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p-ADIC FAMILIES OF MODULAR FORMS
[after Hida, Coleman, and Mazur]

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INTRODUCTION

The theory of p-adic families of modular forms grew out of two highly related traditions in the arithmetic theory of modular forms: the theory of congruences of modular forms (which dates back to work of Ramanujan) and the (more recent) theory of Galois representations attached to modular forms. The first example of a p-adic family of modular forms was the Eisenstein family, considered by Serre in [37]. This is a family of q-expansions, parametrized by the weight k, whose coefficients are p-adically continuous functions of k. Serre's immediate goal in studying this family was to obtain an understanding of the possible congruences between the q-expansion coefficients of modular forms in different weights, especially of the constant terms, since such congruences lead to congruences between special values of ζ-functions.

The papers [22, 23] led to a decisive shift in the theory, placing it at the centre of the arithmetic theory of modular forms. In these papers, Hida constructed p-adic families of cuspforms, varying continuously with the weight k, which were also simultaneous eigenforms for the Hecke operators. Thus, in light of the known construction of Galois representations attached to Hecke eigenforms, one found that associated to these p-adic families of cuspidal eigenforms there were corresponding p-adic families of p-adic Galois representations. The existence of such families led Mazur to develop his general theory of deformations of Galois representations [31], which in turn inspired further developments [45, 43].

Hida's constructions had a certain limitation: if f is a Hecke eigenform of weight $k \geq 1$ and level N prime to p, then f appears in a Hida family if and only if (at least) one of the roots of the pth Hecke polynomial of f is of slope zero (i.e. a p-adic unit). This restriction was removed by the work of Coleman and Mazur [10], who constructed p-adic analytic (more precisely, rigid analytic) curves of eigenforms containing any such form f, whether or not its pth Hecke polynomial admits a unit root; these are the so-called *eigencurves*.

The eigencurves are fundamentally analytic objects. One can also ask whether there is an algebraic family (or more precisely, a scheme) that parametrizes all the f as above, regardless of the slopes of the roots of the pth Hecke polynomial. Indeed, there is such an object; all the eigenforms f (of arbitrary weight but some fixed level N) are parametrized by the $\overline{\mathbb{Z}}_p$ points of $\operatorname{Spec}\mathbb{T}(N)$, where $\mathbb{T}(N)$ is the p-adic Hecke algebra of level N. These points are no longer parametrized by weight; indeed, $\operatorname{Spec}\mathbb{T}(N)$ is (at least conjecturally) of relative dimension three over $\operatorname{Spec}\mathbb{Z}_p$. It is conjectured that every continuous, two-dimensional, semi-simple odd p-adic Galois representation of $G_{\mathbb{Q}}$ that is unramified outside finitely many primes corresponds to a point of $\operatorname{Spec}\mathbb{T}(N)$ for some appropriate value of N. This is one of the main motivations for the study of the families $\operatorname{Spec}\mathbb{T}(N)$, and the related p-adic families of eigenforms constructