Theorem 5.7.7 then shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V_i, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ is a generic $G_{\Sigma_0}$-representation. The same is thus true of its subrepresentation $\mathrm{Hom}_G(\overline{\pi}_i, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$. A consideration of the exact sequence (6.3.23) then shows that $\mathrm{Hom}_G(\overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$ is generic.

Suppose, finally, that $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is an extension $0 \to \chi \to \overline{\rho} \to \chi\overline{\varepsilon}^{-1} \to 0$, for some character $\chi$. Assumption 3.3.1 guarantees that this extension is non-split. The representation $\overline{\pi}$ then sits in a short exact sequence $0 \to \overline{\pi}_1 \to \overline{\pi} \to \overline{\pi}_2 \to 0$, where $\overline{\pi}_1$ itself sits in a short exact sequence $0 \to (\chi \circ \det) \otimes_k \mathrm{St} \to \overline{\pi}_1 \to \chi \circ \det \to 0$, while $\overline{\pi}_2 = \mathrm{Ind}_B^G\,\chi\overline{\varepsilon}^{-1} \otimes \chi\overline{\varepsilon}$. Now $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}\,\overline{\pi}_2$ contains a single weight which does not lie in $W(\overline{\rho})$, and hence $\mathrm{Hom}_G(\overline{\pi}_2, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) = 0$. Thus restriction induces an injection

$$\mathrm{Hom}_G(\overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \hookrightarrow \mathrm{Hom}_G(\overline{\pi}_1, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]),$$

and it suffices to show that the latter $G_{\Sigma_0}$-representation is generic.

If $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is a trés ramifieé extension, then $W(\overline{\rho})$ contains no one-dimensional representations. Consequently, $\mathrm{Hom}_G(\chi \circ \det, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) = 0$, and so if we let $V = \mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}\big((\chi \circ \det) \otimes \mathrm{St}\big)$, then restriction to $V$ induces an injection

$$\mathrm{Hom}_G(\overline{\pi}_1, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]).$$

If the target of this map is non-zero, then $V$ is a weight in $W^{\mathrm{gl}}(\overline{\rho})$, automatically good because $\overline{\rho}$ is trés ramifieé. Thus the proposition in this case again follows from part 3 of Theorem 5.7.7.

If $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is peu ramifieé, then we may find a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant splitting of the surjection $\overline{\pi}_1 \to \chi \circ \det$. If $V$ denotes the image of this splitting, then $V$ generates $\overline{\pi}_1$ over $G$, and so restriction to $V$ induces an injection

$$\mathrm{Hom}_G(\overline{\pi}_1, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]).$$

If the target of this map is non-zero, then $V$ is a weight in $W^{\mathrm{gl}}(\overline{\rho})$, and $V$ is good, since it is one-dimensional. A last application of part 3 of Theorem 5.7.7 proves the proposition in this case. $\square$

6.3.24. **Corollary.** *If $\overline{\rho}$ satisfies Assumption* 5.7.3, *then the $G_{\Sigma_0}$-representation $(X(\theta)/\varpi X(\theta))[\mathfrak{m}]$ is generic, in the sense of Definition* 4.1.3.

*Proof.* The corollary follows from the existence of the embedding (6.3.11), together with Proposition 6.3.22. $\square$

6.4. **Evaluation maps.** If Conjecture 6.1.6 holds, then the isomorphism (6.1.7) gives rise to an embedding of $\pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})$ as a saturated, coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\pi^{\mathrm{m}}_{\Sigma})$, and (6.1.7) can in turn be recovered by restricting the natural evaluation map

$$\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi^{\mathrm{m}}_{\Sigma} \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} X(\pi^{\mathrm{m}}_{\Sigma}) \to \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$$

to $\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi^{\mathrm{m}}_{\Sigma} \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma_0}(\rho^{\mathrm{m}}_{\Sigma})$. Thus, in order to study Conjecture 6.1.6, it is natural to study the coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodules of $X(\pi^{\mathrm{m}}_{\Sigma})$, and the corresponding evaluation maps. In fact, as in the last section, we work in a more general context, in which we fix a deformation $\theta$ of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$ satisfying Assumption 6.3.1, and study submodules of $X(\theta)$ and the associated evaluation maps. This will allow us to establish cases of the weaker conjecture 6.1.9, in situations where we are not able to prove Conjecture 6.1.6 itself.

6.4.1. **Definition.** If $Y$ is a coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$, then we let $\mathrm{ev}_Y$ denote the restriction of the natural evaluation map

$$\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} X(\theta) \to \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$$

to $\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} Y$. (Note that, by Lemma C.48, the inclusion of $Y$ into $X(\theta)$ induces an embedding $\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} Y \hookrightarrow \rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} X(\theta)$.) We introduce further notation for various maps induced by $\mathrm{ev}_Y$.

(1) We denote by

$$\mathrm{ev}_{E,Y} : E \otimes_{\mathcal{O}} \rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} Y \to \widehat{H}^1_{E,\overline{\rho},\Sigma}$$

the map obtained by tensoring $\mathrm{ev}_Y$ with $E$ over $\mathcal{O}$.

(2) If $K_{\Sigma_0}$ is an allowable level for $\overline{\rho}$, then $\mathrm{ev}_Y$ induces a map

$$\mathrm{ev}(K_{\Sigma_0})_Y : \rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \hat{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} Y^{K_{\Sigma_0}} \to \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}},$$

as well as a map

$$\mathrm{ev}(K_{\Sigma_0})_{E,Y} : E \otimes_{\mathcal{O}} \rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \hat{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} Y^{K_{\Sigma_0}} \to \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}.$$

(3) If $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is a closed point, then $\mathrm{ev}_Y$ induces a map

$$\mathrm{ev}(\mathfrak{p})_Y : \rho(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \theta(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} (E \otimes_{\mathcal{O}} Y[\mathfrak{p}]) \to \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}].$$

(4) Reducing $\mathrm{ev}_Y$ modulo $\varpi$ and passing to $\mathfrak{m}$-torsion parts, and taking into account Lemma C.45, we obtain a map

$$\mathrm{ev}(\mathfrak{m})_Y : \overline{\rho} \otimes_k \overline{\pi} \otimes_k (Y/\varpi Y)[\mathfrak{m}] \to H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]$$

as the composite

$$\overline{\rho} \otimes_k \overline{\pi} \otimes_k (Y/\varpi Y)[\mathfrak{m}] \to \overline{\rho} \otimes_k \overline{\pi} \otimes_k (X(\theta)/\varpi X(\theta))[\mathfrak{m}] \hookrightarrow \overline{\rho} \otimes_k \overline{\pi} \otimes_k \mathrm{Hom}_{k[G_{\mathbb{Q}} \times G]}(\overline{\rho} \otimes_k \overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \to H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}],$$

where the first arrow is induced by the inclusion of $Y$ in $X(\theta)$, the second arrow is induced by the embedding (6.3.11), and the third arrow is the natural evaluation map.

The following result gives an important characterization of faithful and saturated coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodules of $X(\theta)$.

6.4.2. **Proposition.** *If $Y$ is a saturated coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$, then the following conditions are equivalent:*

1. *$Y$ is a faithful $\mathbb{T}_{\overline{\rho},\Sigma}$-module.*
2. *For each allowable level $K_{\Sigma_0} \subset G_{\Sigma_0}$ for $\overline{\rho}$, the submodule of invariants $Y^{K_{\Sigma_0}}$ is a faithful $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$-module.*
3. *For each classical closed point $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ which is allowable in the sense of Definition* 5.4.7*, the inclusion $E \otimes_{\mathcal{O}} Y[\mathfrak{p}] \subset E \otimes_{\mathcal{O}} X(\theta)[\mathfrak{p}]$ is an equality.*
4. *For each allowable level $K_{\Sigma_0} \subset G_{\Sigma_0}$ for $\overline{\rho}$, and for each classical closed point $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}[1/p]$ which is allowable in the sense of Definition* 5.4.7*, the inclusion $E \otimes_{\mathcal{O}} Y^{K_{\Sigma_0}}[\mathfrak{p}] \subset E \otimes_{\mathcal{O}} X(\theta)^{K_{\Sigma_0}}[\mathfrak{p}]$ is an equality.*
5. *If $X(\theta)_{\mathrm{ctf}}$ denotes the maximal $\mathbb{T}_{\overline{\rho},\Sigma}$-cotorsion free submodule of $X(\theta)$ (following Definition* C.39*), then $X(\theta)_{\mathrm{ctf}} \subset Y$.*

*Proof.* Note that if $Y$ is saturated in $X(\theta)$, then $Y$ is cofinitely generated over $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$, by Theorem 6.3.12 and Corollary C.34. It is clear that 2 implies 1, and that 3 and 4 are equivalent. Corollary 6.3.17 shows that for any point $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}$ satisfying the requirements of condition 3, the $G_{\Sigma_0}$-representation $E \otimes_{\mathcal{O}} X(\theta)[\mathfrak{p}]$ is irreducible. Thus for any such point $\mathfrak{p}$, we have that $E \otimes_{\mathcal{O}} Y[\mathfrak{p}] \neq 0$ if and only if $E \otimes_{\mathcal{O}} Y[\mathfrak{p}] = E \otimes_{\mathcal{O}} X(\theta)[\mathfrak{p}]$. Proposition C.36, together with the facts that $\mathbb{T}_{\overline{\rho},\Sigma}$ is reduced, and that the allowable closed points $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ are Zariski dense in $\mathrm{Spec}\, \mathbb{T}_{\overline{\rho},\Sigma}$ (by Lemma 5.4.9), then shows that 1 and 3 are equivalent. If condition 4 holds, then Corollary 6.3.17 shows that $Y^{K_{\Sigma_0}}[\mathfrak{p}] \neq 0$ for each allowable closed point $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$. Since these points are Zariski dense in $\mathrm{Spec}\, \mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ (again by Lemma 5.4.9), and since $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$ is reduced, it then follows from Proposition C.22 that 4 implies 2.

It remains to show that 5 is equivalent to the first four conditions. This is a consequence of the following

*Claim:* $X(\theta)_{\mathrm{ctf}}$ is the unique saturated, coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G]$-submodule of $X(\theta)$ which is both faithful and cotorsion free over $\mathbb{T}_{\overline{\rho},\Sigma}$.

Indeed, Proposition C.40 shows that $Y_{\mathrm{ctf}}$ satisfies condition 1 if $Y$ does. The claim then implies that $Y_{\mathrm{ctf}} = X(\theta)_{\mathrm{ctf}}$, and hence that $X(\theta)_{\mathrm{ctf}} \subset Y$. Thus 1 implies 5. As for the converse, the claim implies that $X(\theta)_{\mathrm{ctf}}$ satisfies condition 1. Hence, if $X(\theta)_{\mathrm{ctf}} \subset Y$, we deduce that $Y$ also satisfies condition 1. Thus 5 implies 1.

It remains to prove the claim. Since $X(\theta)$ evidently satisfies condition 4, it must also satisfy 1. Proposition C.40 then implies that $X(\theta)_{\mathrm{ctf}}$ also satisfies condition 1, and so is indeed a faithful $\mathbb{T}_{\overline{\rho},\Sigma}$-cotorsion free $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$. Now if $Y$ is any saturated, coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$ which is faithful and cotorsion free over $\mathbb{T}_{\overline{\rho},\Sigma}$, then certainly $Y \subset X(\theta)_{\mathrm{ctf}}$. Also, since we have already shown that 1 implies 3, we see (taking into account Lemma 5.4.9) that the inclusion $E \otimes_{\mathcal{O}} Y[\mathfrak{p}] \subset E \otimes_{\mathcal{O}} X(\theta)_{\mathrm{ctf}}[\mathfrak{p}]$ is an equality for a Zariski dense set of points $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$. Proposition C.41 thus implies that $Y = X(\theta)_{\mathrm{ctf}}$. □

6.4.3. **Remark.** As we already noted in the proof of Proposition 6.4.2, $X(\theta)$ itself satisfies condition 4 of the proposition, and thus satisfies all the equivalent conditions of the proposition. In particular, if $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ is a closed point, then Proposition C.36 shows that $E \otimes_{\mathcal{O}} X(\theta)[\mathfrak{p}] \neq 0$. From the isomorphism (6.3.10), we conclude (taking, e.g., $\theta = \pi_{\Sigma}^{\mathrm{m}}$, which certainly satisfies Assumption 6.3.1) that for any such $\mathfrak{p}$, there is a non-zero $G_{\mathbb{Q}} \times G$-equivariant homomorphism $\rho(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \pi(\mathfrak{p}) \to \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]$.

We give an alternative description of $X(\theta)_{\mathrm{ctf}}$, after making a preliminary definition.

6.4.4. **Definition.** We will say that an ideal $I \subset \mathbb{T}_{\overline{\rho},\Sigma}$ is allowable if it is of the form $I := \bigcap_{i=1}^n \mathfrak{p}_i$, where each $\mathfrak{p}_i$ is an allowable closed point of $\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$, in the sense of Definition 5.4.7.

Clearly the intersection of any two allowable ideals of $\mathbb{T}_{\overline{\rho},\Sigma}$ is again an allowable ideal. Thus the allowable ideals form a directed set, ordered by inclusion. Thus $\bigcup_{I \text{ allowable}} X(\theta)[I]$ is a $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$ (the union, as indicated, being taken over all allowable ideals).

6.4.5. **Lemma.** *The closure* (*in the sense of Definition* C.28) *of* $\bigcup\limits_{I \text{ allowable}} X(\theta)[I]$ *in* $X(\theta)$ *coincides with* $X(\theta)_{\mathrm{ctf}}$.

*Proof.* Let $I$ be an allowable ideal of $\mathbb{T}_{\overline{\rho},\Sigma}$, and as in Definition 6.4.4, write $I := \bigcap_{i=1}^n \mathfrak{p}_i$, where each $\mathfrak{p}_i$ is a classical closed point of $\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ for which $\rho(\mathfrak{p}_i)_{|G_{\mathbb{Q}_p}}$ is crystabelline and absolutely irreducible, and not exceptional. If $Y$ is a saturated, coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$, which is faithful as a $\mathbb{T}_{\overline{\rho},\Sigma}$-module, then Proposition 6.4.2 shows that $\sum_{i=1}^n X(\theta)[\mathfrak{p}_i] \subset Y$. Since $X(\theta)[I]$ is equal to the saturation of $\sum_{i=1}^n X(\theta)[\mathfrak{p}_i]$ in $X(\theta)$, we see that in fact $X(\theta)[I] \subset Y$. Thus, if we let $W$ denote the closure $\bigcup_{I \text{ allowable}} X(\theta)[I]$ in $X(\theta)$, in the sense of Definition C.28, then we find that $W \subset Y$. On the other hand, $W$ is a saturated, coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$ (by Lemmas C.30 and C.31), which is faithful as a $\mathbb{T}_{\overline{\rho},\Sigma}$-module (by Lemma 5.4.9), and so $W$ is the minimal such submodule of $X(\theta)$. In particular, $W = W_{\mathrm{ctf}}$, and so $W$ is cotorsion free over $\mathbb{T}_{\overline{\rho},\Sigma}$. Since $W$ satisfies condition 1 of Proposition 6.4.2, it must also satisfy condition 5 of that proposition, and so $X(\theta)_{\mathrm{ctf}} \subset W$. We conclude that $W = X(\theta)_{\mathrm{ctf}}$, as claimed. □

We will need the following strengthening of Lemma 6.3.3.

6.4.6. **Proposition.** *Suppose that there exists a saturated coadmissible* $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-*submodule* $Y$ *of* $X(\theta)$, *satisfying the equivalent conditions of Proposition* 6.4.2, *and for which* $\mathrm{ev}_Y$ *is injective. Then if* $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ *is a closed point for which* $\rho(\mathfrak{p})$ *is an extension* (*possibly trivial*) *of distinct characters, there is an isomorphism* $\theta(\mathfrak{p}) \cong \pi(\mathfrak{p})$.

*Proof.* Write $\rho(\mathfrak{p})$ as an extension $0 \to \chi_1 \to \rho(\mathfrak{p}) \to \chi_2 \to 0$, with $\chi_1 \neq \chi_2$. Since $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ satisfies Assumption 3.3.1, we see that $\chi_1\chi_2^{-1} \neq \varepsilon^{-1}$, while if $\chi_1\chi_2^{-1} = \varepsilon$, then

$p \geq 5$ and $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is a twist of a non-split extension of $\overline{\varepsilon}^{-1}$ by 1. In particular, in this latter case $\overline{\rho}_{|G_{\mathbb{Q}_p}}$, and so also $\overline{\pi}$, has trivial endomorphisms, the deformation functor $\mathrm{Def}(\overline{\pi})$ is representable, hence $\theta \cong \pi^{\mathrm{m}}_{\Sigma}$ (see Remark 6.3.2), and, in particular, $\theta(\mathfrak{p}) \cong \pi(\mathfrak{p})$. Thus we may assume that $\chi_1\chi_2^{-1} \neq \underline{1}, \varepsilon^{\pm 1}$.

Choose an allowable level $K_{\Sigma_0}$ for $\overline{\rho}$ so that $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}} \xrightarrow{\sim} \mathbb{T}_{\overline{\rho},\Sigma}$, and hence such that $Y^{K_{\Sigma_0}}$ is faithful as a $\mathbb{T}_{\overline{\rho},\Sigma}$-module (by Proposition 6.4.2 (2)). The evaluation map $\mathrm{ev}_Y$ is assumed to be an embedding, and thus so is the evaluation map $\mathrm{ev}(K_{\Sigma_0})_Y$. This latter map in turn induces an embedding

$$\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} (\theta/\mathfrak{p}^n\theta) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} Y^{K_{\Sigma_0}}[\mathfrak{p}^n] \hookrightarrow \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}[\mathfrak{p}^n].$$

Since $\mathrm{Ord}_B$ is a left-exact functor, Corollary 5.6.13 then implies that the action of $G_{\mathbb{Q}_p}$ on the cokernel of the embedding

$$\frac{\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \mathrm{Ord}_B\big((\theta/\mathfrak{p}^n\theta) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}^n} Y^{K_{\Sigma}}[\mathfrak{p}^n]\big)}{\Big(\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \mathrm{Ord}_B\big((\theta/\mathfrak{p}^n\theta) \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}^n} Y^{K_{\Sigma}}[\mathfrak{p}^n]\big)\Big)^{\mathrm{ab},S}}$$

factors through $G^{\mathrm{ab}}_{\mathbb{Q}_p}$. The claim of the proposition now follows from Proposition 3.6.4. □

We now begin our study of evaluation maps.

6.4.7. **Theorem.** *Let $Y$ be a saturated coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$ satisfying the equivalent conditions of Proposition* 6.4.2*, and suppose furthermore either that $\theta = \pi^{\mathrm{m}}_{\Sigma}$ and $Y = X(\pi^{\mathrm{m}}_{\Sigma})$, or that $\mathrm{ev}_Y$ is injective. Then the map $\mathrm{ev}_{Y,E}$ is surjective (or equivalently, for each allowable level $K_{\Sigma_0} \subset G_{\Sigma_0}$, the map $\mathrm{ev}_Y(K_{\Sigma_0})_E$ is surjective).*

*Proof.* The equivalence of the two conditions is seen by alternately passing to $K_{\Sigma_0}$-invariants, and passing to the inductive limit over all $K_{\Sigma_0}$. Suppose that they hold. We claim that the image of $\mathrm{ev}_{Y,E}$ contains $\bigoplus_{\mathfrak{p}\in\mathcal{C}} \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\mathrm{l.alg}}$, where $\mathcal{C}$ is the set of closed points of $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}(K_{\Sigma_0})_{\overline{\rho},\Sigma}[1/p]$ which are classical, and whose associated Galois representations are crystalline at $p$. Since $\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\mathrm{l.alg}}$ is an irreducible $G_{\mathbb{Q}} \times G_p \times G_{\Sigma_0}$-representation, by [38, Thm. 7.4.2], it suffices in fact to show that the image of $\mathrm{ev}(\mathfrak{p})_Y$ contains a non-zero locally algebraic vector, for each $\mathfrak{p} \in \mathcal{C}$.

If the Galois representation $\rho(\mathfrak{p})$ associated to $\mathfrak{p} \in \mathcal{C}$ is irreducible locally at $p$, and is not exceptional at $p$ (in the sense of Definition 3.3.18), then the isomorphism (6.3.19) of Corollary 6.3.17, together with the equality $E \otimes_{\mathcal{O}} Y[\mathfrak{p}] = E \otimes_{\mathcal{O}} X(\theta)[\mathfrak{p}]$ given by part 3 of Proposition 6.4.2, shows directly that the image of $\mathrm{ev}(\mathfrak{p})_Y$ contains $\widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\mathrm{l.alg}}$. Suppose next either that $\rho(\mathfrak{p})$ is irreducible locally at $p$, but is exceptional at $p$ (note that, as a special case of the Tate conjecture, this is in fact conjectured never to occur; see e.g. [21]), or else that $\rho(\mathfrak{p})$ is reducible, but non-split. Note that in the latter case, since $\mathfrak{p}$ is classical, the two characters of which $\rho(\mathfrak{p})$ is an extension must be distinct. The hypothesis of the theorem, together with Proposition 6.4.6, shows that $\theta(\mathfrak{p}) \cong \pi(\mathfrak{p})$. Theorem 3.3.21 then implies that $\theta(\mathfrak{p})$ contains locally algebraic vectors, and Proposition C.36 implies that $E \otimes_{\mathcal{O}} Y[\mathfrak{p}] \neq 0$. Taking into account Proposition 6.2.2 (which does apply, since, as already noted, in the case when $\rho(\mathfrak{p})$ is an extension of two characters, the characters are distinct), we conclude that the image of $\mathrm{ev}(\mathfrak{p})_Y$ contains a non-zero locally algebraic vector.

Suppose now that $\rho(\mathfrak{p})$ is the direct sum of two characters. The representation $\theta(\mathfrak{p}) \cong \pi(\mathfrak{p})$ may then be decomposed as a direct sum $\theta(\mathfrak{p}) \cong \pi_1 \bigoplus \pi_2$, where $\pi_1$

is equal to the universal unitary completion (in the sense of [37]) of the locally algebraic representation $\widetilde{\pi}_p(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}})$ attached to $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}$ as in the statement of [38, Conj. 3.3.1 (7)]. (See the discussion of Subsection 3.4.) If $\mathrm{ev}_Y$ is an injection, then the same is true of the induced evaluation map $\mathrm{ev}(\mathfrak{p})_Y$. Since Proposition C.36 implies that $E \otimes_{\mathcal{O}} Y[\mathfrak{p}] \neq 0$, we see again in this case that the image of $\mathrm{ev}(\mathfrak{p})_Y$ contains a non-zero locally algebraic vector. Finally, we consider the case when $Y = X(\pi_\Sigma^{\mathrm{m}})$. Since $\pi_1$ is the universal unitary completion of $\widetilde{\pi}_p(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}})$, we see from [38, Thm. 7.4.2] that there is a non-zero $G_{\mathbb{Q}} \times G$-equivariant embedding $\rho(\mathfrak{p}) \otimes_E \pi_1 \hookrightarrow \widehat{H}^1_{E,\rho,\Sigma}[\mathfrak{p}]$. Composing with the projection $\rho(\mathfrak{p}) \otimes_E \pi(\mathfrak{p}) \to \rho(\mathfrak{p}) \otimes_E \pi_1$, we obtain an element of $E \otimes_{\mathcal{O}} X(\pi_\Sigma^{\mathrm{m}})[\mathfrak{p}]$ whose image contains a non-zero locally algebraic vector. Thus the image of $\mathrm{ev}(\mathfrak{p})_{X(\pi_\Sigma^{\mathrm{m}})}$ contains a non-zero locally algebraic vector in this case as well.

Since, by Corollary 5.4.5, the direct sum $\bigoplus_{\mathfrak{p} \in \mathcal{C}} \widehat{H}^1_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\mathrm{l.alg}}$ is dense in $\widehat{H}^1_E$, we have proved that the image of $\mathrm{ev}_{Y,E}$ is dense in $\widehat{H}^1_E$. On the other hand, it follows from Theorem 6.3.12, together with Lemma 3.1.16, that $\rho_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \,\hat{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} (X(\theta))^{K_{\Sigma_0}}$ is a $\varpi$-adically admissible representation of $G$ over $\mathbb{T}_{\overline{\rho},\Sigma}$ for any allowable level $K_{\Sigma_0}$. The same is true of $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\Sigma}$, by Lemma 5.3.5. It follows from Proposition 3.1.3 that the image of $\mathrm{ev}(K_{\Sigma_0})_{E,Y}$ is closed. Passing to the limit over all $K_{\Sigma_0}$, we find that the image of $\mathrm{ev}_{E,Y}$ is closed. This proves the theorem. □

6.4.8. **Remark.** As remarked in footnote 9, in the case when $p > 2$, Paškūnas [68] has extended the results of [3] to cover the exceptional cases. Thus in this case it is possible to avoid the appeal to Theorem 3.3.21 in the proof of the preceding theorem. (Note that in the case when $\rho(\mathfrak{p})$ is reducible it is easy to show directly that $\pi(\mathfrak{p})$ (and hence $\theta(\mathfrak{p})$) contains non-zero locally algebraic vectors, whether or not $\rho(\mathfrak{p})$ is split).

On the other hand, it is possible to replace the application of the density results of Subsection 5.4 in the proof of Theorem 6.4.7 by instead making a wholesale appeal to Theorem 3.3.21. Indeed, if $\mathfrak{p}$ is a classical closed point of $\mathrm{Spec}\, \mathbb{T}(K_{\Sigma_0})_{\overline{\rho},\Sigma}[1/p]$ corresponding to a modular form of weight two, then Theorem 3.3.21, together with Remark 3.3.23, shows that $\pi(\mathfrak{p})$ contains non-zero smooth vectors, and so, arguing just as in the proof given above, one finds that the image of $\mathrm{ev}_{Y,E}$ contains all of the smooth vectors of $\widehat{H}^1_{E,\overline{\rho},\Sigma}$. These are dense in $\widehat{H}^1_{E,\overline{\rho},\Sigma}$ by construction, and hence the argument can again be concluded via an appeal to Proposition 3.1.3.

6.4.9. **Theorem.** *Let $Y$ be a coadmissible $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\Sigma_0}]$-submodule of $X(\theta)$. The following are equivalent:*

(1) *The map $\mathrm{ev}_Y$ is an isomorphism.*
(2) *$Y$ is a faithful $\mathbb{T}_{\overline{\rho},\Sigma}$-module, and the map $\mathrm{ev}_Y(\mathfrak{m})$ is injective.*

*If $Y$ satisfies these equivalent conditions, then furthermore:*

(3) *$Y = X(\theta)_{\mathrm{ctf}}$.*

*Proof.* Suppose first that $\mathrm{ev}_Y$ is an isomorphism. Since $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$ is a faithful $\mathbb{T}_{\overline{\rho},\Sigma}$-module, we see that $Y$ must be a faithful $\mathbb{T}_{\overline{\rho},\Sigma}$-module. Reducing the map $\mathrm{ev}_Y$ modulo $\varpi$, and then passing to the $\mathfrak{m}$-torsion in the source and target, we find that $\mathrm{ev}_Y(\mathfrak{m})$ is again an isomorphism, and so in particular, is injective. Thus 1 implies 2.

Conversely, if $\mathrm{ev}_Y(\mathfrak{m})$ is injective, then by Lemma C.46, we see that $\mathrm{ev}_Y$ is injective, with saturated image. Lemma C.52 then implies that $Y$ must be saturated in $X(\theta)$. If $Y$ is furthermore faithful as a $\mathbb{T}_{\overline{\rho},\Sigma}$-module, then Theorem 6.4.7 shows that $\mathrm{ev}_{E,Y}$ is surjective. It follows that $\mathrm{ev}_Y$ is in fact an isomorphism, and so 2 implies 1.

Suppose now that $\mathrm{ev}_Y$ satisfies conditions 1 and 2. As we have already noted, it follows that $Y$ is saturated in $X(\theta)$, and so condition 5 of Proposition 6.4.2 shows that

$$X(\theta)_{\mathrm{ctf}} \subset Y. \tag{6.4.10}$$

Since $X(\theta)_{\mathrm{ctf}}$ is saturated in $X(\theta)$, and so in $Y$, we conclude that the induced map

$$\big(X(\theta)_{\mathrm{ctf}}/\varpi X(\theta)_{\mathrm{ctf}}\big)[\mathfrak{m}] \to (Y/\varpi Y)[\mathfrak{m}]$$

is an embedding, and thus that $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}(\mathfrak{m})$ is injective, since $\mathrm{ev}_Y(\mathfrak{m})$ is. The equivalence of 1 and 2 of the present proposition, applied now to $X(\theta)_{\mathrm{ctf}}$, then shows that $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}$ is an isomorphism. Thus the inclusion (6.4.10) induces an isomorphism

$$\pi_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_\Sigma^{\mathrm{m}} \hat{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} X(\theta)_{\mathrm{ctf}} \xrightarrow{\sim} \pi_\Sigma^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_\Sigma^{\mathrm{m}} \hat{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} Y.$$

It follows from Lemma C.51 that the inclusion (6.4.10) is an equality, i.e. that part 3 holds. □

6.4.11. **Theorem.** *Suppose that $\overline{\rho}$ satisfies Assumption* 5.7.3. *If $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}$ is an isomorphism, then there is an isomorphism $X(\theta)_{\mathrm{ctf}} \cong \pi_{\Sigma_0}(\rho_\Sigma^{\mathrm{m}})$.*

*Proof.* We will let $\Pi$ denote the set of closed points $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ that are allowable in the sense of Definition 5.4.7, and will show that $X(\theta)_{\mathrm{ctf}}$ and $\Pi$ together satisfy the conditions of Theorem 4.4.1. That proposition will then show that $X(\theta)_{\mathrm{ctf}} \xrightarrow{\sim} \pi_{\Sigma_0}(\rho)$.

Since $X(\theta)_{\mathrm{ctf}}$ is saturated in $X(\theta)$, there is a natural embedding

$$X(\theta)_{\mathrm{ctf}}/\varpi X(\theta)_{\mathrm{ctf}} \hookrightarrow X(\theta)/\varpi X(\theta),$$

and hence also an embedding

$$\big(X(\theta)_{\mathrm{ctf}}/\varpi X(\theta)_{\mathrm{ctf}}\big)[\mathfrak{m}] \hookrightarrow \big(X(\theta)/\varpi X(\theta)\big)[\mathfrak{m}].$$

A consideration of the embedding (6.3.11), together with Proposition 6.3.22, then shows that $\big(X(\theta)_{\mathrm{ctf}}/\varpi X(\theta)_{\mathrm{ctf}}\big)[\mathfrak{m}]$ is a generic representation of $G_{\Sigma_0}$.

Our assumption on $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}$, together with the equivalence of conditions 1 and 2 in Theorem 6.4.9, shows that the map $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}(\mathfrak{m})$ induces an embedding

$$\overline{\rho} \otimes_k \overline{\pi} \otimes_k \big(X(\theta)_{\mathrm{ctf}}/\varpi X(\theta)_{\mathrm{ctf}}\big)[\mathfrak{m}] \hookrightarrow H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]. \tag{6.4.12}$$

Choose $V$ to be a good weight in $W^{\mathrm{gl}}(\overline{\rho})$ (in the sense of Definition 5.7.5). It follows from part 1 of Theorem 5.7.7 that $V$ lies in $W(\overline{\rho})$. Lemma 3.5.5 gives an isomorphism

$$\mathrm{m}(V,\overline{\rho}) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V,\overline{\pi}) \xrightarrow{\sim} \operatorname{Hom}_G(c\text{–}\operatorname{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)}^G V, \overline{\pi}).$$

Thus, applying the functor $F_{\Sigma_0} \operatorname{soc}_{\mathcal{H}(V)} \operatorname{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, -)$ to the embedding (6.4.12) yields an embedding

$$\begin{aligned}\overline{\rho} \otimes_k \operatorname{soc}_{\mathcal{H}(V)} \mathrm{m}(V,\overline{\rho}) \otimes_k F_{\Sigma_0}\Big(\big(X(\theta)_{\mathrm{ctf}}/\varpi X(\theta)_{\mathrm{ctf}}\big)[\mathfrak{m}]\Big) & \\ \hookrightarrow F_{\Sigma_0} \operatorname{soc}_{\mathcal{H}(V)} \operatorname{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) & \\ \xrightarrow{\sim} \overline{\rho} \otimes_k \operatorname{soc}_{\mathcal{H}(V)} \mathrm{m}(V,\overline{\rho}), &\end{aligned}$$

where the isomorphism is provided by part 2 of Theorem 5.7.7. We conclude that $F_{\Sigma_0}\Big(\big(X(\theta)_{\mathrm{ctf}}/\varpi X(\theta)_{\mathrm{ctf}}\big)[\mathfrak{m}]\Big)$ is at most one-dimensional. Combined with the result of the preceding paragraph, this shows that condition 1 of Theorem 4.4.1 holds.

Since each point of $\Pi$ is a classical closed point, the associated local representations $\rho(\mathfrak{p})_{|G_{\mathbb{Q}_\ell}}$ ($\ell \in \Sigma_0$) are always generic [51, p. 354]. Corollary 6.3.17 and Lemma 6.4.5 together show that for each $\mathfrak{p} \in \Pi$, we have a $G_{\Sigma_0}$-equivariant isomorphism (and so in particular a $G_{\Sigma_0}$-equivariant map) $\pi_{\Sigma_0}(\rho(\mathfrak{p})) \xrightarrow{\sim} E \otimes_{\mathcal{O}} X(\theta)_{\mathrm{ctf}}[\mathfrak{p}]$. Lemma 6.4.5 also shows that the saturation of $\sum_{\mathfrak{p}} X(\theta)_{\mathrm{ctf}}[\mathfrak{p}]$ (where $\mathfrak{p}$ ranges over all elements of $\Pi$) is dense in $X(\theta)_{\mathrm{ctf}}$. Thus conditions 2 (a), (b), and (c) of Theorem 4.4.1 all hold. As already noted, this completes the proof of the theorem. □

6.4.13. **Corollary.** *If there is an isomorphism of $\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}} \times G \times G_{\Sigma_0}]$-modules*

$$\rho_{\Sigma}^{\mathrm{m}} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \,\widehat{\otimes}_{\mathbb{T}_{\overline{\rho},\Sigma}} \pi_{\Sigma_0}(\rho_{\Sigma}^{\mathrm{m}}) \xrightarrow{\sim} \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}, \tag{6.4.14}$$

*then $X(\theta)_{\mathrm{ctf}} \cong \pi_{\Sigma_0}(\rho_{\Sigma}^{\mathrm{m}})$, and the evaluation map $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}$ is an isomorphism. Conversely, if $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}$ is an isomorphism, and if $\overline{\rho}$ satisfies Assumption* 5.7.3, *then $X(\theta)_{\mathrm{ctf}} \cong \pi_{\Sigma_0}(\rho_{\Sigma}^{\mathrm{m}})$, and there is an isomorphism of the form* (6.4.14).

*Proof.* If there is an isomorphism of the form (6.4.14) then it gives rise to an embedding $\pi_{\Sigma_0}(\rho_{\Sigma}^{\mathrm{m}}) \hookrightarrow X(\theta)$, such that, letting $Y$ denote the image of this map, the evaluation map $\mathrm{ev}_Y$ is an isomorphism. Theorem 6.4.9 then shows that $Y = X(\theta)_{\mathrm{ctf}}$.

Conversely, if $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}$ is an isomorphism, and if $\overline{\rho}$ satisfies Assumption 5.7.3, then Theorem 6.4.11 shows that $X(\theta)_{\mathrm{ctf}} \cong \pi_{\Sigma_0}(\rho_{\Sigma}^{\mathrm{m}})$, and so $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}$ gives rise to an isomorphism of the form (6.4.14). □

The preceding results are most easily applied in the case when $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ has only trivial endomorphisms (in which case, as we noted in Remark 6.3.2, we necessarily have that $\theta \cong \rho_{\Sigma}^{\mathrm{m}}$).

We first prove a lemma.

6.4.15. **Lemma.** *Suppose that $\overline{\pi}_1$ and $\overline{\pi}_2$ are smooth representations of $G$ over $k$, that $U$ is a $k$-vector space, and that $f : \overline{\pi}_1 \otimes_k U \to \overline{\pi}_2$ is a $G$-equivariant $k$-linear map (the $G$-action on the source being defined via its action on the first factor in the tensor product). If furthermore $\overline{\pi}_1$ is admissible and* $\mathrm{soc}(\overline{\pi}_1)$ *(the $G$-socle of $\overline{\pi}_1$) is multiplicity free, and if for every non-zero element $u \in U$, the map $\overline{\pi}_1 \xrightarrow{\sim} \overline{\pi}_1 \otimes_k (k \cdot u) \to \overline{\pi}_2$ induced by $f$ is an embedding, then $f$ itself is an embedding.*

*Proof.* Write $\mathrm{soc}(\overline{\pi}_1) = \bigoplus_{i=1}^s \overline{\pi}_{1,i}$, where the $\overline{\pi}_{1,i}$ are pair-wise non-isomorphic irreducible admissible smooth $G$-representations. We have the evident isomorphism

$$\bigoplus_{i=1}^{s} \overline{\pi}_{1,i} \otimes_k U \xrightarrow{\sim} \mathrm{soc}(\overline{\pi}_1) \otimes_k U \xrightarrow{\sim} \mathrm{soc}(\overline{\pi}_1 \otimes_k U).$$

If $f$ has a non-zero kernel, then this kernel has a non-zero socle, and so has non-zero intersection with $\overline{\pi}_{1,i} \otimes_k U$ for at least one value of $i$ (since the $\overline{\pi}_{1,i}$ are pair-wise non-isomorphic), and so contains $\overline{\pi}_{1,i} \otimes_k (k \cdot u)$ for some non-zero $u \in U$. Consequently, the map $\overline{\pi}_1 \otimes_k (k \cdot u) \to \overline{\pi}_2$ is not injective, contradicting our assumption. □

We now prove our main theorem in the case when $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ has only trivial endomorphisms.

6.4.16. **Theorem.** *If* $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ *has only trivial endomorphisms, then the evaluation map* $\mathrm{ev}_{X(\pi_\Sigma^{\mathfrak{m}})}$ *is an isomorphism, and hence Conjecture* 6.1.6 *holds.*

*Proof.* If we show that $\mathrm{ev}_{X(\pi_\Sigma^{\mathfrak{m}})}$ is an isomorphism, then part 3 of Theorem 6.4.9 and Corollary 6.4.13 together imply that Conjecture 6.1.6 holds. To prove that $\mathrm{ev}_{X(\pi_\Sigma^{\mathfrak{m}})}$ is an isomorphism, it suffices, by 2 of Theorem 6.4.9, to prove that the map $\mathrm{ev}(\mathfrak{m})_{X(\pi_\Sigma^{\mathfrak{m}})}$ is injective. For this, it suffices in turn to prove that the evaluation map

$$\overline{\rho}\otimes_k \overline{\pi}\otimes_k \mathrm{Hom}_{k[G_{\mathbb{Q}}\times G]}(\overline{\rho}\otimes_k\overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])\to H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}] \tag{6.4.17}$$

is injective. Since $\overline{\rho}$ is absolutely irreducible, the evaluation map

$$\overline{\rho}\otimes_k \mathrm{Hom}_{k[G_{\mathbb{Q}}]}(\overline{\rho}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])\to H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]$$

is injective, and so to prove the injectivity of (6.4.17), it suffices in turn to prove that the induced map

$$\overline{\pi}\otimes_k \mathrm{Hom}_{k[G_{\mathbb{Q}}\times G]}(\overline{\rho}\otimes_k\overline{\pi}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])\to \mathrm{Hom}_{k[G_{\mathbb{Q}}]}(\overline{\rho}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]) \tag{6.4.18}$$

is injective.

By Lemma 6.4.15, to prove the injectivity of (6.4.18), it suffices to prove that any non-zero $G$-equivariant map

$$\overline{\pi}\to \mathrm{Hom}_{k[G_{\mathbb{Q}}]}(\overline{\rho}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])$$

is injective. Since

$$\mathrm{Hom}_{k[G_{\mathbb{Q}}]}(\overline{\rho}, H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}])\hookrightarrow \overline{\rho}^{\vee}\otimes_k H^1_{k,\overline{\rho},\Sigma}\xrightarrow{\sim} H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]\oplus H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]$$

(the latter $G$-equivariant isomorphism being obtained by choosing a $k$-basis for $\overline{\rho}$), we see that it in fact suffices to show that any non-zero $G$-equivariant map

$$\overline{\pi}\to H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}] \tag{6.4.19}$$

is injective. If $\overline{\rho}_{|G_{\mathbb{Q}_p}}$, and hence $\overline{\pi}$, is irreducible, this is clear. If not, then write $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ as an extension

$$0\to\chi_1\to\overline{\rho}\to\chi_2\to 0.$$

Since $\overline{\rho}$ is assumed to have only trivial endomorphisms, we have $\chi_1\neq\chi_2$, and this extension is non-split. By Assumption 3.3.1, we also have $\chi_1\chi_2^{-1}\neq\overline{\varepsilon}^{-1}$. Thus if $\chi_1\chi_2^{-1}\neq\overline{\varepsilon}$, then $\overline{\pi}$ is an extension of $\mathrm{Ind}_B^G\chi_2\otimes\chi_1\overline{\varepsilon}$ by the non-isomorphic representation $\mathrm{Ind}_B^G\chi_1\otimes\chi_2\overline{\varepsilon}$, with each of these principal series representations being irreducible. The $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of the first of these principal series consists of a single Serre weight, which does not line in $W(\overline{\rho})$. Taking into account part 1 of Theorem 5.7.7, we find that there is no embedding of $\mathrm{Ind}_B^G\chi_2\otimes\chi_1\overline{\varepsilon}$ into $H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}]$, and thus that indeed any non-zero map of the form (6.4.19) is injective.

If $\chi_1\chi_2^{-1}=\overline{\varepsilon}$, then, writing $\chi:=\chi_1$, we find that $\overline{\pi}$ is a non-split extension of $\mathrm{Ind}_B^G\chi\overline{\varepsilon}^{-1}\otimes\chi\overline{\varepsilon}$ (which is irreducible) by a representation $\overline{\pi}_1$ which itself sits in a non-split short exact sequence

$$0\to(\chi\circ\det)\otimes_k\mathrm{St}\to\overline{\pi}_1\to\chi\circ\det\to 0.$$

The unique Serre weight in the $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of $\mathrm{Ind}_B^G\chi\overline{\varepsilon}^{-1}\otimes\chi\overline{\varepsilon}$ does not lie in $W(\overline{\rho})$, and so a non-zero map of the form (6.4.19) cannot contain all of $\overline{\pi}_1$ in its kernel. On the other hand, if it contains just $(\chi\circ\det)\otimes_k\mathrm{St}$ in its kernel, then we find that $H^1_{k,\overline{\rho},\Sigma}$ contains a one-dimensional $G$-invariant subrepresentation, which is impossible, by Ihara's Lemma. Thus any non-zero map of the form (6.4.19) must in fact be injective. This completes the argument. □

When $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is the sum of two distinct characters, we can establish a slightly weaker result.

6.4.20. **Theorem.** *If* $\overline{\rho}_{\,|G_{\mathbb{Q}_p}}$ *is isomorphic to the direct sum of distinct characters, then Conjecture* 6.1.9 *holds.*

The somewhat lengthy proof of the theorem is presented in the following subsection.

6.5. **Proof of Theorem 6.4.20.** Throughout this subsection we suppose that $\overline{\rho} = \chi_1 \bigoplus \chi_2$, with $\chi_1\chi_2^{-1} \neq \underline{1}, \overline{\varepsilon}^{\pm 1}$. (The arguments typically apply more generally, but we have no need for them other than in this case.) We introduce the following hypothesis on a point $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$:

6.5.1. **Assumption.** *The closed point* $\mathfrak{p}$ *is allowable, in the sense of Definition* 5.4.7*, and* $E \xrightarrow{\sim} \kappa(\mathfrak{p})$ (*or, equivalently,* $\mathcal{O} \xrightarrow{\sim} \mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}$).

The following result lies at the heart of our argument.

6.5.2. **Proposition.** *If* $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ *satisfies Assumption* 6.5.1*, then there is an isomorphism of* $\mathcal{O}[G_{\Sigma_0} \times G]$*-modules*

$$\rho(\mathfrak{p})^\circ \otimes_{\mathcal{O}} \pi(\mathfrak{p})^\circ \xrightarrow{\sim} F_{\Sigma_0}(M(\mathfrak{p})_{\mathcal{O}})$$

(*where we refer to Definition* 6.3.14 *for the definition of* $M(\mathfrak{p})_{\mathcal{O}}$).

*Proof.* Applying the functor $F_{\Sigma_0}$ to the isomorphism of part 1 of Proposition 6.3.15, we obtain an isomorphism

$$F_{\Sigma_0}(M(\mathfrak{p})_E) \xrightarrow{\sim} \rho(p) \otimes_E \pi(\mathfrak{p}). \tag{6.5.3}$$

Thus $F_{\Sigma_0}(M(\mathfrak{p})_{\mathcal{O}})$ is a bounded open $G_{\mathbb{Q}} \times G$-invariant lattice in $\rho(p) \otimes_E \pi(\mathfrak{p})$. Since $\overline{\rho}$ is absolutely irreducible, any such lattice is of the form

$$\rho(\mathfrak{p})^\circ \otimes_{\mathcal{O}} \pi^\circ,$$

for some bounded open $G$-invariant lattice $\pi^\circ$ in $\pi(\mathfrak{p})$. In particular, we may write

$$F_{\Sigma_0}(M(\mathfrak{p})_{\mathcal{O}}) \xrightarrow{\sim} \rho(\mathfrak{p})^\circ \otimes_{\mathcal{O}} \pi^\circ,$$

for some lattice $\pi^\circ$. Our goal is to show that $\pi^\circ \xrightarrow{\sim} \pi(\mathfrak{p})^\circ$.

Since $M(\mathfrak{p})_{\mathcal{O}}$ is saturated in $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$, we see that the induced embedding

$$\rho(\mathfrak{p})^\circ \otimes_{\mathcal{O}} \pi^\circ \xrightarrow{\sim} F_{\Sigma_0}(M(\mathfrak{p})_{\mathcal{O}})) \hookrightarrow F_{\Sigma_0}(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})$$

is saturated, and hence that reduction modulo $\varpi^n$ induces an embedding

$$(\rho(\mathfrak{p})^\circ/\varpi^n\rho(\mathfrak{p})) \otimes_{\mathcal{O}/\varpi^n} (\pi^\circ/\varpi^n\pi^\circ) \hookrightarrow F_{\Sigma_0}(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^n\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}),$$

for any $n \geq 0$. Taking ordinary parts, we obtain an embedding

$$\begin{aligned}(\rho(\mathfrak{p})^\circ/\varpi^n\rho(\mathfrak{p})) \otimes_{\mathcal{O}/\varpi^n} \operatorname{Ord}_B(\pi^\circ/\varpi^n\pi^\circ) &\hookrightarrow \operatorname{Ord}_B\big(F_{\Sigma_0}(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^n\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})\big)\\ &\xrightarrow{\sim} F_{\Sigma_0}\big(\operatorname{Ord}_B(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}/\varpi^n\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})\big).\end{aligned}$$

From Theorem 5.6.11, we conclude that the action of $G_{\mathbb{Q}_p}$ on the quotient

$$\frac{(\rho(\mathfrak{p})^\circ/\varpi^n\rho(\mathfrak{p})) \otimes_{\mathcal{O}/\varpi^n} \operatorname{Ord}_B(\pi^\circ/\varpi^n\pi^\circ)}{(\rho(\mathfrak{p})^\circ/\varpi^n\rho(\mathfrak{p})) \otimes_{\mathcal{O}/\varpi^n} \operatorname{Ord}_B(\pi^\circ/\varpi^n\pi^\circ)^{\mathrm{ab},S}}$$

factors through $G^{\mathrm{ab}}_{\mathbb{Q}_p}$. The theorem is now seen to follow from Proposition 3.6.3. □

Before stating our next results, we introduce some notation.

6.5.4. **Definition.** We write $\mathbb{T}'_{\overline{\rho},\Sigma} := \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(\mathbb{T}'_{\overline{\rho},\Sigma})$ to denote the space of $\mathcal{O}$-linear, $\mathfrak{m}$-adically continuous maps from $\mathbb{T}_{\overline{\rho},\Sigma}$ to $\mathcal{O}$.

6.5.5. **Definition.** Suppose that $I$ is an allowable ideal in $\mathbb{T}_{\overline{\rho},\Sigma}$, and write $I = \bigcap_{i=1}^n \mathfrak{p}_i$, where the $\mathfrak{p}_i$ are distinct allowable closed points of $\mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$. There is an embedding $\bigoplus_{i=1}^n M(\mathfrak{p}_i)_E \hookrightarrow \widehat{H}^1_{E,\overline{\rho},\Sigma_0}$. Since

$$E\otimes_{\mathcal{O}}(\mathbb{T}_{\overline{\rho},\Sigma}/I) \xrightarrow{\sim} \prod_{i=1}^n E\otimes_{\mathcal{O}}(\mathbb{T}_{\overline{\rho},\Sigma}/\mathfrak{p}_i),$$

we see that the image of this embedding may be identified with the closure of the space $\widehat{H}^1_{E,\overline{\rho},\Sigma}[I]_{\mathrm{l.alg}}$ of locally algebraic elements of $\widehat{H}^1_{E,\overline{\rho},\Sigma}$ that are annihilated by $I$. Generalizing the notation of Definition 6.3.14, we denote the image of this embedding by $M(I)_E$, and write $M(I)_{\mathcal{O}} := M(I)_E \bigcap \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$.

Following Proposition 5.5.3, we make the factorization

$$\rho^{\mathrm{m}}_{\Sigma}\otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} U_{\Sigma} \xrightarrow{\sim} \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma},$$

where $U_{\Sigma} := \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}]}(\rho^{\mathrm{m}}_{\Sigma}, \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})$ is a $G\times G_{\Sigma_0}$-representation. This factorization induces a corresponding factorization

$$\rho^{\mathrm{m}}_{\Sigma}\otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}]}\big(\rho^{\mathrm{m}}_{\Sigma}, M(I)_{\mathcal{O}}\big) \xrightarrow{\sim} M(I)_{\mathcal{O}}.$$

6.5.6. **Definition.** If $I\subset \mathbb{T}_{\overline{\rho},\Sigma}$ is an allowable ideal, then we write

$$W_I := F_{\Sigma_0}\Big(\mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}]}\big(\rho^{\mathrm{m}}_{\Sigma}, M(I)_{\mathcal{O}}\big)\Big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}]}\Big(\rho^{\mathrm{m}}_{\Sigma}, F_{\Sigma_0}\big(M(I)_{\mathcal{O}}\big)\Big),$$

and

$$\theta_I := \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}}\big((\mathbb{T}_{\overline{\rho},\Sigma}/I)', W_I\big)$$

(where, following the notation introduced in Subsection 3.7, we write $(\mathbb{T}_{\overline{\rho},\Sigma}/I)' := \mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\overline{\rho},\Sigma}/I, \mathcal{O})$), so that there is a natural evaluation map

$$(\mathbb{T}_{\overline{\rho},\Sigma}/I)'\otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_I \to W_I. \tag{6.5.7}$$

Both $W_I$ and $\theta_I$ are naturally $G$-representations over $\mathbb{T}_{\overline{\rho},\Sigma}/I$, and (6.5.7) is a map of $(\mathbb{T}_{\overline{\rho},\Sigma}/I)[G]$-modules.

6.5.8. **Lemma.** *If $J\subset I$ is an inclusion of allowable ideals in $\mathbb{T}_{\overline{\rho},\Sigma}$, then there is a natural isomorphism $W_J \xrightarrow{\sim} W_I[J]$.*

*Proof.* Evidently the inclusion $M(J)_{\mathcal{O}}\subset M(I)_{\mathcal{O}}$ induces an isomorphism

$$M(J)_{\mathcal{O}} \xrightarrow{\sim} M(I)_{\mathcal{O}}[J].$$

The lemma follows directly from this and from the definition of $W_I$ and $W_J$ in terms of $M(I)_{\mathcal{O}}$ and $M(J)_{\mathcal{O}}$. □

We immediately deduce the following corollary from Proposition 6.5.2.

6.5.9. **Corollary.** *If $\mathfrak{p}\in \mathrm{Spec}\,\mathbb{T}_{\overline{\rho},\Sigma}[1/p]$ satisfies Assumption 6.5.1, then there is a $G$-equivariant isomorphism $W_{\mathfrak{p}}\cong \pi(\mathfrak{p})^{\circ}$.*

Our goal is now to prove a partial generalization of the preceding corollary for a general allowable ideal. In the course of the argument it will be necessary to make an extension of scalars (so that we may put ourselves in a situation where the prime ideals under consideration satisfy Assumption 6.5.1), and so we begin by introducing some notation, and proving a lemma, related to such an extension.

Suppose that $E'$ is a finite extension of $E$, let $\mathcal{O}'$ be the integral closure of $\mathcal{O}$ in $E'$, and let $k'$ be the residue field of $\mathcal{O}'$ (a finite extension of $k$). Write $\overline{\rho}' := k' \otimes_k \rho$. We define $\widehat{H}^1_{E',\overline{\rho}',\Sigma}$, $\widehat{H}^1_{\mathcal{O}',\overline{\rho}',\Sigma}$, and $\mathbb{T}_{\overline{\rho}',\Sigma}$ in analogy with $\widehat{H}^1_{E,\overline{\rho},\Sigma}$, $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$, and $\mathbb{T}_{\overline{\rho},\Sigma}$. We furthermore define $\rho^{\mathrm{m}\prime}_{\Sigma}$ and $\pi^{\mathrm{m}\prime}_{\Sigma}$ in analogy with $\rho^{\mathrm{m}}_{\Sigma}$ and $\pi^{\mathrm{m}}_{\Sigma}$. Thus $\rho^{\mathrm{m}\prime}_{\Sigma}$ denotes the universal promodular deformation of $\overline{\rho}'$ over $\mathbb{T}_{\overline{\rho}',\Sigma}$, while $\pi^{\mathrm{m}\prime}_{\Sigma}$ is defined so that $\mathrm{MF}(\pi^{\mathrm{m}\prime}_{\Sigma}) \xrightarrow{\sim} \rho^{\mathrm{m}\prime}_{\Sigma|G_{\mathbb{Q}_p}}$, with $\pi^{\mathrm{m}\prime}_{\Sigma|G_{\mathbb{Q}_p}}$ admitting the central character $\det(\rho^{\mathrm{m}\prime}_{\Sigma|G_{\mathbb{Q}_p}})\varepsilon$. There are natural isomorphisms $E' \otimes_E \widehat{H}^1_{E,\overline{\rho},\Sigma} \xrightarrow{\sim} \widehat{H}^1_{E',\overline{\rho}',\Sigma}$, $\mathcal{O}' \otimes_{\mathcal{O}} \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma} \xrightarrow{\sim} \widehat{H}^1_{\mathcal{O}',\overline{\rho}',\Sigma}$, $\mathcal{O}' \otimes_{\mathcal{O}} \mathbb{T}_{\overline{\rho},\Sigma} \xrightarrow{\sim} \mathbb{T}_{\overline{\rho}',\Sigma}$, $\mathcal{O}' \otimes_{\mathcal{O}} \rho^{\mathrm{m}}_{\Sigma} \xrightarrow{\sim} \rho^{\mathrm{m}\prime}_{\Sigma}$, and $\mathcal{O}' \otimes_{\mathcal{O}} \pi^{\mathrm{m}}_{\Sigma} \xrightarrow{\sim} \pi^{\mathrm{m}\prime}_{\Sigma}$.

6.5.10. **Lemma.** *If $I$ is an allowable ideal* (*in the sense of Definition* 6.4.4) *in* $\mathbb{T}_{\overline{\rho},\Sigma}$, *then $I' := \mathcal{O}' \otimes_{\mathcal{O}} I$ is an allowable ideal in $\mathbb{T}_{\overline{\rho}',\Sigma}$.*

*Proof.* Write $I = \bigcap_{i=1}^n \mathfrak{p}_n$ as in Definition 6.4.4. If we tensor the embedding

$$\mathbb{T}_{\overline{\rho},\Sigma}/I \hookrightarrow \prod_{i=1}^n \kappa(\mathfrak{p}_i)$$

with $\mathcal{O}'$ over $\mathcal{O}$, we obtain an embedding

$$\mathcal{O}' \otimes_{\mathcal{O}} (\mathbb{T}_{\overline{\rho},\Sigma}/I) \hookrightarrow \prod_{i=1}^n E' \otimes_E \kappa(\mathfrak{p}_i).$$

Since $E'$ and each $\kappa(\mathfrak{p}_i)$ are fields of characteristic zero, the target of this embedding is reduced. Thus so is the source. From the isomorphism

$$\mathbb{T}_{\overline{\rho}',\Sigma}/I' \xrightarrow{\sim} \mathcal{O}' \otimes_{\mathcal{O}} (\mathbb{T}_{\overline{\rho},\Sigma}/I),$$

we conclude that $I'$ is a radical ideal, say $I' = \bigcap_{j=1}^r \mathfrak{q}_j$. Each $\mathfrak{q}_j$ contains one of the $\mathfrak{p}_i$, and since $\mathbb{T}_{\overline{\rho}',\Sigma}$ is a finite extension of $\mathbb{T}_{\overline{\rho},\Sigma}$, the corresponding residue field extension $\kappa(\mathfrak{q}_j)/\kappa(\mathfrak{p}_i)$ is finite. Thus $\mathfrak{q}_j$ is a classical closed point of $\mathbb{T}_{\overline{\rho}',\Sigma}$, and there is an isomorphism $\rho(\mathfrak{q}_j) \xrightarrow{\sim} \kappa(\mathfrak{q}_j) \otimes_{\kappa(\mathfrak{p}_i)} \rho(\mathfrak{p}_i)$. (If $\mathfrak{p}'_i$ corresponds to a cuspidal Hecke eigenform defined over $\kappa(\mathfrak{p}_i)$, then $\mathfrak{q}_j$ corresponds to the same Hecke eigenform, but regarded as being defined over $\kappa(\mathfrak{q}_j)$.) Thus $I'$ is allowable. □

We now state and prove our generalization of Corollary 6.5.9.

6.5.11. **Proposition.** *If $I$ is an allowable ideal, then $\theta_I$ is a deformation of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma}/I$, and the evaluation map* (6.5.7) *is an isomorphism.*

*Proof.* Write $I = \bigcap_{i=1}^n \mathfrak{p}_i$, as in Definition 6.4.4. Choose $E'$ to be a finite Galois extension of $E$, sufficiently large so that each of the residue fields $\kappa(\mathfrak{p}_i)$ admits an embedding into $E'$. Lemma 6.5.10 shows that $I' := \mathcal{O}' \otimes_{\mathcal{O}} I$ is an allowable ideal of $\mathbb{T}_{\overline{\rho}',\Sigma}$, and if we write $I' = \bigcap_{j=1}^r \mathfrak{q}_j$, then by virtue of our choice of $E'$, each of the residue fields $\kappa(\mathfrak{q}_j)$ is equal to $E'$.

Clearly there are natural isomorphisms $W_{I'} \xrightarrow{\sim} \mathcal{O}' \otimes_{\mathcal{O}} W_I$ and $\theta_{I'} := \mathcal{O}' \otimes_{\mathcal{O}} \theta_I$, and the evaluation map

$$(\mathbb{T}_{\overline{\rho}',\Sigma_0'}/I')' \otimes_{\mathbb{T}_{\overline{\rho}',\Sigma_0}} \theta_{I'} \to W_{I'}. \tag{6.5.12}$$

is obtained from the map (6.5.7) by tensoring with $\mathcal{O}'$ over $\mathcal{O}$. Thus it suffices to prove the proposition with $E$, $\overline{\rho}$, $I$, etc., replaced by $E'$, $\overline{\rho}'$, $I'$, etc., and hence, replacing $E$ by $E'$, $\overline{\rho}$ by $\overline{\rho}'$, $I$ by $I'$, and so on, it is no loss of generality to assume from the beginning that for each $\mathfrak{p}_i$ appearing in the decomposition of $I$, the residue field $\kappa(\mathfrak{p}_i)$ is equal to $E$, and we do so from now on.

We will now prove the statement of the theorem (under the additional assumption that $\kappa(\mathfrak{p}_i) = E$ for each $i = 1, \ldots, n$) by induction on $n$. The base case $n = 1$ follows from Corollary 6.5.9. Suppose now that $n > 1$, and write $J := \bigcap_{i=1}^{n-1} \mathfrak{p}_i$, $\mathfrak{p} := \mathfrak{p}_n$. By induction, $\theta_J$ is a deformation of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma}/J$, and the evaluation map $(\mathbb{T}_{\overline{\rho},\Sigma}/J)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_J \to W_J$ is an isomorphism.

Lemma 6.5.8 provides isomorphisms $W_I[J] \xrightarrow{\sim} W_J$ and $W_I[\mathfrak{p}] \xrightarrow{\sim} W_{\mathfrak{p}}$, and so it follows from the induction hypothesis, Theorem 5.7.7, Lemma 3.5.5 (2), and Proposition 3.7.11 that $W_I := (\mathbb{T}_{\overline{\rho},\Sigma}/I)' \otimes_{T_{\overline{\rho},\Sigma}} X$ for some deformation $X$ of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma}/I$. It then follows from Lemma 3.7.10 that $X \xrightarrow{\sim} \theta_I$, showing that $\theta_I$ is a deformation of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma}/I$, and that the evaluation map (6.5.7) is an isomorphism. □

We note one particular corollary of the preceding result.

6.5.13. **Corollary.** *If $J \subset I$ is an inclusion of allowable ideals, then there is a natural isomorphism $\theta_I/J\theta_I \xrightarrow{\sim} \theta_J$.*

*Proof.* Proposition 6.5.11 shows that the natural evaluation maps provide isomorphisms

$$(\mathbb{T}_{\overline{\rho},\Sigma}/I)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_I \xrightarrow{\sim} W_I \tag{6.5.14}$$

and

$$(\mathbb{T}_{\overline{\rho},\Sigma}/J)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_J \xrightarrow{\sim} W_J. \tag{6.5.15}$$

Now consider the composite of the sequence of isomorphisms:

$$\begin{aligned}(\mathbb{T}_{\overline{\rho},\Sigma}/J)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_I/J\theta_I \xrightarrow{\sim} (\mathbb{T}_{\overline{\rho},\Sigma}/J)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_I \xrightarrow{\sim} \big((\mathbb{T}_{\overline{\rho},\Sigma}/I)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_I\big)[J] \\ \xrightarrow{\sim} W_I[J] \xrightarrow{\sim} W_J \xrightarrow{\sim} \mathbb{T}'_{\overline{\rho},\Sigma}[J] \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_J,\end{aligned}$$

of which the first is evident, the second is obtained by applying part 3 of Lemma B.6 with $A$ taken to be $\mathbb{T}_{\overline{\rho},\Sigma}/I$, $M$ taken to be $\mathbb{T}_{\overline{\rho},\Sigma}/J$, and $N$ taken to be $(\mathbb{T}_{\overline{\rho},\Sigma}/I)'$, the third is (6.5.14), the fourth is obtained from Lemma 6.5.8, and the fifth is the inverse of (6.5.15). Applying $\mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}}(\mathbb{T}'_{\overline{\rho},\Sigma}[J], -)$ to the source and target of this composite isomorphism, and taking into account Lemma 3.7.10, we obtain the required isomorphism of deformations $\theta_I/J\theta_I \xrightarrow{\sim} \theta_J$. □

We are now ready to give the:

*Proof of Theorem 6.4.20.* Corollary 6.5.13 allows us to form the projective limit $\theta := \varprojlim_I \theta_I$, the projective limit being taken over all allowable ideals $I$ of $\mathbb{T}_{\overline{\rho},\Sigma}$, to obtain a deformation $\theta$ of $\overline{\pi}$ over $\mathbb{T}_{\overline{\rho},\Sigma} \xrightarrow{\sim} \varprojlim_I \mathbb{T}_{\overline{\rho},\Sigma}/I$. By construction, $\theta/I\theta \xrightarrow{\sim} \theta_I$ for any allowable ideal $I \subset \mathbb{T}_{\overline{\rho},\Sigma}$. In particular, if $\mathfrak{p}$ is an allowable prime ideal, we compute that

$$\theta(\mathfrak{p}) \xrightarrow{\sim} E \otimes_{\mathcal{O}} \theta_{\mathfrak{p}} \xrightarrow{\sim} F_{\Sigma_0}\big(\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho(\mathfrak{p}), M(I)_E)\big) \xrightarrow{\sim} \pi(\mathfrak{p}),$$

the final isomorphism following from part 1 of Proposition 6.3.15. Thus $\theta$ satisfies Assumption 6.3.1. Our goal is to prove that $\mathrm{ev}_{X(\theta)_{\mathrm{ctf}}}$ is an isomorphism; the claims of the theorem will follow from this, together with Corollary 6.4.13 and Proposition 6.4.6.

Proposition 6.4.2 shows that $X(\theta)_{\mathrm{ctf}}$ is a faithful $\mathbb{T}_{\overline{\rho},\Sigma}$-module, and so by Theorem 6.4.9, it suffices to prove that $\mathrm{ev}(\mathfrak{m})_{X(\theta)_{\mathrm{ctf}}}$ is injective. Since, by Lemma 6.4.5, $X(\theta)_{\mathrm{ctf}}$ is the closure of $\bigcup_I X(\theta)[I]$, with $I$ ranging over the allowable ideals in $\mathbb{T}_{\overline{\rho},\Sigma}$, we see that $\big(X(\theta)/\varpi X(\theta)\big)[\mathfrak{m}] \xrightarrow{\sim} \bigcup_I \big(X(\theta)[I]/\varpi X(\theta)[I]\big)[\mathfrak{m}]$, and thus it suffices to prove that the evaluation map

$$\overline{\rho} \otimes_k \overline{\pi} \otimes_k \big(X(\theta)[I]/\varpi X(\theta)[I]\big)[\mathfrak{m}] \to H^1_{k,\overline{\rho}}$$

is injective, for each allowable ideal $I$. By Lemma 4.1.4 and Corollary 6.3.24, for this it suffices in turn to show that

$$\overline{\rho} \otimes_k \overline{\pi} \otimes_k F_{\Sigma_0}\big(X(\theta)[I]/\varpi X(\theta)[I]\big)[\mathfrak{m}] \to F_{\Sigma_0}(H^1_{k,\overline{\rho}})$$

is injective, which itself will follow in turn if we show that, for each $I$, the evaluation map

$$\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} F_{\Sigma_0}\big(X(\theta)[I]\big) \to F_{\Sigma_0}(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}) \tag{6.5.16}$$

is injective, with saturated image.

Now

$$\begin{aligned} X(\theta)[I] &= \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}\times G]}(\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta, \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})[I] \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}\times G]}(\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta/I\theta, \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}). \end{aligned} \tag{6.5.17}$$

If we write $I = \bigcap_{i=1}^n \mathfrak{p}_i$, where the $\mathfrak{p}_i$ are distinct allowable prime ideals, then since

$$M(I)_{\mathcal{O}} := \big(\oplus_{i=1}^n M(\mathfrak{p}_i)_E\big) \cap \widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma},$$

we conclude from the isomorphisms (6.3.16) (applied to $\mathfrak{p} = \mathfrak{p}_1, \ldots, \mathfrak{p}_n$) and (6.5.17) that we have an isomorphism

$$X(\theta)[I] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}\times G]}\big(\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta/I\theta, M(I)_{\mathcal{O}}\big),$$

which in turn induces an isomorphism

$$F_{\Sigma_0}\big(X(\theta)[I]\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G_{\mathbb{Q}}\times G]}\Big(\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta/I\theta, F_{\Sigma_0}\big(M(I)_{\mathcal{O}}\big)\Big). \tag{6.5.18}$$

Since by construction $M(I)_{\mathcal{O}}$ is a saturated $\mathcal{O}$-submodule of $\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma}$, we see that $F_{\Sigma_0}\big(M(I)_{\mathcal{O}}\big)$ is saturated in $F_{\Sigma_0}(\widehat{H}^1_{\mathcal{O},\overline{\rho},\Sigma})$. Hence, in order to verify that (6.5.16) is injective with saturated image, it suffices to verify that the evaluation map

$$\rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta/I\theta \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} F_{\Sigma_0}\big(X(\theta)[I]\big) \to F_{\Sigma_0}\big(M(I)_{\mathcal{O}}\big), \tag{6.5.19}$$

which arises from the description of $F_{\Sigma_0}\big(X(\theta)[I]\big)$ provided by (6.5.18), is injective, with saturated image.

Now $F_{\Sigma_0}\big(M(I)_{\mathcal{O}}\big) \xrightarrow{\sim} \rho^{\mathrm{m}}_{\Sigma} \otimes_{\mathbb{T}_{\Sigma}} W_I$, and so we may rewrite (6.5.18) as an isomorphism

$$F_{\Sigma_0}\big(X(\theta)[I]\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G]}(\theta/I\theta, W_I). \tag{6.5.20}$$

This in turn gives rise to an evaluation map

$$\theta/I\theta \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} F_{\Sigma_0}\big(X(\theta)[I]\big) \to W_I, \tag{6.5.21}$$

from which (6.5.19) is obtained by tensoring with $\rho_\Sigma^{\mathrm{m}}$ over $\mathbb{T}_{\overline{\rho},\Sigma}$. It thus suffices to show that (6.5.21) is injective, with saturated image.

As already noted, there is an isomorphism $\theta/I\theta \xrightarrow{\sim} \theta_I$, while Proposition 6.5.11 gives an isomorphism $(\mathbb{T}_{\overline{\rho},\Sigma}/I)' \otimes_{\mathbb{T}_{\Sigma,\overline{\rho}}} \theta_I \xrightarrow{\sim} W_I$. Thus we may rewrite the isomorphism (6.5.20) as

$$F_{\Sigma_0}\big(X(\theta)[I]\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G]}\big(\theta_I, (\mathbb{T}_{\overline{\rho},\Sigma}/I)' \otimes_{\mathbb{T}_{\Sigma,\overline{\rho}}} \theta_I\big),$$

and hence rewrite the evaluation map (6.5.21) as the evaluation map

$$\theta_I \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G]}\big(\theta_I, (\mathbb{T}_{\overline{\rho},\Sigma}/I)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_I\big) \xrightarrow{\sim} (\mathbb{T}_{\overline{\rho},\Sigma}/I)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_I. \tag{6.5.22}$$

It follows from Lemma 3.1.19 that the natural map

$$(\mathbb{T}_{\overline{\rho},\Sigma}/I)' \to \mathrm{Hom}_{\mathbb{T}_{\overline{\rho},\Sigma}[G]}\big(\theta_I, (\mathbb{T}_{\overline{\rho},\Sigma}/I)' \otimes_{\mathbb{T}_{\overline{\rho},\Sigma}} \theta_I\big)$$

is in fact an isomorphism, and hence that (6.5.22) is also an isomorphism. In particular it is injective with saturated image, and so we are done. ☐

# 7. Applications

**7.1. Applications to the Fontaine–Mazur conjecture.** In this subsection, we prove the following theorem, which is a rephrasing of part 2 of Corollary 1.2.2.

7.1.1. **Theorem.** *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be a pro-modular lift of the absolutely irreducible representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$, unramified outside of the finite set of primes $S \cup \{p\}$, and suppose that*

(1) $\overline{\rho}_{|G_{\mathbb{Q}_p}} \not\sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$;
(2) *If $\rho_{|G_{\mathbb{Q}_p}}$ is isomorphic to the direct sum of two characters, then these characters have non-isomorphic reductions modulo $\varpi$;*
(3) *$\rho_{|G_{\mathbb{Q}_p}}$ is de Rham with distinct Hodge–Tate weights.*

*Then $\rho$ is a twist of the Galois representation attached to a classical cuspidal eigenform of weight $k \geq 2$.*

*Proof.* It follows from Theorem 1.2.1 (1) that there is a $G$-equivariant embedding

$$B(\rho) \hookrightarrow \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \widehat{H}^1_E).$$

By Theorem 3.3.21, the space $B(\rho)_{\mathrm{l.alg}}$ is non-zero, and thus $\mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \widehat{H}^1_{E,\mathrm{l.alg}})$ is non-zero. The theorem is thus seen to follow from [38, Thm. 7.4.2]. ☐

**7.2. Applications to a conjecture of Kisin.** In this subsection, we prove the following theorem, which is a rephrasing of part 1 of Corollary 1.2.2.

7.2.1. **Theorem.** *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be a pro-modular lift of the absolutely irreducible representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$, unramified outside of the finite set of primes $S \cup \{p\}$, and suppose that*

(1) $\overline{\rho}_{|G_{\mathbb{Q}_p}} \not\sim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon} \end{pmatrix}$;
(2) *$\rho_{|G_{\mathbb{Q}_p}}$ is trianguline.*

*Then $\rho$ is a twist of the Galois representation attached to an overconvergent $p$-adic cuspidal eigenform of finite slope.*

*Proof.* We begin by recalling some constructions from [38, §7.5]. For any tame level $K^p$, the locally analytic Jacquet module $J_B\big((\widehat{H}^1(K^p)_E)_{\mathrm{an}}\big)$ is an essentially admissible locally analytic representation of $T$, corresponding to a rigid analytic coherent sheaf $\mathcal{E}(K^p)$ on the space $\widehat{\mathrm{T}}$ of locally analytic characters of $T$. Since $J_B$ is a functor, the Hecke algebra $\mathbb{T}(K^p)$ acts on $J_B\big((\widehat{H}^1(K^p)_E)_{\mathrm{an}}\big)$, and so on $\mathcal{E}(K^p)$, and generates a coherent sheaf of endomorphisms of $\mathcal{E}(K^p)$, which we denote by $\mathcal{A}(K^p)$. Suppose now that (in the notation of Subsection 5.2), the tame level $K^p$ is of the form $K_{\Sigma_0}K_0^{\Sigma}$, where $K_{\Sigma_0}$ is a compact open subgroup of $G_{\Sigma_0}$. We then write $\mathcal{E}(K_{\Sigma_0}) := \mathcal{E}(K^p)$ and $\mathcal{A}(K_{\Sigma_0}) := \mathcal{A}(K^p)$. If $K_{\Sigma_0}$ is chosen to be allowable for $\overline{\rho}$, then $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}$ is a direct summand of $\widehat{H}^1(K_{\Sigma_0})_E$, and so $J_B\big((\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}})_{\mathrm{an}}\big)$ is a direct summand of $J_B\big((\widehat{H}^1(K^p)_E)_{\mathrm{an}}\big)$. We write $\mathcal{E}(K_{\Sigma_0})_{\overline{\rho}}$ and $\mathcal{A}(K_{\Sigma_0})_{\overline{\rho}}$ for the associated direct summands of $\mathcal{E}(K_{\Sigma_0})$ and $\mathcal{A}(K_{\Sigma_0})$.

Let $\mathfrak{p} \in \operatorname{Spec} \mathbb{T}_{\overline{\rho},\Sigma}$ denote the kernel of the map $\mathbb{T}_{\overline{\rho},\Sigma} \to E$ corresponding to $\rho$. Theorem 1.2.1 (1) shows that there is a non-zero $G$-equivariant map

$$B(\rho) \to \operatorname{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}]),$$

for a sufficiently small choice of $K_{\Sigma_0}$. Passing to locally analytic Jacquet modules for $B$, we obtain a $T$-equivariant map

$$J_B\big(B(\rho)_{\mathrm{an}}\big) \to \operatorname{Hom}_{E[G_{\mathbb{Q}}]}\Big(\rho, J_B\big((\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}])_{\mathrm{an}}\big)\Big).$$

Since $\rho$ is trianguline, the locally analytic vectors in each topological Jordan–Hölder factor of $B(\rho)$ have a non-vanishing Jacquet module (see [38, §§5.2 and 6]), and so (by left-exactness of the functor $J_B\big((-)_{\mathrm{an}}\big)$) this map is again non-zero. Thus we conclude that

$$J_B\big((\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}})_{\mathrm{an}}\big)[\mathfrak{p}] = J_B\big((\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}}[\mathfrak{p}])_{\mathrm{an}}\big) \neq 0,$$

and hence that the system of Hecke eigenvalues attached to $\rho$ appears in the space $\operatorname{Spec} \mathcal{A}(K^p)_{\overline{\rho}}$. It then follows from [38, Thm. 7.5.8] that $\rho$ is a twist of the Galois representation attached to a finite slope overconvergent $p$-adic cuspidal eigenform.

We note that the proof of [38, Thm. 7.5.8] is only sketched, and so the above argument may rightly be regarded as incomplete. However, we can avoid the appeal to [38, Thm. 7.5.8] as follows: As is explained in the beginning of the proof of [38, Thm. 7.5.8], to conclude that every point of $\operatorname{Spec} \mathcal{A}(K_{\Sigma_0})_{\overline{\rho}}$ (and so in particular, the Galois representation $\rho$) arises as the twist of a point corresponding to a finite slope overconvergent $p$-adic cuspidal eigenform, it suffices to show that $\operatorname{Spec} \mathcal{A}(K_{\Sigma_0})_{\overline{\rho}}$ is equidimensional of dimension two. For this, it suffices in turn to show that the support of $\mathcal{E}(K_{\Sigma_0})_{\overline{\rho}}$ is equidimensional of dimension two, and that $\mathcal{E}(K_{\Sigma_0})_{\overline{\rho}}$ has no embedded primes as a module over its support, i.e. that the support of every non-zero section of $\mathcal{E}(K_{\Sigma_0})_{\overline{\rho}}$ has dimension two. Since $\mathcal{E}(K_{\Sigma_0})_{\overline{\rho}}$ is the coherent sheaf on $\widehat{\mathrm{T}}$ associated to $J_B\big((\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho}})_{\mathrm{an}}\big)$, this follows from Corollary 5.3.19 and [35, Prop. 4.2.36]. (In fact, as stated, this last result shows merely that the support of $\mathcal{E}(K_{\Sigma_0})_{\overline{\rho}}$ is of dimension two. But in fact the considerations in its proof suffice to establish the stronger result that each non-zero element of $\mathcal{E}(K_{\Sigma_0})_{\overline{\rho}}$ has support of dimension two.) □

7.3. **Promodularity.** In this subsection we present the:

*Proof of Theorem 1.2.3.* As we noted prior to stating this theorem, its proof is a synthesis of results and methods due to Böckle, Diamond–Flach–Guo, and Khare–Wintenberger and Kisin. (In particular, the $p$-adic Langlands correspondence plays no role in its proof!)

As in the statement of the theorem, let $V$ be a continuous odd irreducible representation of $G_{\mathbb{Q}}$ over $E$, whose reduction $\overline{V}$ satisfies the given conditions 1 and 2. Serre's conjecture (proved by Khare–Wintenberger and Kisin [53, 54, 55, 60]) implies that $\overline{V}$ is modular, and thus we may apply the results (and more generally the methods) of Böckle [4] to study the deformation space of $\overline{V}$, parametrizing deformations of $\overline{V}$ unramified outside $\Sigma$.

In the case when $\overline{V}_{|G_{\mathbb{Q}_p}}$ is absolutely reducible, Theorem 1.2.3 follows from [4, Cor. 3.8]. (Note that our condition 2 ensures that the condition "$\overline{\mu}_\ell \neq \overline{\chi}_\ell$" of that result is satisfied.) Thus we may assume that $\overline{V}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible.

If $\overline{V}_{|G_{\mathbb{Q}_p}}$ is furthermore finite flat (up to a twist), then Theorem 1.2.3 again follows from [4, Cor. 3.8]. In this (the finite flat) case, the proof of [4, Cor. 3.8] depends on two ingredients: the main theorem of Wiles [86] and Taylor–Wiles [83], as strengthened by Diamond [30], which identifies the universal deformation ring of $\overline{V}$ parametrizing deformations that are finite flat locally at $p$ with an appropriate (weight two) Hecke algebra, and the result of Ramakrishna [70] showing that the universal deformation ring of $\overline{V}_{|G_{\mathbb{Q}_p}}$ parametrizing finite flat deformations of fixed determinant is formally smooth of relative dimension one over $\mathcal{O}$.

Both of these ingredients have since been extended to the case of an arbitrary $\overline{V}$ for which $\overline{V}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible, by Diamond, Flach, and Guo [31]. More precisely, twisting $\overline{V}$ if necessary, we may assume that $\overline{V}$ is in the image of the Fontaine–Laffaille functor [45], i.e. arises from a two-dimensional Fontaine–Laffaille module with coefficients in $k$, having filtration indices 0 and $k-1$, with $2 \leq k \leq p$. In [31] it is proved that the universal deformation ring of $\overline{V}$, parametrizing deformations over objects $A$ of $\mathrm{Art}(\mathcal{O})$ that locally at $p$ arise from a rank two Fontaine–Laffaille module with coefficients in $A$ having filtration indices 0 and $k-1$, may be identified with an appropriate (weight $k-1$) Hecke algebra (see [31, Thm. 3.6] and its proof; note that although this result includes the hypothesis $k < p$, the proof is in fact valid in the case $k = p$ also). Furthermore, the corresponding universal deformation ring of $\overline{V}_{|G_{\mathbb{Q}_p}}$ is shown to be formally smooth of relative dimension one over $\mathcal{O}$ [31, Cor. 2.3]. With these two ingredients at hand, the argument proving [4, Cor. 3.8] in the finite flat case extends to handle the case of arbitrary $\overline{V}$ for which $\overline{V}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible, which in turn suffices to prove Theorem 1.2.3 for such $\overline{V}$. □

7.4. **Locally algebraic vectors.** In this subsection we present the:

*Completion of the proof of Theorem 3.3.22.* Colmez has proved the theorem in the case when $V_E$ is crystabelline or the twist of a semi-stable representation [25, Thm. VI.6.50]. Thus we assume that $V_E$ is potentially crystalline but not crystabelline. Let $D_{\mathrm{pcrys}}(V_E)$ denote the potentially crystalline Dieudonné module of $V_E$, and let $a > b$ denote the Hodge–Tate weights of $V_E$.

Colmez has shown [25, Thm. VI.6.42] that $B(V_E)_{\mathrm{l.alg}}$ depends only on $D_{\mathrm{pcrys}}(V_E)$ as a $(\varphi, N, G_{\mathbb{Q}_p})$-module, together with the pair of weights $a, b$ (i.e. it is independent of the Hodge filtration on $D_{\mathrm{dR}}(V_E)$). Equivalently, it depends only on the Weil–Deligne representation $\mathrm{WD}(V_E)$ attached to the $(\varphi, N, G_{\mathbb{Q}_p})$-module $D_{\mathrm{pcrys}}(V_E)$, together with the pair of weights $a$ and $b$. Equivalently again, it depends only on the admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\pi_p(V_E)$ attached to $\mathrm{WD}(V_E)$ via the classical local Langlands correspondence, together with the pair of weight $a$ and $b$. Thus in order to prove the theorem, it suffices to construct a single two-dimensional continuous representation $V'_E$ of $G_{\mathbb{Q}_p}$ over $E$, with Hodge–Tate weight $a$ and $b$, such that $\pi_p(V'_E) = \pi_p(V_E)$, and such that

$$B(V'_E)_{\mathrm{l.alg}} \xrightarrow{\sim} \mathrm{Sym}^{a-b-1} E^2 \otimes_E \det{}^b \otimes_E \pi_p(V_E).$$

Indeed, it suffices to do this after making an extension of scalars from $E$ to some finite extension $E'$, since the formation of locally algebraic vectors is compatible with the extension of scalars. We may also replace $V_E$ by an unramified twist, since the formation of locally algebraic vectors is compatible with such twisting.

We now set about constructing such a $V'_E$. Note that our assumption that $V_E$ is potentially crystalline, but not crystabelline, implies (indeed, is equivalent to requiring) that $\pi_p(V_E)$ be cuspidal. Let $\sigma$ be the minimal type of $\pi_p(V_E)$ [49], and let $\sigma_0$ be a $\mathrm{GL}_2(\mathbb{Z}_p)$-invariant $\mathcal{O}_E$-lattice in $\sigma$. Write $\tau_0 := \mathrm{Sym}^{a-b-1}\mathcal{O}_E^2 \otimes_{\mathcal{O}_E} \det^b \otimes \sigma_0$ (thought of as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation) and $\tau := E \otimes_{\mathcal{O}_E} \tau_0$, and let $W$ be a Jordan–Hölder constituent of the reduction $\tau_0/\varpi\tau_0$.

Choose a modular $\overline{\rho}$ such that $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is irreducible and such that $W \in W(\overline{\rho})$, i.e. $W$ is a Serre weight of $\overline{\rho}$. (It is easy to find such a $\overline{\rho}$, e.g. as the reduction of the induction of a Grössen-character of the appropriate weight, for some quadratic imaginary extension of $\mathbb{Q}$ in which $p$ is inert.) Let $\Sigma$ be the set of primes at which $\overline{\rho}$ is ramified, together with $p$, and let $K(\Sigma_0)$ be an allowable level for $\overline{\rho}$, chosen so that $GL_2(\mathbb{Z}_p)K(\Sigma_0)K_0^{\Sigma}$ is neat. We use the notation of Section 5; thus we let $\mathfrak{m}$ denote the maximal ideal in $\mathbb{T}(K_{\Sigma_0})$ corresponding to $\overline{\rho}$. Since $W$ is a Serre weight of $\overline{\rho}$, the weight part of Serre's conjecture shows that there is a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant embedding $W \hookrightarrow H^1(K_{\Sigma_0})_{k,\overline{\rho},\Sigma}[\mathfrak{m}]$. It then follows from Proposition 5.3.15 that we may lift this to a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant map $\tau_0 \to \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}_E,\overline{\rho},\Sigma}$, which in turn gives rise to a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant map $\tau \to \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\Sigma}$, which is necessarily an embedding, since $\tau$ is irreducible as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation over $E$. The space $\mathrm{Hom}_{E[GL_2(\mathbb{Z}_p)]}(\tau, \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\Sigma})$ is finite-dimensional over $E$, and is a $\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}$-module, and hence we may find $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{T}(K_{\Sigma_0})_{\overline{\rho}}[1/p]$ such that $\mathrm{Hom}_{E[GL_2(\mathbb{Z}_p)]}(\tau, \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\Sigma})[\mathfrak{p}]$ is non-zero, or, in other words, such that there is an embedding

$$\tau \hookrightarrow \widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\Sigma}[\mathfrak{p}]. \tag{7.4.1}$$

Since $\widehat{H}^1(K_{\Sigma_0})_{E,\overline{\rho},\Sigma}[\mathfrak{p}]$ contains non-zero locally $\mathrm{Sym}^{a-b-1}E^2 \otimes_E \det^b$-algebraic vectors (e.g. those in the image of the embedding (7.4.1)), we see that $\mathfrak{p}$ is a classical point of $\mathrm{Spec}\,\mathbb{T}(K_{\Sigma_0})$, corresponding (up to a cyclotomic twist) to a classical Hecke eigenform defined over $E' := \kappa(\mathfrak{p})$, a finite extension of $E$. The local factor at $p$ of the corresponding automorphic representation $\pi$ then has $\sigma$ as a type, and so must coincide with $E' \otimes_E \pi_p(V_E)$, up to an unramified twist. To ease notation, we replace $E$ with $E'$ and $V_E$ with an appropriate unramified twist (which, as was noted above, we may do without changing the problem), and thus assume that $\kappa(\mathfrak{p}) = E$ and that the local factor at $p$ is equal to $\pi_p(V_E)$.

If we let $N$ denote the conductor of $\pi$, and let

$$K_1^p(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \prod_{\ell \neq p} \mathrm{GL}_2(\mathbb{Z}_\ell) \mid c \equiv 0 \bmod N, d \equiv 1 \bmod N\},$$

then it follows from [38, Thm. 7.4.2] that there is a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism

$$\widehat{H}^1\big(K_1^p(N)\big)_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\text{l.alg}} \xrightarrow{\sim} \rho(\mathfrak{p}) \otimes_E \mathrm{Sym}^{a-b-1} E^2 \otimes_E \det{}^b \otimes_E \pi_p(V_E).$$

On the other hand, Theorem 1.2.1 shows that there is a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism

$$\rho(\mathfrak{p}) \otimes_E B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big) \xrightarrow{\sim} \widehat{H}^1\big(K_1^p(N)\big)_{E,\overline{\rho},\Sigma}[\mathfrak{p}].$$

Hence

$$\begin{aligned} \rho(\mathfrak{p}) \otimes_E B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big)_{\text{l.alg}} &\xrightarrow{\sim} \widehat{H}^1\big(K_1^p(N)\big)_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\text{l.alg}} \\ &\xrightarrow{\sim} \rho(\mathfrak{p}) \otimes_E \mathrm{Sym}^{a-b-1} E^2 \otimes_E \det{}^b \otimes_E \pi_p(V_E). \end{aligned}$$

As remarked above, this serves to complete the proof of the theorem. □

7.4.2. **Remark.** We don't need the full strength of Theorem 1.2.1 in order to prove the preceding result. Indeed, it suffices to have a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map

$$\rho(\mathfrak{p}) \otimes_E B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big) \to \widehat{H}^1\big(K_1^p(N)\big)_{E,\overline{\rho},\Sigma}[\mathfrak{p}]$$

(which will then necessarily be an embedding, by Proposition 3.3.24), the existence of which is guaranteed by Remark 6.4.3. Such an embedding induces a corresponding embedding

$$\begin{aligned} \rho(\mathfrak{p}) \otimes_E B\big(\rho(\mathfrak{p})_{|G_{\mathbb{Q}_p}}\big)_{\text{l.alg}} &\hookrightarrow \widehat{H}^1\big(K_1^p(N)\big)_{E,\overline{\rho},\Sigma}[\mathfrak{p}]_{\text{l.alg}} \\ &\xrightarrow{\sim} \rho(\mathfrak{p}) \otimes_E \mathrm{Sym}^{a-b-1} E^2 \otimes_E \det{}^b \otimes_E \pi_p(V_E). \end{aligned}$$

Since the target of this embedding is irreducible, and the source is non-zero [25, Thm. VI.6.18], it is then necessarily an isomorphism.

## Appendix A. Banach modules

Let $A$ be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$, and let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal of $A$. We will define the notion of a Banach module over $A_{\mathfrak{p}}$, after first making some preliminary definitions.

A.1. **Definition.** If $V$ is an $A_{\mathfrak{p}}$-module, then we say that two $A$-submodules $W_1$ and $W_2$ of $V$ are commensurable if there exists $a \in A \setminus \mathfrak{p}$ such that $aW_1 \subset W_2 \subset a^{-1}W_1$.

One immediately checks that commensurability is an equivalence relation on the set of $A$-submodules of any given $A_{\mathfrak{p}}$-module.

A.2. **Definition.** If $V$ is an $A_{\mathfrak{p}}$-module, we say that an $A$-submodule $W$ of $V$ is an $A$-lattice in $V$ if $W$ generates $V$ as an $A_{\mathfrak{p}}$-module, or, equivalently, if the natural map $A_{\mathfrak{p}} \otimes_A W \to V$ is an isomorphism.

A.3. **Definition.** A Banach module over $A_{\mathfrak{p}}$ consists of an $A_{\mathfrak{p}}$-module $V$, together with the choice of a non-empty commensurability class of $\mathfrak{m}$-adically complete and separated $A$-lattices in $V$. (Note that we are not suggesting that any lattice commensurable to a given $\mathfrak{m}$-adically complete and separated $A$-lattice in $V$ is itself necessarily complete and separated. Rather, we are considering commensurability as a relation on the set of complete and separated $A$-lattices in $V$.)

If $V$ is a Banach module over $A_{\mathfrak{p}}$, and if $W$ is a member of the chosen commensurability class of $\mathfrak{m}$-adically complete and separated $A$-lattices in $V$, then we will sometimes refer to $W$ as a choice of unit ball in $V$.

A.4. **Remark.** If $A = \mathcal{O}$ and $\mathfrak{p}$ is the zero ideal, so that $A_{\mathfrak{p}} = E$, then the notion of Banach module over $E$ reduces to the standard notion of a Banach space over $E$. (Here, and throughout the paper, by a Banach space over $E$ we mean a complete topological vector space whose topology can be defined by a norm; we do not regard a Banach space as having a fixed choice of norm defining its topology, and thus its unit ball is well-defined only up to commensurability.)

## Appendix B. Orthonormalizable and pro-free modules

Let $A$ be a complete Noetherian local ring.

B.1. **Definition.** We say that an $A$-module $V$ is orthonormalizable if $V$ is $\mathfrak{m}$-adically complete and separated, and if the quotient $V/\mathfrak{m}^i V$ is free over $A/\mathfrak{m}^i$, for each $i \geq 0$.

We begin by showing that an $A$-module $V$ is orthonormalizable if and only if its isomorphic to the $\mathfrak{m}$-adic completion of a free $A$-module, or equivalently, if and only if its admits an orthonormal basis in a sense analogous to that used in the theory of $p$-adic Banach spaces.

B.2. **Lemma.** *Let $V$ be an orthonormalizable $A$-module, let $\{\bar{e}_i\}_{i\in I}$ be a basis of $V/\mathfrak{m}V$, and let $\{e_i\}_{i\in I}$ denote some choice of lifts of these basis elements to elements of $V$. If $F$ denotes the $A$-submodule of $V$ generated by $\{e_i\}_{i\in I}$, then:*

1. *$F$ is freely generated by $\{e_i\}_{i\in I}$, i.e. $F$ is a free $A$-module with $\{e_i\}_{i\in I}$ as a basis;*
2. *For each $r \geq 0$, the natural map*

(B.3)
$$F/\mathfrak{m}^r F \to V/\mathfrak{m}^r V$$

*is an isomorphism. In particular, if $\widehat{F}$ denotes the $\mathfrak{m}$-adic completion of $F$, then the natural map $\widehat{F} \to V$ is an isomorphism.*

*Proof.* Evidently, $V = F + \mathfrak{m}V$. Iterating this, we find that $V = F + \mathfrak{m}^r V$ for any $r \geq 0$. Thus the map (B.3) is surjective. This proves a part of 2.

Now consider the map $\imath : A^I \to V$ defined by $\imath_I\big((a_i)\big) = \sum_{i\in I} a_i e_i$. We intend to show that $\imath$ is an embedding. If $r > 0$, then reducing $\imath$ mod $\mathfrak{m}^r$ yields a map $\imath_r : (A/\mathfrak{m}^r)^I \to V/\mathfrak{m}^r V$ between free $A/\mathfrak{m}^r$-modules, which when reduced mod $\mathfrak{m}$ becomes an embedding. One easily concludes that $\imath_r$ is itself an embedding. Since $r > 0$ was arbitrary, we find that $\imath$ is an embedding, and thus that the image $F$ of $\imath$ is free. This proves 1.

Finally, we note that the preceding argument shows that the map $F/\mathfrak{m}^r F \to V/\mathfrak{m}^r V$ is injective for any $r > 0$. This completes the proof of 2. □

B.4. **Definition.** In the context of Lemma B.2, we refer to $\{e_i\}_{i\in I}$ as an orthonormal basis for the orthonormalizable $A$-module $V$. (The terminology is motivated by the obvious analogy with the usual notion of an orthonormal basis of a $p$-adic Banach space.)

B.5. **Remark.** If $F$ is the free $A$-module generated by a basis $\{e_i\}_{i\in I}$, then the $\mathfrak{m}$-adic completion $\widehat{F}$ of $F$ admits the following concrete description:

$$\widehat{F} \cong \{(a_i)_{i\in I} \in A^I \mid a_i \to 0 \ \mathfrak{m}\text{-adically } \},$$

where we write $a_i \to 0$ $\mathfrak{m}$-adically to denote that for any $r \geq 0$, there exists a finite subset $J \subset I$ such $a_i \in \mathfrak{m}^r$ if $i \in I \setminus J$. Consequently, if $\{e_i\}_{i\in I}$ is an orthonormal basis for an orthonormalizable $A$-module $V$, then $V$ admits the same concrete description.

The following lemmas establish some elementary facts regarding orthonormalizable modules.

B.6. **Lemma.** *Let $V$ be an orthonormalizable $A$-module.*

(1) *$V$ is flat.*
(2) *If $M$ is a finitely generated $A$-module, then $M \otimes_A V$ is $\mathfrak{m}$-adically complete.*
(3) *If $M$ and $N$ are $A$-modules, with $M$ being finitely generated, then the natural map*

$$\mathrm{Hom}_A(M, N) \otimes_A V \to \mathrm{Hom}_A(M, N \otimes_A V)$$

*is an isomorphism.*
(4) *If $B$ is a finite local $A$-algebra, than $B \otimes_A V$ is an orthonormalizable $B$-module.*

*Proof.* Fix an orthonormal basis $\{e_i\}$ for $V$, and if $M$ is any finitely generated $A$-module, define

$$F(M) := \{(m_i)_{i\in I} \in M^I \mid m_i \to 0 \ \mathfrak{m}\text{-adically } \}.$$

The formation of $F(M)$ is functorial on the category of finitely generated $A$-modules, and it follows from the Artin–Rees lemma that it is an exact functor. Since $F(A) \xrightarrow{\sim} V$ (see Remark B.5), we conclude that $F(M) \xrightarrow{\sim} M \otimes_A V$ for any finitely generated $A$-module $M$. (Apply each of the right exact functors $F$ and $- \otimes_A V$ to a presentation $A^s \to A^r \to M \to 0$ for $M$.) Thus $- \otimes_A V$ is exact on the category of finitely generated $A$-modules, and hence on the category of all $A$-modules. (Since tensor products commute with inductive limits, any $A$-module is the inductive limit of finitely generated $A$-submodules, and the formation of inductive limits is exact.) Claim 1 follows. Obviously $F(M)$ is $\mathfrak{m}$-adically complete, and so 2 also follows.

We turn to proving 3, and so fix $A$-modules $M$ and $N$, with $M$ finitely generated. If we note that the functors $\mathrm{Hom}_A(M, -)$ and $- \otimes_A V$ are compatible with the formation of inductive limits (the former since $M$ is finitely generated over $A$, and the latter as a general property of tensor products), we see (writing $N$ as an inductive limit of finitely generated $A$-modules) that we may in addition assume that $N$ is finitely generated over $A$. Since $\mathrm{Hom}_A(M, N) \otimes_A V$ and $N \otimes_A V$ are both $\mathfrak{m}$-adically complete, by 2, we see that the natural map $\mathrm{Hom}_A(M, N) \otimes_A V \to \mathrm{Hom}_A(M, N \otimes_A V)$ may be computed as the projective limit over $r$ of the maps

$$\big(\mathrm{Hom}_A(M, N)/\mathfrak{m}^r \,\mathrm{Hom}_A(M, N)\big) \otimes_{A/\mathfrak{m}^r} (V/\mathfrak{m}^r V) \\ \to \mathrm{Hom}_A\big(M/\mathfrak{m}^r M, (N/\mathfrak{m}^r N) \otimes_{A/\mathfrak{m}^r} (V/\mathfrak{m}^r V)\big).$$

We may factor each of these maps in the form

$$\begin{aligned}\big(\mathrm{Hom}_A(M,N)/\mathfrak{m}^r\,\mathrm{Hom}_A(M,N)\big)\otimes_{A/\mathfrak{m}^r}(V/\mathfrak{m}^rV)\\ \to \mathrm{Hom}_A(M/\mathfrak{m}^rM, N/\mathfrak{m}^rN)\otimes_{A/\mathfrak{m}^r}(V/\mathfrak{m}^rV)\\ \to \mathrm{Hom}_A\big(M/\mathfrak{m}^rM,(N/\mathfrak{m}^rN)\otimes_{A/\mathfrak{m}^r}(V/\mathfrak{m}^rV)\big).\end{aligned}$$

The second map in this factorization is evidently an isomorphism, since $V/\mathfrak{m}^rV$ is free over $A/\mathfrak{m}^r$ for each $r>0$, and the first of these maps yields an isomorphism after passing to the projective limit over $r$, since by the Artin–Rees lemma, the map of projective systems

$$\{\mathrm{Hom}_A(M,N)/\mathfrak{m}^r\,\mathrm{Hom}_A(M,N)\}_r \to \{\mathrm{Hom}_A(M/\mathfrak{m}^rM, N/\mathfrak{m}^rN)\}_r$$

is an equivalence, in the sense that in both its kernel and cokernel, the composition of sufficiently many consecutive transition maps vanishes. This proves 3.

To prove 4, we note that, if $\mathfrak{n}$ denotes the maximal ideal of $B$, then the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies on $B$ coincide. Thus to show that $B\otimes_A V$ is orthonormal, it suffices to show that $(B\otimes_A V)/\mathfrak{m}^r(B\otimes_A V)$ is free over $B/\mathfrak{m}^rB$ for each $r>0$. This follows from the isomorphism $(B\otimes_A V)/\mathfrak{m}^r(B\otimes_A V)\xrightarrow{\sim}(B/\mathfrak{m}^rB)\otimes_{A/\mathfrak{m}^r}(V/\mathfrak{m}^rV)$, and the fact that $V/\mathfrak{m}^rV$ is free over $A/\mathfrak{m}^r$ by assumption. □

B.7. **Lemma.** *Let $U$ and $V$ be orthonormalizable $A$-modules. If $M$ is any finitely generated $A$-module, then the natural map*

$$M\otimes_A \mathrm{Hom}_A(U,V)\to \mathrm{Hom}_A(U,M\otimes_A V)$$

*is an isomorphism.*

*Proof.* Choose an orthonormal basis of $U$, and let $F\subset U$ be the free $A$-submodule spanned by this orthonormal basis, so that $U$ is the $\mathfrak{m}$-adic completion of $F$. Since $V$ and $M\otimes_A V$ are $\mathfrak{m}$-adically complete (the latter by part 2 of Lemma B.6), there are isomorphisms $\mathrm{Hom}_A(U,V)\xrightarrow{\sim}\mathrm{Hom}_A(F,V)$ and $\mathrm{Hom}_A(U,M\otimes_A V)\xrightarrow{\sim}\mathrm{Hom}_A(F,M\otimes_A V)$. Thus it suffices to prove that

$$M\otimes_A \mathrm{Hom}_A(F,V)\to \mathrm{Hom}_A(F,M\otimes_A V)$$

is an isomorphism.

Applying the exact functor $\mathrm{Hom}_A(F,\text{–})$ to the tensor product with $V$ over $A$ of a presentation $A^s\to A^r\to M\to 0$ of $M$ yields an exact sequence

$$A^s\otimes_A \mathrm{Hom}_A(F,V)\to A^r\otimes_A \mathrm{Hom}_A(F,V)\to \mathrm{Hom}_A(F,M\otimes_A V)\to 0.$$

On the other hand, tensoring this presentation of $M$ with $\mathrm{Hom}_A(F,V)$ over $A$ yields an exact sequence

$$A^s\otimes_A \mathrm{Hom}_A(F,V)\to A^r\otimes_A \mathrm{Hom}_A(F,V)\to M\otimes_A \mathrm{Hom}_A(F,V)\to 0.$$

Comparing the two exact sequences yields the desired isomorphism. □

B.8. **Lemma.** *If $A$ is flat over $\mathcal{O}$, and if $V$ is a non-zero orthonormalizable $A$-module, then $A=(E\otimes_{\mathcal{O}}A)\bigcap \mathrm{End}_A(V)$, where the intersection takes place in $\mathrm{End}_{E\otimes_{\mathcal{O}}A}(E\otimes_{\mathcal{O}}V)$.*

*Proof.* Choose an orthonormal basis for $V$, and let $F$ be the free $A$-submodule $F$ of $V$ spanned by this orthonormal basis, so that $V$ is the $\mathfrak{m}$-adic completion of $F$. If $a/\varpi^i\in(E\otimes_{\mathcal{O}}A)\bigcap\mathrm{End}_A(V)$, with $a\in A$ and $i\geq 0$, then $aF\subset F\bigcap\varpi^iV=\varpi^iF$, and hence (since $F$ is non-zero and free over $A$) we see that in fact $a/\varpi^i \in A$. The lemma follows. $\square$

The next lemma establishes a certain faithfulness result for completed tensor products with orthonormalizable modules.

B.9. **Lemma.** *Let $M$ be an $A$-module which is $\varpi$-adically separated, and with the property that any element of $M/\varpi M$ is annihilated by some power of $\mathfrak{m}$. If the $\varpi$-adically completed tensor product $M\hat{\otimes}_A V$ vanishes for some orthonormalizable $A$-module $V$, then either $V = 0$ or $M = 0$.*

*Proof.* We will suppose $M\hat{\otimes}_A V = 0$ and that $V \neq 0$, and deduce that $M = 0$. Since $M$ is $\varpi$-adically separated, by assumption, it suffices to show that $M/\varpi M = 0$. If we consider the composite

$$(M\hat{\otimes}_A V)/\varpi(M\hat{\otimes}_A V) \to (M \otimes_A V)/\varpi(M \otimes_A V) \xrightarrow{\sim} (M/\varpi M) \otimes_A V,$$

the first arrow being the natural surjection, we deduce that $(M/\varpi M) \otimes_A V = 0$. Also by assumption $M/\varpi M \xrightarrow{\sim} \varinjlim_i (M/\varpi M)[\mathfrak{m}^i]$, and so (since tensor product commute with inductive limits) we have an isomorphism

$$0 = (M/\varpi M) \otimes_A V \xrightarrow{\sim} \varinjlim_i (M/\varpi M)[\mathfrak{m}^i] \otimes_A V.$$

Since $V$ is $A$-flat, by Lemma B.6, the transition maps in the indicated inductive system are injective, and so we conclude that $(M/\varpi M)[\mathfrak{m}^i] \otimes_A V = 0$, for each $i \geq 0$. There is an evident isomorphism

$$(M/\varpi M)[\mathfrak{m}^i] \otimes_A V \xrightarrow{\sim} (M/\varpi M)[\mathfrak{m}^i] \otimes_{A/\mathfrak{m}^i} (V/\mathfrak{m}^i V),$$

and thus $(M/\varpi M)[\mathfrak{m}^i] \otimes_{A/\mathfrak{m}^i} (V/\mathfrak{m}^i V) = 0$, for each $i \geq 0$. Since $V$ is a non-zero orthonormalizable $A$-module, its quotient $V/\mathfrak{m}^i V$ is a non-zero free $A/\mathfrak{m}^i$-module. We conclude that $(M/\varpi M)[\mathfrak{m}^i] = 0$ for each $i \geq 0$, and passing to the inductive limit over $i$, that $M/\varpi M = 0$, as required. $\square$

Finally, we give a "dual" description of orthonormalizable $A$-modules.

B.10. **Definition.** We say that a profinite $A$-module $M$ is pro-free if it is topologically isomorphic to a direct product of copies of $A$, each factor being equipped with its $\mathfrak{m}$-adic topology.

B.11. **Proposition.** (1) *If $V$ is an orthonormalizable $A$-module, then*

$$\mathrm{Hom}_A(V, A)/\mathfrak{m}^i \, \mathrm{Hom}_A(V, A) \xrightarrow{\sim} \mathrm{Hom}_{A/\mathfrak{m}^i}(V/\mathfrak{m}^i V, A/\mathfrak{m}^i),$$

*for each $i > 0$.*

(2) *If for any orthonormalizable $A$-module $V$ we equip $\mathrm{Hom}_A(V, A)$ with the topology of point-wise convergence, with respect to the $\mathfrak{m}$-adic topology on $A$, then the functor $V \mapsto \mathrm{Hom}_A(V, A)$ induces an equivalence of categories between the category of orthonormalizable $A$-modules and the category of pro-free profinite $A$-modules.*

*Proof.* Part 1 follows from Lemma B.7, applied with $M = A/\mathfrak{m}^i$ and $U = A$. Part 2 is easily verified by the reader. A quasi-inverse functor is provided by $M \mapsto \mathrm{Hom}_{A-\mathrm{cont}}(M, A)$, where the target is the space of continuous $A$-linear maps from $M$ to $A$ (where $A$ is given its $\mathfrak{m}$-adic topology). $\square$

## Appendix C. Coadmissible representations

Throughout this appendix we let $\Gamma$ denote a topological group which contains a profinite open subgroup whose pro-order if prime-to-$p$. The example we have in mind is $\Gamma = G_{\Sigma_0}$, where $\Sigma_0$ is a finite set of primes distinct from $p$, and, as in the body of the paper, we have written $G_{\Sigma_0} := \prod_{\ell\in\Sigma}\mathrm{GL}_2(\mathbb{Q}_\ell)$. However, in view of possible future applications, it seems sensible to write this appendix in the natural level of generality with respect to which its results hold.

Our goal in this appendix is to introduce, for any object $A$ of $\mathrm{Comp}(\mathcal{O})$, the notion of a coadmissible smooth representation of $\Gamma$ over $A$, and to describe some simple results and constructions pertaining to such representations.

Coadmissible smooth representations are dual to admissible smooth representations, in an appropriate sense, and so, in order to define them, it is necessary to first define a class of $A$-modules that are dual to finitely generated $\mathcal{O}$-torsion free $A$-modules.

C.1. **Definition.** We say that an $A$-module $X$ is cofinitely generated if it satisfies the following conditions:

(1) $X$ is $\varpi$-adically complete and separated.
(2) $X$ is $\mathcal{O}$-torsion free.
(3) The action map $A \times X \to X$ induced by the $A$-module structure on $X$ is continuous, when $A$ is given its $\mathfrak{m}$-adic topology, and $X$ is given its $\varpi$-adic topology.
(4) $(X/\varpi X)[\mathfrak{m}]$ is finite-dimensional over $k$.

C.2. **Remark.** (1) Condition 3 of Definition C.1 is equivalent to the requirement that for each $i > 0$, and each element $\overline{x} \in X/\varpi^i X$, there is a $j > 0$ such that $\mathfrak{m}^j$ annihilates $\overline{x}$.
(2) From the fact that $A/\mathfrak{m}^j$ has finite length, for any $j > 0$, one easily infers that condition 4 of the Definition C.1 is equivalent to the apparently stronger condition that for every $i, j > 0$, the $\mathcal{O}$-module $(X/\varpi^i X)[\mathfrak{m}^j]$ is of finite length.

As already intimated, an important fact about cofinitely generated $A$-modules is that they are dual (in an appropriate sense) to finitely generated $\mathcal{O}$-torsion free $A$-modules. In order to prove this, and to exploit it to prove further results about cofinitely generated $A$-modules, we must recall some facts about duality for $\varpi$-adically complete and separated, $\mathcal{O}$-torsion free modules.

If $X$ is any $\mathcal{O}$-torsion free, $\varpi$-adically complete and separated $\mathcal{O}$-module, then its $\mathcal{O}$-dual $\mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})$ has a natural profinite topology, namely the topology of pointwise convergence. If we write $M := \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})$, then we have the following natural isomorphisms:

$$M/\varpi^i M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(X/\varpi^i X, \mathcal{O}/\varpi^i\mathcal{O}), \tag{C.3}$$

for each $i > 0$. Each of the spaces $M/\varpi^i M$ is again equipped with a natural profinite topology of pointwise convergence, and there is a topological isomorphism

$$M \xrightarrow{\sim} \varprojlim_i M/\varpi^i M. \tag{C.4}$$

The functor $X \mapsto \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})$ induces an anti-equivalence of categories between the category of $\mathcal{O}$-torsion free, $\varpi$-adically complete and separated $\mathcal{O}$-modules and the category of $\mathcal{O}$-torsion free, profinite linear-topological[16] $\mathcal{O}$-modules, with a quasi-inverse functor given by $M \mapsto \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(M, \mathcal{O})$, where $\mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(M, \mathcal{O})$ denotes the space of continuous $\mathcal{O}$-linear maps from $M$ to $\mathcal{O}$. (This is proved in the course of proving [77, Thm. 1.2].)

Suppose now that $X$ satisfies conditions 1, 2, and 3 of Definition C.1. The $A$-action on $X$ induces a transposed $A$-action on $M := \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$, making $M$ a profinite linear-topological $A$-module. The reader may easily verify that the functor $X \mapsto \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ induces an anti-equivalence of categories between the category of $A$-modules satisfying conditions 1, 2, and 3 of Definition C.1 and the category of $\mathcal{O}$-torsion free profinite linear-topological $A$-modules; of course, a quasi-inverse functor is again provided by $M \mapsto \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(X, \mathcal{O})$.

We now give the promised description of cofinitely generated $A$-modules in terms of their duals.

C.5. **Proposition.** *If $X$ is an $A$-module satisfying conditions 1, 2, and 3 of Definition C.1, then $X$ is cofinitely generated over $A$ if and only if $\mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ is finitely generated over $A$. Furthermore, the functor $X \mapsto \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ induces an anti-equivalence between the category of cofinitely generated $A$-modules, and the category of $\mathcal{O}$-torsion free finitely generated $A$-modules, with a quasi-inverse functor being provided by $M \mapsto \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(M, \mathcal{O})$ (where $M$ is equipped with its $\mathfrak{m}$-adic topology).*

*Proof.* Note that any profinite linear-topological $A$-module that is finitely generated over $A$ is necessarily equipped with its $\mathfrak{m}$-adic topology. Thus, in light of the above discussion, it suffices to show that $(X/\varpi X)[\mathfrak{m}]$ is finite-dimensional over $k$ if and only if $M := \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ is finitely generated over $A$.

Since any profinite linear-topological $A$-module is necessarily $\mathfrak{m}$-adically complete, we see that $M$ is finitely generated over $A$ if and only if $M/\mathfrak{m}M$ is finite-dimensional over $k$. The isomorphism (C.3) (in the case $i = 1$) induces an isomorphism between $\mathrm{Hom}_{\mathcal{O}}\big((X/\varpi X)[\mathfrak{m}], k\big)$ and $M/\mathfrak{m}M$. Thus $(X/\varpi X)[\mathfrak{m}]$ is indeed finite-dimensional over $k$ if and only if $M/\mathfrak{m}M$ is finite-dimensional over $k$. The proposition follows. □

C.6. **Definition.** As usual, we say that a submodule $Y$ of an $\mathcal{O}$-torsion free $\mathcal{O}$-module $X$ is saturated if the quotient $X/Y$ is $\mathcal{O}$-torsion free. If $X$ is an $\mathcal{O}$-module, and $Y$ is an $\mathcal{O}$-submodule of $X$, then we let $\mathrm{sat}_X(Y)$ denote the saturation of $Y$ in $X$, i.e. $\mathrm{sat}_X(Y) := \{x \in X \mid \varpi^i x \in Y \text{ for some } i \geq 0\}$.

We note the following simple lemma.

C.7. **Lemma.** *If $X$ is a $\varpi$-adically complete and torsion free $\mathcal{O}$-module, and if $Y$ is a saturated $\mathcal{O}$-submodule of $X$, then the $\varpi$-adic closure of $Y$ in $X$ is again saturated.*

*Proof.* Let $\overline{Y}$ denote the $\varpi$-adic closure of $Y$ in $X$, and suppose that $\varpi^i x \in \overline{Y}$ for some $x \in X$. For any $j \geq i$, there exists $x_j \in X$ such that $\varpi^i x - \varpi^j x_j \in Y$. Since $Y$ is saturated in $X$, we infer that $x - \varpi^{j-i} x_j \in Y$. Letting $j \to \infty$, we find that $x \in \overline{Y}$, and thus $\overline{Y}$ is saturated in $X$, as claimed. □

The following results record some additional properties of cofinitely generated $A$-modules, and of maps between them.

[16] I.e. having a neighbourhood basis of the origin consisting of open $\mathcal{O}$-submodules.

C.8. **Proposition.** *If $\phi : Y \to X$ is an $A$-linear map between cofinitely generated $A$-modules, then both $\phi(Y)$ and $\mathrm{sat}_X\big(\phi(Y)\big)$ are cofinitely generated $A$-modules, as are the kernel of $\phi$ and the quotient $X/\mathrm{sat}_X\big(\phi(Y)\big)$ (which is the maximal $\mathcal{O}$-torsion free quotient of the cokernel of $\phi$). Furthermore, the $\mathcal{O}$-torsion submodule of the cokernel of $\phi$ is of bounded exponent.*

*Proof.* We begin by noting that morphisms in the category of $A$-modules satisfying conditions 1, 2, and 3 of Definition C.1 admit kernels, cokernels, images, and coimages. If $\phi : Y \to X$ is a morphism in this category, then the categorical kernel of $\phi$ coincides with the usual module-theoretic kernel of $\phi$, and the categorical coimage of $\phi$ coincides with the usual module-theoretic image. The categorical image of $\phi$ coincides with the $\varpi$-adic closure of $\mathrm{sat}_X\big(\phi(Y)\big)$ in $X$, and the categorical cokernel of $\phi$ coincides with the quotient of $X$ by the categorical image of $\phi$. Similarly, the category of $\mathcal{O}$-torsion free, profinite linear-topological $A$-modules admits kernels, cokernels, images, and coimages (as it must, in light of the anti-equivalence described above), which may be described in the same manner, namely: if $\phi : M \to N$ is a morphism in this category, then its categorical kernel coincides with its usual module-theoretic kernel, its categorical coimage coincides with its usual module-theoretic image (equipped with the quotient topology), its categorical image coincides with the closure in $N$ of $\mathrm{sat}_N\big(\phi(M)\big)$, and its categorical cokernel coincides with the quotient of $N$ by its categorical image. Of course, the anti-equivalence $X \mapsto \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ interchanges categorical kernels and cokernels, and categorical images and coimages.

Suppose now that $\phi : M \to N$ is a morphism of finitely generated $A$-modules. The Artin–Rees Lemma shows that $\mathrm{sat}_N\big(\phi(M)\big)$ is closed in $N$, and so coincides with the categorical image of $\phi$. Furthermore, the quotient $\mathrm{sat}_N\big(\phi(M)\big)/\phi(M)$, which is torsion as an $\mathcal{O}$-module, is finitely generated over $A$, and so of finite exponent. Thus $\varpi^i \mathrm{sat}_N\big(\phi(M)\big) \subset \phi(M) \subset \mathrm{sat}_N\big(\phi(M)\big)$ for some $i \geq 0$. Both $\mathrm{sat}_N\big(\phi(M)\big)$ and $\phi(M)$ (which is the categorical coimage of $\phi$) are finitely generated over $A$, being submodules of $N$. The kernel of $\phi$ is also finitely generated over $A$, being a submodule of $M$, as is the quotient $N/\mathrm{sat}_N\big(\phi(M)\big)$ (which is equal to the categorical cokernel of $\phi$).

Applying the anti-equivalence of categories of Proposition C.5, we find that if $\phi : Y \to X$ is an $A$-linear morphism of cofinitely generated $A$-modules, then the categorical coimage of $\phi$, the categorical image of $\phi$, the categorical kernel of $\phi$, and the categorical cokernel of $\phi$ are all cofinitely generated over $A$; that is, $\phi(Y)$, the $\varpi$-adic closure in $X$ of $\mathrm{sat}_X\big(\phi(Y)\big)$, the kernel of $\phi$, and the quotient of $X$ by the $\varpi$-adic closure in $X$ of $\mathrm{sat}_X\big(\phi(Y)\big)$ are all cofinitely generated over $A$. Furthermore, if we write $Z$ to denote the $\varpi$-adic closure in $X$ of $\mathrm{sat}_X\big(\phi(Y)\big)$, then we find that $\varpi^i Z \subset \phi(Y) \subset Z$ for some $i \geq 0$. This shows that in fact $Z \subset \mathrm{sat}_X\big(\phi(Y)\big)$, and thus that $Z = \mathrm{sat}_X\big(\phi(Y)\big)$, i.e. that $\mathrm{sat}_X\big(\phi(Y)\big)$ is already $\varpi$-adically closed in $X$, and is cofinitely generated over $A$. Furthermore, we see that $\mathrm{sat}_X\big(\phi(Y)\big)/\phi(Y)$ is of bounded exponent. This proves the proposition. ∎

C.9. **Proposition.** *If $X$ is a cofinitely generated $A$-module, and if $Y$ is an $A$-submodule of $X$, then the following are equivalent:*

(1) *$Y$ is a cofinitely generated $A$-module.*
(2) *$Y$ is $\varpi$-adically closed in $X$, and the $\mathcal{O}$-torsion submodule of the quotient $X/Y$ is of bounded exponent.*

(3) *$Y$ is $\varpi$-adically closed in $X$, and the $\varpi$-adic topology on $X$ induces the $\varpi$-adic topology on $Y$.*

*Proof.* The reader may easily check that if $X$ is any $\mathcal{O}$-torsion free $\mathcal{O}$-module, and if $Y$ is a submodule of $X$, then the $\varpi$-adic topology on $X$ induces the $\varpi$-adic topology on $Y$ if and only if the torsion submodule of $X/Y$ is of bounded exponent. Indeed, the following conditions are equivalent (for some $i \geq 0$): $\varpi^{i+1}X \bigcap Y \subset \varpi Y$; $\varpi^{i+j}X \bigcap Y \subset \varpi^j Y$ for all $j > 0$; any $\mathcal{O}$-torsion element of $X/Y$ is annihilated by $\varpi^i$. In particular, we conclude that conditions 2 and 3 of the proposition are equivalent.

Suppose now that condition 1 holds. Applying Proposition C.8 to the inclusion $Y \subset X$, we find that $X/Y$ has bounded exponent. The observation of the preceding paragraph shows that the $\varpi$-adic topology on $X$ induces the $\varpi$-adic topology on $Y$. Since $Y$ is $\varpi$-adically complete (being cofinitely generated over $A$), we see that $Y$ is $\varpi$-adically closed in $X$. Thus condition 2 (and so also condition 3) holds.

Suppose conversely that conditions 2 and 3 hold. Since $\mathrm{sat}_X(Y)/Y$ is equal to the $\mathcal{O}$-torsion submodule of $X/Y$, we see that

$$\varpi^i \mathrm{sat}_X(Y) \subset Y \subset \mathrm{sat}_X(Y) \tag{C.10}$$

for some $i \geq 0$. From this we conclude that $\mathrm{sat}_X(Y)$ is also $\varpi$-adically closed in $X$, and of course, since $\mathrm{sat}_X(Y)$ is saturated in $X$, by construction, the $\varpi$-adic topology on $X$ induces the $\varpi$-adic topology on $\mathrm{sat}_X(Y)$. Thus $\mathrm{sat}_X(Y)$ satisfies conditions 1, 2, and 3 of Definition C.1. Furthermore, the natural map $\big(\mathrm{sat}_X(Y)\big)/\varpi\big(\mathrm{sat}_X(Y)\big)[\mathfrak{m}] \hookrightarrow (X/\varpi X)[\mathfrak{m}]$ is injective, and thus the source of this map is finite-dimensional over $k$, since the target is so. This shows that $\mathrm{sat}_X(Y)$ is cofinitely generated over $A$.

Now condition 3 shows that $Y$ is $\varpi$-adically complete, and thus $Y$ also satisfies conditions 1, 2, and 3 of Definition C.1. From (C.10), we obtain an embedding

$$\mathrm{Hom}_{\mathcal{O}}(Y, \mathcal{O}) \hookrightarrow \mathrm{Hom}_{\mathcal{O}}(\varpi^i \mathrm{sat}_X(Y), \mathcal{O}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathrm{sat}_X(Y), \mathcal{O}).$$

Since $\mathrm{sat}_X(Y)$ is cofinitely generated over $A$, Proposition C.5 shows that the target of this embedding is finitely generated over $A$, and thus so is the source. Another application of the same proposition then shows that $Y$ is cofinitely generated over $A$, i.e. that condition 1 holds. □

C.11. **Proposition.** *If $M$ is a finitely generated $A$-module and $X$ is a cofinitely generated $A$-module, then $\mathrm{Hom}_A(M, X)$ is a cofinitely generated $A$-module. Furthermore, there is a natural isomorphism*

$$\big(M \otimes_A \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})\big)/\big(M \otimes_A \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})\big)[\varpi^\infty] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{Hom}_A(M, X), \mathcal{O}\big).$$

*Proof.* The fact that $\mathrm{Hom}_A(M, X)$ is cofinitely generated over $A$ can be verified directly from the definition. Indeed, the fact that $\mathrm{Hom}_A(M, X)$ is $\varpi$-adically complete and separated and $\mathcal{O}$-torsion free follows directly from the corresponding properties of $X$. The continuity condition on the $A$-action is also easily verified. Finally, there is a natural embedding

$$\big(\mathrm{Hom}_A(M, X)/\varpi\, \mathrm{Hom}_A(M, X)\big)[\mathfrak{m}] \hookrightarrow \mathrm{Hom}_k(M/\mathfrak{m}M, (X/\varpi X)[\mathfrak{m}]).$$

Since $M$ is finitely generated over $A$, the quotient $M/\mathfrak{m}M$ is finite-dimensional over $k$, while $(X/\varpi X)[\mathfrak{m}]$ is finite-dimensional over $k$ by assumption. The target of this embedding is thus finite-dimensional over $k$, and hence so is the source.

We now give another proof of the cofinite generation of $\mathrm{Hom}_A(M, X)$, which also establishes the claimed description of $\mathrm{Hom}_{\mathcal{O}}\big(\mathrm{Hom}_A(M,X),\mathcal{O}\big)$. To this end, choose a finite presentation $A^s \to A^r \to M \to 0$ of $M$. Applying the functor $\mathrm{Hom}_A(-, X)$, we obtain an exact sequence

$$0 \to \mathrm{Hom}_A(M, X) \to X^r \to X^s.$$

It now follows from Proposition C.8 that $\mathrm{Hom}_A(M, X)$ is cofinitely generated, since it is the kernel of an $A$-linear map between cofinitely generated $A$-modules. On the other hand, tensoring this presentation of $M$ with $\mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ over $A$ yields an exact sequence

$$\mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})^s \to \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})^r \to M \otimes_A \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O}) \to 0,$$

and the same proposition (or, more precisely, its proof), shows that there is an isomorphism between $\mathrm{Hom}_{\mathcal{O}}\big(\mathrm{Hom}_A(M,X),\mathcal{O}\big)$ and the quotient of $\mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})^r$ by the saturation of the image of $\mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})^s$, or, more canoncically, that the natural map

$$M \otimes_A \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{Hom}_A(M,X),\mathcal{O}\big)$$

induces an isomorphism

$$\big(M \otimes_A \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})\big)/\big(M \otimes_A \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})\big)[\varpi^\infty] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{Hom}_A(M,X),\mathcal{O}\big),$$

as claimed. □

C.12. **Lemma.** *If $A \to B$ is a finite morphism between objects of* $\mathrm{Comp}(\mathcal{O})$*, and if $X$ is a $B$-module, then $X$ is cofinitely generated as a $B$-module if and only if $X$ is cofinitely generated as an $A$-module.*

*Proof.* This is easily verified directly, and also follows from the fact that the $B$-module $\mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})$ is finitely generated over $B$ if and only if its finitely generated over $A$. □

C.13. **Proposition.** *Let $A \to B$ be a finite morphism between objects of* $\mathrm{Comp}(\mathcal{O})$*, and let $X$ be a cofinitely generated $A$-module.*

1. $\mathrm{Hom}_A(B,X)$ *is a cofinitely generated $B$-module.*
2. *If $X$ is faithful as an $A$-module, and if $B$ is reduced, then* $\mathrm{Hom}_A(B,X)$ *is faithful as a $B$-module.*

*Proof.* Proposition C.11 shows that $\mathrm{Hom}_A(B,X)$ is cofinitely generated as an $A$-module, and Lemma C.12 then shows that it is also cofinitely generated as a $B$-module. This proves (1).

To prove (2), note that if $M = \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})$ and $N = \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{Hom}_A(B,X),\mathcal{O}\big)$, then by Proposition C.5, there is an isomorphism $X \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(M,\mathcal{O})$ (resp. $\mathrm{Hom}_A(B,X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(N,\mathcal{O})$), and hence $X$ (resp. $\mathrm{Hom}_A(B,X)$) is faithful as an $A$-module (resp. $B$-module) if and only if the same is true of $M$ (resp. $N$). Also, by Proposition C.11, there is an isomorphism

$$(B \otimes_A M)/(B \otimes_A M)[\varpi^\infty] \xrightarrow{\sim} N.$$

Thus in order to establish (2), it suffices to show that if $M$ is a faithful $A$-module, then $(B \otimes_A M)/(B \otimes_A M)[\varpi^\infty]$ is a faithful $B$-module. Since $B$ is reduced, it suffices for this to show this quotient has non-zero localization at each minimal prime of $B$. Since $B$ is flat over $\mathcal{O}$, the localization of $B$ at each of its minimal primes is torsion free as an $\mathcal{O}$-module, and thus the localization of this quotient at such a prime coincides with the localization of $B \otimes_A M$ itself at that prime. Since $B$ is finite over $A$, and since $M$ is faithful as an $A$-module, this localization is indeed non-zero, and (2) follows. $\square$

Since $A$ is a complete local ring, its localization $A[1/p]$ is a Jacobson ring. A point $\mathfrak{p} \in \operatorname{Spec} A[1/p] \subset \operatorname{Spec} A$ is closed if and only if $A/\mathfrak{p}$ is a finite $\mathcal{O}$-algebra, or equivalently, if and only if $\kappa(\mathfrak{p})$ $(= A/\mathfrak{p}[1/p]$, the fraction field of $A/\mathfrak{p})$ is a finite extension of $E$.

C.14. **Lemma.** *Let $X$ be a cofinitely generated $A$-module. If $\mathfrak{p} \in \operatorname{Spec} A[1/p]$ is a closed point, then $E \otimes_{\mathcal{O}} X[\mathfrak{p}]$ and $\kappa(\mathfrak{p}) \otimes_A \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})$ are dual finite-dimensional $\kappa(\mathfrak{p})$-vector spaces.*

*Proof.* There is a natural isomorphism $X[\mathfrak{p}] \xrightarrow{\sim} \operatorname{Hom}_A(A/\mathfrak{p}, X)$, and so Proposition C.13 (1) shows that $X[\mathfrak{p}]$ is cofinitely generated over $A/\mathfrak{p}$. The equivalence of categories of Proposition C.5 (applied with $A$ replaced by $A/\mathfrak{p}$) implies that $X[\mathfrak{p}] \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}-\mathrm{cont}}(M, \mathcal{O})$ for some finite generated torsion free $A/\mathfrak{p}$-module $M$. However, the ring $A/\mathfrak{p}$ is finite over $\mathcal{O}$, and hence $M$ is in fact finitely generated as an $\mathcal{O}$-module. The same is thus true of $\operatorname{Hom}_{\mathcal{O}-\mathrm{cont}}(M, \mathcal{O})$, and hence of $X[\mathfrak{p}]$. This immediately implies that $E \otimes_{\mathcal{O}} X[\mathfrak{p}]$ is finite-dimensional over $E$, and so also over $\kappa(\mathfrak{p})$.

Proposition C.11 also provides an isomorphism

$$(A/\mathfrak{p}) \otimes_A \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O}) / \big((A/\mathfrak{p}) \otimes_A \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})\big)[\varpi^\infty] \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}}(X[\mathfrak{p}], \mathcal{O}),$$

and hence an isomorphism

$$\kappa(\mathfrak{p}) \otimes_A \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O}) \xrightarrow{\sim} \operatorname{Hom}_E(E \otimes_{\mathcal{O}} X[\mathfrak{p}], E).$$

Linear algebra provides a $\kappa(\mathfrak{p})$-linear isomorphism

$$\operatorname{Hom}_E(E \otimes_{\mathcal{O}} X[\mathfrak{p}], E) \xrightarrow{\sim} \operatorname{Hom}_{\kappa(\mathfrak{p})}\big(E \otimes_{\mathcal{O}} X[\mathfrak{p}], \kappa(\mathfrak{p})\big),$$

and so we see that $E \otimes_{\mathcal{O}} X[\mathfrak{p}]$ and $\kappa(\mathfrak{p}) \otimes_A \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})$ are dual $\kappa(\mathfrak{p})$-vector spaces, as claimed. $\square$

We now prove some results regarding the completed tensor products of orthonormalizable and cofinitely generated $A$-modules.

C.15. **Lemma.** *Let be $f : Y \to X$ be a $A$-linear map of cofinitely generated $A$-modules, and let $V$ be an orthonormalizable $A$-module.*

(1) *If $f$ is an embedding, then the induced map*
$$V \hat{\otimes}_A Y \to V \hat{\otimes}_A X$$
*of $\varpi$-adically completed tensor products is again an embedding.*

(2) *If $f$ furthermore has saturated image, then the same is true of the induced map*
$$V \hat{\otimes}_A Y \to V \hat{\otimes}_A X.$$

*Proof.* Suppose given an embedding $f : Y \hookrightarrow X$, with cokernel $C$. We then have a short exact sequence

$$0 \to Y \to X \to C \to 0,$$

which upon tensoring by $V$ over $A$ yields a short exact sequence

$$0 \to V \otimes_A Y \to V \otimes_A X \to V \otimes_A C \to 0, \qquad \text{(C.16)}$$

since since $V$ is $A$-flat, by Lemma B.6. Proposition C.9 shows that $\mathcal{O}$-torsion submodule of $C$ has bounded exponent, say $\varpi^i$ for some $i \geq 0$, and thus (again using the fact that $V$ is flat over $A$) the $\mathcal{O}$-torsion submodule of $V \otimes_A C$ has the same bounded exponent. Thus passing to $\varpi$-adic completions in (C.16) preserves exactness, i.e. we obtain a short exact sequence

$$0 \to V \hat{\otimes}_A Y \to V \hat{\otimes}_A X \to V \hat{\otimes}_A C \to 0. \tag{C.17}$$

In particular, the second arrow is injective, and so part 1 is proved.

If $C$ is $\mathcal{O}$-flat (i.e. if we can take $i = 0$), then the preceding discussion shows that the same is true of $V \otimes_A C$, and hence of its $\varpi$-adic completion $V \hat{\otimes}_A C$. A consideration of (C.17) then shows that the second arrow is not only injective, but has saturated image, proving 2. □

C.18. **Lemma.** *Let be $f : Y \to X$ be a $A$-linear map of cofinitely generated $A$-modules, and let $V$ be a non-zero orthonormalizable $A$-module.*

1. *If the induced map $V \hat{\otimes}_A Y \to V \hat{\otimes}_A X$ of $\varpi$-adically completed tensor products is an embedding (with saturated image), then $f$ is an embedding (with saturated image).*
2. *If the induced map $V \hat{\otimes}_A Y \to V \hat{\otimes}_A X$ of $\varpi$-adically completed tensor products is an isomorphism, then $f$ is an isomorphism.*

*Proof.* Consider the short exact sequence

$$0 \to K \to Y \to X \to C \to 0$$

(so $K$, resp. $C$, is the kernel, resp. cokernel, of $f$). Tensoring with the $A$-flat (by Lemma B.6) module $V$, we obtain an exact sequence

$$0 \to V \otimes_A K \to V \otimes_A Y \to V \otimes_A X \to V \otimes_A C \to 0. \tag{C.19}$$

Now $K$, $Y$, and $X$ are $\mathcal{O}$-torsion free, while Proposition C.9 shows that the $\mathcal{O}$-torsion submodule of $C$ has bounded exponent, and hence the same is true of the $\mathcal{O}$-torsion submodule of $V \otimes_A C$ (since $V$ is $A$-flat). Thus $\varpi$-adically completing (C.19) yields an exact sequence

$$0 \to V \hat{\otimes}_A K \to V \hat{\otimes}_A Y \to V \hat{\otimes}_A X \to V \hat{\otimes}_A C \to 0. \tag{C.20}$$

Proposition C.9 shows that $K$ is cofinitely generated over $A$, and so in particular, any element of $K/\varpi K$ is annihilated by some power of $\mathfrak{m}$. The same is true of the elements of $C/\varpi C$ (since this is a quotient of $X/\varpi X$, and $X$ is cofinitely generated over $A$). Consequently Lemma B.9 shows that $V \hat{\otimes}_A K$ (resp. $V \hat{\otimes}_A C$) vanishes if and only if $K$ (resp. $C$) vanishes, and also that $V \hat{\otimes}_A C$ is $\mathcal{O}$-torsion free if and only if $C$ is. These facts, taken together with the exact sequence (C.20), serve to prove the lemma. □

We now introduce the notion of cosupport for cofinitely generated $A$-modules.

C.21. **Definition.** If $X$ is a cofinitely generated $A$-module, then we define the cosupport of $X$ (which is a subset of $\operatorname{Spec} A$) to be the support of its dual $\operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})$ (which is a finitely generated $A$-module).

Thus if $I$ denotes the annihilator ideal in $A$ of $X$, so that $I$ is simultaneously the annihilator ideal in $A$ of the finitely generated $A$-module $\operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})$, then the cosupport of $X$ is equal to the closed subset $\operatorname{Spec} A/I$ of $\operatorname{Spec} A$.

C.22. **Proposition.** *If $X$ is a cofinitely generated $A$-module, then a closed point $\mathfrak{p} \in \operatorname{Spec} A[1/p]$ is contained in the cosupport of $X$ if and only if $X[\mathfrak{p}] \neq 0$, or equivalently, if and only if $E \otimes_{\mathcal{O}} X[\mathfrak{p}] \neq 0$.*

*Proof.* By definition, $\mathfrak{p}$ lies in the cosupport of $X$ if and only if $\mathfrak{p}$ lies in the support of $\operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})$. Since this latter module is finitely generated over $A$, Nakayama's lemma shows that $\mathfrak{p}$ lies in its support if and only if $\kappa(\mathfrak{p}) \otimes_A \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})$ is non-zero. The proposition then follows from Lemma C.14. □

We now introduce the class of coadmissible smooth $\Gamma$-representations over $A$.

C.23. **Definition.** We say that a smooth $\Gamma$-representation on an $\mathcal{O}$-torsion free $A$-module $X$ is coadmissible if for each open subgroup $K_{\Sigma_0}$ of $\Gamma$, the space of invariants $X^{K_{\Sigma_0}}$ is a cofinitely generated $A$-module.

C.24. **Definition.** If $X$ is a smooth representation of $\Gamma$ on an $\mathcal{O}$-torsion free $A$-module, then we write $\widetilde{X}$ to denote the smooth contragredient to $X$, i.e.

$$\widetilde{X} := \{\phi \in \operatorname{Hom}(X, \mathcal{O}) \mid \phi \text{ is fixed by some compact open subgroup of } \Gamma\}.$$

(Here $\Gamma$ acts on $\operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})$ via the contragredient action.)

C.25. **Lemma.** *Let $X$ be a smooth representation of $\Gamma$ over $A$, that satisfies the following conditions:*

(1) *$X$ is $\mathcal{O}$-torsion free.*
(2) *For each (or equivalently, for a cofinal collection of) open subgroups $H$ of $\Gamma$, the space of invariants $X^H$ is $\varpi$-adically complete and separated.*
(3) *The action map $A \times X \to X$ induced by the $A$-module structure on $X$ is continuous, when $A$ is given its $\mathfrak{m}$-adic topology, and $X$ is given its $\varpi$-adic topology.*

*Then the smooth $\Gamma$-representation $X$ is coadmissible if and only if $(X/\varpi X)[\mathfrak{m}]$ (which is a smooth representation of $\Gamma$ over $k$) is admissible.*

*Proof.* Let $H$ be a open subgroup of $\Gamma$. Conditions 1, 2, and 3 of the lemma imply that $X^H$ satisfies conditions 1, 2, and 3 of Definition C.1. Now choose $H$ to be a compact open subgroup of $\Gamma$, and so small that its pro-order is prime-to-$p$. Reduction modulo $\mathfrak{m}$ commutes with the formation of $H$-invariants, and thus we see that $X^H$ satisfies condition 4 of Definition C.1 for every such $H$ if and only if $(X/\varpi X)[\mathfrak{m}]$ is admissible. □

C.26. **Lemma.** *The functor $X \mapsto \widetilde{X}$ induces an anti-equivalence of categories between the category of coadmissible smooth representations of $\Gamma$ on $\mathcal{O}$-torsion free $A$-modules, and the category of admissible smooth representations of $\Gamma$ on $\mathcal{O}$-torsion free $A$-modules.*

*Proof.* If $X$ is a coadmissible smooth representation of $\Gamma$ over $A$, and if $H$ is a compact open subgroup of $\Gamma$, chosen so that the pro-order of $H$ is prime-to-$p$, then

$$\widetilde{X}^H := \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})^H \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}}(X^H, \mathcal{O}).$$

Thus Proposition C.5 shows that $\widetilde{X}^H$ is finitely generated over $A$, and hence $\widetilde{X}$ is an admissible smooth representation of $\Gamma$ over $A$. The claimed anti-equivalence of categories is easily verified by the reader, taking into account the anti-equivalence of categories of Proposition C.5. □

C.27. **Lemma.** *If $X$ is a coadmissible smooth $A[\Gamma]$-module and $Y$ is a coadmissible $A[\Gamma]$-submodule of $X$, then $\mathrm{sat}_X(Y)$ is again a coadmissible $A[\Gamma]$-module.*

*Proof.* If $H$ is any compact open subgroup of $\Gamma$, then $\mathrm{sat}_X(Y^H) = \mathrm{sat}_X(Y)^H$. The present lemma is thus seen to follow from Proposition C.8. □

C.28. **Definition.** If $X$ is a coadmissible smooth $A[\Gamma]$-module, we say that an $A[\Gamma]$-submodule $Y$ of $X$ is closed if $Y^H$ is $\varpi$-adically closed in $X^H$ for any compact open subgroup (or equivalently, for a cofinal collection of compact open subgroups) $H$ of $\Gamma$. If $Y$ is any $A[\Gamma]$-submodule of $X$, then we define the closure of $Y$ to be the intersection of all closed $A[\Gamma]$-submodules of $X$ (in the preceding sense) containing $Y$. (More concretely, the closure of $Y$ is equal to the union, over all compact open subgroups $H$ of $\Gamma$, of the $\varpi$-adic closure of $Y^H$ in $X^H$.)

C.29. **Remark.** If $X$ is a coadmissible smooth $A[\Gamma]$-module, then there is a natural isomorphism $\varinjlim_H X^H \xrightarrow{\sim} X$, where $H$ runs over all (or equivalently, a cofinal collection of) compact open subgroups of $\Gamma$. Equipping each $X^H$ with its $\varpi$-adic topology, we may then endow $X$ with the resulting $\mathcal{O}$-linear inductive limit topology.[17] One may then prove that an $\mathcal{O}$-submodule $Y$ of $X$ is then closed in $X$, with respect to this topology, if and only if $Y^H$ is $\varpi$-adically closed in $X^H$ for each $H$. This gives some justification for the preceding definition.

C.30. **Lemma.** *If $X$ is a coadmissible smooth $A[\Gamma]$-module, and if $Y$ is a saturated $A[\Gamma]$-submodule of $X$, then the closure of $Y$ in $X$ is again saturated.*

*Proof.* If we let $\overline{Y}$ denote the closure of $Y$ in $X$, then it suffices to show that $\overline{Y}^H$ is saturated in $X^H$, for each compact open subgroup $H$ of $\Gamma$. Since $\overline{Y}^H$ coincides with the $\varpi$-adic closure of $Y^H$ in $X^H$, and since the assumption that $Y$ is saturated in $X$ implies that $Y^H$ is saturated in $X^H$, this follows from Lemma C.7. □

C.31. **Lemma.** *Let $X$ be a coadmissible smooth $A[\Gamma]$-module. If $Y$ is a saturated $A[\Gamma]$-submodule $Y$ of $X$, then $Y$ is coadmissible if and only if $Y$ is closed in $X$, in the sense of Definition C.28.*

*Proof.* This follows directly from Proposition C.9. □

C.32. **Definition.** A coadmissible smooth $A[\Gamma]$-module $X$ is called cofinitely generated over $A[\Gamma]$ if $\widetilde{X}$ is finitely generated over $A[\Gamma]$.

C.33. **Proposition.** *A coadmissible smooth $A[\Gamma]$-module $X$ is cofinitely generated over $A[\Gamma]$ if and only if $(X/\varpi X)[\mathfrak{m}]$ is finitely generated over $k[\Gamma]$.*

*Proof.* Recall that an admissible smooth representation $W$ of $\Gamma$ over $k$ is finitely generated if and only if it is of finite length. Since $W$ is of finite length if and only if the same is true of its smooth contragredient $\widetilde{W}$, we conclude that $W$ is finitely generated if and only if $\widetilde{W}$ is. In particular, since $(X/\varpi X)[\mathfrak{m}]$ is the smooth contragredient to $\widetilde{X}/\mathfrak{m}\widetilde{X}$, we see that $(X/\varpi X)[\mathfrak{m}]$ is finitely generated over $k[\Gamma]$ if and only if $\widetilde{X}/\mathfrak{m}\widetilde{X}$ is. The proposition thus follows from the fact that the admissible smooth $A[\Gamma]$-representation $\widetilde{X}$ is finitely generated over $A[\Gamma]$ if and only if $\widetilde{X}/\mathfrak{m}\widetilde{X}$ is finitely generated over $k[\Gamma]$ [43]. □

[17] I.e. we take as a basis of neighbourhoods of the origin in $X$ those $\mathcal{O}$-submodules $Y$ such that $Y^H$ is $\varpi$-adically open in $X^H$ for each $H$.

C.34. **Corollary.** *If $X$ is a coadmissible smooth $A[\Gamma]$-module that is cofinitely generated over $A[\Gamma]$, and if $Y$ is a saturated coadmissible $A[\Gamma]$-submodule of $X$, then $Y$ is again cofinitely generated over $A[\Gamma]$.*

*Proof.* The assumption that $Y$ is saturated in $X$ implies that the natural map $Y/\varpi Y \to X/\varpi X$ is injective. The corollary then follows from the preceding proposition, and the fact that a subrepresentation of a finitely generated admissible smooth $\Gamma$-representation over $k$ is again finitely generated. (Indeed, as was already noted in the proof of the preceding result, finitely generated admissible smooth $\Gamma$-representations over $k$ are necessarily of finite length.) □

C.35. **Lemma.** *If $X$ is a coadmissible smooth $\Gamma$-representation over $A$, and if $\mathfrak{p} \in \operatorname{Spec} A[1/p]$ is a closed point, then $E \otimes_{\mathcal{O}} X[\mathfrak{p}]$ is naturally isomorphic to the smooth contragredient of the admissible smooth $\Gamma$-representation $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}$ over $\kappa(\mathfrak{p})$.*

*Proof.* As already noted in the proof of Lemma C.26, if $H$ is a compact open subgroup of $\Gamma$ of pro-order prime-to-$p$, then

$$\widetilde{X}^H := \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O})^H \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}}(X^H, \mathcal{O}).$$

Lemma C.14 then yields the second of the pairs of isomorphisms

$$\big(\kappa(\mathfrak{p}) \otimes_A \widetilde{X}\big)^H \xrightarrow{\sim} \kappa(\mathfrak{p}) \otimes \widetilde{X}^H \xrightarrow{\sim} \operatorname{Hom}_{\kappa(\mathfrak{p})}\big(E \otimes_{\mathcal{O}} X^H[\mathfrak{p}], \kappa(\mathfrak{p})\big).$$

(The first follows from the fact that $H$ has pro-order prime-to-$p$.) Letting $H$ run over a cofinal set of compact open subgroups of $\Gamma$, we find that $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}$ and $E \otimes_{\mathcal{O}} X[\mathfrak{p}]$ are indeed mutually dual admissible smooth representations of $\Gamma$ over $\kappa(\mathfrak{p})$. □

C.36. **Proposition.** *If the ring $A$ is reduced, and if $X$ is a coadmissible smooth $\Gamma$-representation over $A$ which is cofinitely generated over $A[\Gamma]$, then the following are equivalent:*

1. *$X[\mathfrak{p}] \neq 0$, or equivalently, $E \otimes_{\mathcal{O}} X[\mathfrak{p}] \neq 0$, for every closed point $\mathfrak{p} \in \operatorname{Spec} A[1/p]$.*
2. *$X[\mathfrak{p}] \neq 0$, or equivalently, $E \otimes_{\mathcal{O}} X[\mathfrak{p}] \neq 0$, for a Zariski dense subset of closed points $\mathfrak{p} \in \operatorname{Spec} A[1/p]$.*
3. *$X$ is a faithful $A$-module.*

*Proof.* Clearly the annihilator of $X$ in $A$ coincides with the annihilator of $\widetilde{X}$ in $A$, and Lemma C.35 shows that $E \otimes_{\mathcal{O}} X[\mathfrak{p}] \neq 0$ if and only if $\kappa(\mathfrak{p}) \otimes_A \widetilde{X} \neq 0$, for any closed point $\mathfrak{p} \in \operatorname{Spec} A[1/p]$. Thus it suffices to verify that (1′) $\kappa(\mathfrak{p}) \otimes_A \widetilde{X} \neq 0$ for every closed point $\mathfrak{p} \in \operatorname{Spec} A[1/p]$ if and only if (2′) $\kappa(\mathfrak{p}) \otimes_A \widetilde{X} \neq 0$ for a Zariski dense subset of closed points $\mathfrak{p} \in \operatorname{Spec} A[1/p]$ if and only if (3′) $\widetilde{X}$ is faithful as an $A$-module.

By assumption, $\widetilde{X}$ is finitely generated over $A[\Gamma]$, and thus is generated over $A[\Gamma]$ by $\widetilde{X}^H$, for some sufficiently small compact open subgroup $H$ of $G$. Shrinking $H$ if necessary, we may assume that the pro-order of $H$ is prime-to-$p$. Since $\widetilde{X}^H$ is finitely generated, the following are equivalent: (1″) $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}^H \neq 0$ for every closed point $\mathfrak{p} \in \operatorname{Spec} A[1/p]$; (2″) $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}^H \neq 0$ for a Zariski dense subset of closed points $\mathfrak{p} \in \operatorname{Spec} A[1/p]$; (3″) $\widetilde{X}^H$ is faithful as an $A$-module.

We now note that $\kappa(\mathfrak{p}) \otimes_A \widetilde{X} \neq 0$ if and only if $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}^H \neq 0$. Indeed, $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}$ is generated over $\kappa(\mathfrak{p})[\Gamma]$ by $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}^H$, and so the only if direction is clear. On the other hand, $\widetilde{X}^H$ is a direct summand of $\widetilde{X}$, since $p$ doesn't divide the pro-order of $H$, and thus $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}^H$ is a direct summand of $\kappa(\mathfrak{p}) \otimes_A \widetilde{X}$. This establishes the if direction. Consequently ($1'$) and ($1''$) are equivalent, as are ($2'$) and ($2''$). Since $\widetilde{X}^H$ generates $\widetilde{X}$ over $A[\Gamma]$, conditions ($3'$) and ($3''$) are also equivalent. This completes the proof of the proposition. □

C.37. **Definition.** Suppose that $A$ is reduced. We say that a coadmissible smooth $A[\Gamma]$-module $X$ is cotorsion free if $\widetilde{X}$ is torsion free as an $A$-module (i.e. if multiplication by any non-zero divisor of $A$ induces an injection on $\widetilde{X}$).

C.38. **Proposition.** *If $A$ is reduced, and if $X$ is a coadmissible smooth $A[\Gamma]$-module, then $X$ contains a unique maximal cotorsion free coadmissible $A[\Gamma]$-submodule, which is furthermore saturated in $X$.*

*Proof.* Let $\widetilde{X}_{\mathrm{tor}}$ denote the maximal $A$-torsion submodule of $\widetilde{X}$, and let $\widetilde{X}_{\mathrm{tf}} := \widetilde{X}/\widetilde{X}_{\mathrm{tor}}$ denote the maximal $A$-torsion free quotient of $\widetilde{X}$. Clearly $\widetilde{X}_{\mathrm{tor}}$ is an $A[\Gamma]$-submodule of $\widetilde{X}$, and is saturated in $\widetilde{X}$. Thus $\widetilde{X}_{\mathrm{tf}}$ is an $A[\Gamma]$-module, and is $\mathcal{O}$-torsion free. It is admissible, since $\widetilde{X}$ is, and corresponds under the anti-equivalence of Lemma C.26 to the sort-after submodule of $\widetilde{X}$. □

C.39. **Definition.** If $A$ is reduced, and if $X$ is a coadmissible smooth $A[\Gamma]$-module, then we let $X_{\mathrm{ctf}}$ denote the maximal cotorsion free coadmissible $A[\Gamma]$-submodule of $X$ whose existence is guaranteed by the preceding proposition.

C.40. **Proposition.** *If $A$ is reduced, and if $X$ is a cofinitely generated coadmissible smooth $\Gamma$-representation over $A$, which is faithful as an $A$-module, then $X_{\mathrm{ctf}}$ is also faithful as an $A$-module.*

*Proof.* Passing from $X$ to $\widetilde{X}$, it suffices to show that if $\widetilde{X}$ is a finitely generated smooth $A[\Gamma]$-module, which is faithful as an $A$-module, then $\widetilde{X}_{\mathrm{tf}}$ (the maximal $A$-torsion free quotient of $\widetilde{X}$) is again faithful as an $A$-module. If we choose a compact open subgroup $H$ of $\Gamma$ such that $\widetilde{X}^H$ generates $\widetilde{X}$ over $A[\Gamma]$, then we find that $\widetilde{X}^H$ must be a faithful $A$-module. Since $\widetilde{X}^H$ is finitely generated over $A$, this implies that $(\widetilde{X}^H)_{\mathrm{tf}}$ is again a faithful $A$-module. The inclusion $\widetilde{X}^H \subset \widetilde{X}$ induces an embedding $(\widetilde{X}^H)_{\mathrm{tf}} \hookrightarrow \widetilde{X}_{\mathrm{tf}}$. Thus $\widetilde{X}_{\mathrm{tf}}$ contains a faithful $A$-submodule, and so is itself faithful as an $A$-module. □

C.41. **Proposition.** *If $A$ is reduced, if $X$ is a coadmissible cotorsion free smooth $A[\Gamma]$-module, and if $Y$ is a saturated coadmissible $A[\Gamma]$-submodule of $X$ with the property that $Y[\mathfrak{p}] = X[\mathfrak{p}]$ (or equivalently, since $Y$ is saturated in $X$, that $E \otimes_{\mathcal{O}} Y[\mathfrak{p}] = E \otimes_{\mathcal{O}} X[\mathfrak{p}]$) for a Zariski dense set of closed points $\mathfrak{p} \in \operatorname{Spec} A[1/p]$, then $Y = X$.*

*Proof.* If $H$ is a compact open subgroup of $\Gamma$ whose pro-order is prime-to-$p$, then $Y^H$ is a saturated cofinitely generated $A$-submodule of the cofinitely generated $A$-module $X^H$, and thus the induced map of finitely generated $A$-modules

$$\mathrm{Hom}_{\mathcal{O}}(X^H, \mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}(Y^H, \mathcal{O}) \tag{C.42}$$

is surjective. By assumption $Y^H[\mathfrak{p}] = X^H[\mathfrak{p}]$ for a Zariski dense set of closed points $\mathfrak{p} \in \operatorname{Spec} A[1/p]$, and so from Lemma C.14, we conclude that the induced surjection

$$\kappa(\mathfrak{p}) \otimes_A \mathrm{Hom}_{\mathcal{O}}(X^H, \mathcal{O}) \to \kappa(\mathfrak{p}) \otimes_A \mathrm{Hom}_{\mathcal{O}}(Y^H, \mathcal{O})$$

is an isomorphism for a Zariski dense set of closed points $\mathfrak{p} \in \operatorname{Spec} A[1/p]$. Thus the kernel of the surjection (C.42) is torsion. The $A$-submodule $X^H$ is a direct summand of $X$, since the pro-order of $H$ is prime-to-$p$, and consequently $\operatorname{Hom}_{\mathcal{O}}(X^H, \mathcal{O})$ is a direct summand of $\widetilde{X}$. Thus $\operatorname{Hom}_{\mathcal{O}}(X^H, \mathcal{O})$ is also torsion free as an $A$-module, and hence so are all of its submodules. In particular, the kernel of (C.42) is simultaneously torsion and torsion free as an $A$-module, and so must vanish. Consequently, the surjection (C.42) is in fact an isomorphism, and thus $Y^H = X^H$. Taking $H$ to be arbitrarily small, we conclude that in fact $Y = X$, as claimed. □

C.43. **Definition.** If $V$ is any $A$-module, and if $X$ is a coadmissible smooth $A[\Gamma]$-module, then we write

$$V \overset{\curlywedge}{\otimes}_A X := \varinjlim_{H} V \hat{\otimes}_A X^H,$$

where $H$ runs over the collection of open subgroups of $\Gamma$, and $V \hat{\otimes}_A X^H$ denotes the $\varpi$-adic completion of the tensor product $V \otimes_A X^H$.

The formation of $V \overset{\curlywedge}{\otimes}_A X$ is evidently functorial in both $V$ and $X$, and the $\Gamma$-action on $X$ induces a corresponding $\Gamma$-action on $V \overset{\curlywedge}{\otimes}_A X$. In the remainder of this appendix, we prove some simple lemmas involving this functor.

C.44. **Lemma.** *If $V$ is an orthonormalizable $A$-module, $X$ is a coadmissible $A[\Gamma]$-module, and $H$ is an open subgroup of $\Gamma$, then the natural map $V \hat{\otimes}_A X^H \to V \overset{\curlywedge}{\otimes}_A X$ is an embedding with saturated image.*

*Proof.* Let $H' \subset H$ be an inclusion of open subgroups. Since $X^H$ is evidently saturated in $X$, it is saturated in $X^{H'}$, and hence, by part 2 of Lemma C.15, the induced map $V \hat{\otimes}_A X^H \to V \hat{\otimes}_A X^{H'}$ is injective, with saturated image. Passing to the inductive limit over all $H'$, the lemma follows. □

C.45. **Lemma.** *If $V$ is an orthonormalizable $A$-module and $X$ is a coadmissible representation of $\Gamma$ over $A$, then there is a natural isomorphism*

$$V/\mathfrak{m}V \otimes_k (X/\varpi X)[\mathfrak{m}] \xrightarrow{\sim} \big((V \overset{\curlywedge}{\otimes}_A X)/\varpi(V \overset{\curlywedge}{\otimes}_A X)\big)[\mathfrak{m}].$$

*Proof.* Since $V \overset{\curlywedge}{\otimes}_A X = \varinjlim_{H} V \hat{\otimes} X^H$, where $H$ runs over the open subgroups of $\Gamma$, we see that

$$(V \overset{\curlywedge}{\otimes}_A X)/\varpi(V \overset{\curlywedge}{\otimes}_A X) \xrightarrow{\sim} \varinjlim_{H} (V \otimes_A X^H)/\varpi V \otimes_A X^H) \xrightarrow{\sim} V/\varpi V \otimes_{A/\varpi A} X/\varpi X$$

(the last isomorphism holding since tensor produces commute with inductive limits). We now apply the functor $(-)[\mathfrak{m}]$, which is isomorphic to the functor $\operatorname{Hom}_{A/\varpi A}(k, -)$, to the composite of these isomorphisms, to obtain the first in the sequence of isomorphisms

$$\big((V \overset{\curlywedge}{\otimes}_A X)/\varpi(V \overset{\curlywedge}{\otimes}_A X)\big)[\mathfrak{m}] \xrightarrow{\sim} (V/\varpi V \otimes_{A/\varpi A} X/\varpi X)[\mathfrak{m}]$$
$$\xrightarrow{\sim} V/\varpi V \otimes_{A/\varpi A} (X/\varpi X)[\mathfrak{m}] \xrightarrow{\sim} V/\mathfrak{m} \otimes_k (X/\varpi X)[\mathfrak{m}];$$

the second isomorphism is given by part 3 of Lemma B.6, which applies, since $V/\varpi V$ is orthonormalizable over $A/\varpi A$, by part 4 of the same lemma, while the third isomorphism is evident. The composite of this sequence of isomorphisms yields the required isomorphism. □

C.46. **Lemma.** *If $V$ is an orthonormalizable $A$-module, $X$ is a coadmissible representation of $\Gamma$ over $A$, and $W$ is an $\mathcal{O}$-flat $A$-module, then an $A$-linear map $V \overset{\curlywedge}{\otimes}_A X \to W$ is injective, with saturated image, if and only if the map $V/\mathfrak{m}V \otimes_k (X/\varpi X)[\mathfrak{m}] \to (W/\varpi W)[\mathfrak{m}]$ obtained by reducing it modulo $\varpi$ and passing to $\mathfrak{m}$-invariants* (*and then applying the isomorphism of Lemma* C.45) *is injective.*

*Proof.* A map $V \overset{\curlywedge}{\otimes}_A X \to W$ is injective, with saturated image, if and only if the induced map

$$(V \overset{\curlywedge}{\otimes}_A X)/\varpi(V \overset{\curlywedge}{\otimes}_A X) \to W/\varpi W \tag{C.47}$$

is injective. Now, as was noted in the proof of the preceding lemma, there is a natural isomorphism

$$(V/\varpi V) \otimes_A (X/\varpi X) \xrightarrow{\sim} (V \overset{\curlywedge}{\otimes}_A X)/\varpi(V \overset{\curlywedge}{\otimes}_A X).$$

Since each element of $(X/\varpi X)$ is annihilated by some power of $\mathfrak{m}$, the same is thus true of each element of the source of this isomorphism, and thus also of each element of its target. An easy argument then shows that (C.47) is injective if and only if the induced map on $\mathfrak{m}$-torsion submodules is injective. This proves the lemma. □

C.48. **Lemma.** *Let $Y \hookrightarrow X$ be an $A$-linear, $\Gamma$-equivariant embedding of coadmissible $A[\Gamma]$-modules. If $V$ is an orthonormalizable $A$-module, then the induced map*

$$V \overset{\curlywedge}{\otimes}_A Y \to V \overset{\curlywedge}{\otimes}_A X \tag{C.49}$$

*is again an embedding.*

*Proof.* Write $Y \xrightarrow{\sim} \varinjlim_H Y^H$, $X \xrightarrow{\sim} \varinjlim_H X^H$, where $H$ runs over the open subgroups of $\Gamma$. The map (C.49) is the inductive limit of the maps $V \hat{\otimes}_A Y \to V \hat{\otimes}_A X$, which are injective, by Lemma C.15. Thus (C.49) is also injective, as claimed. □

C.50. **Lemma.** *Let $H$ be a compact open subgroup of $\Gamma$ whose pro-order is prime-to-$p$. If $V$ is a $\varpi$-adically complete $A$-module and if $X$ is a coadmissible smooth $A[\Gamma]$-module, then the induced map $V \hat{\otimes} X^H \to (V \overset{\curlywedge}{\otimes} X)^H$ is an isomorphism.*

*Proof.* We may write $X \xrightarrow{\sim} \varinjlim_N X^N$, where $N$ runs over normal open subgroups of $H$, and that $V \overset{\curlywedge}{\otimes}_A X \xrightarrow{\sim} \varinjlim_N V \hat{\otimes}_A X^N$. Passing to the $H$-invariants, we find that

$$(V \overset{\curlywedge}{\otimes}_A X)^H \xrightarrow{\sim} \varinjlim_N (V \hat{\otimes}_A X^N)^H \xrightarrow{\sim} \varinjlim_N V \hat{\otimes}_A X^H \xrightarrow{\sim} V \hat{\otimes}_A X^H,$$

as required. □

C.51. **Lemma.** *Let $Y \to X$ be an $A$-linear, $\Gamma$-equivariant map between coadmissible smooth $A[\Gamma]$-modules, and let $V$ be an orthonormalizable $A$-module. If the induced map $V \overset{\curlywedge}{\otimes}_A Y \to V \overset{\curlywedge}{\otimes}_A X$ is an isomorphism, then the same is true of the given map $Y \to X$.*

*Proof.* Let $H$ be a compact open subgroup of $\Gamma$ whose pro-order is prime-to-$p$. Lemma C.50 shows that the map $V \hat{\otimes}_A Y^H \to V \hat{\otimes}_A X^H$ is an isomorphism. Part 2 of Lemma C.18 then shows that the map $Y^H \to X^H$ is an isomorphism. Passing to the inductive limit over all $H$, we find that $Y \xrightarrow{\sim} X$, as claimed. □

C.52. **Lemma.** *Let $V$ be an orthonormalizable $A$-module, let $Y \hookrightarrow X$ be an embedding of coadmissible $A[\Gamma]$-modules, let $W$ be an $A$-module, and suppose give an $A$-linear map $V\overset{\curlywedge}{\otimes}_A X \to W$. If the composite*

$$V\overset{\curlywedge}{\otimes}_A Y \to V\overset{\curlywedge}{\otimes}_A X \to W$$

*has saturated image, then the embedding $Y \hookrightarrow X$ has saturated image.*

*Proof.* It suffices to show that the induced embedding $Y^H \hookrightarrow X^H$ has a saturated image, for each open subgroup $H$ of $\Gamma$. Now Lemma C.44 shows that the natural map $V\hat{\otimes}_A Y^H \to V\overset{\curlywedge}{\otimes}_A Y$ is an embedding, with a saturated image. Thus so is the composite

$$V\hat{\otimes}_A Y^H \to V\hat{\otimes}_A X^H \to V\overset{\curlywedge}{\otimes}_A X \to W,$$

and hence so is the first of these maps. Part 1 of Lemma C.18 then implies that the embedding $Y^H \hookrightarrow X^H$ has a saturated image, as required. □

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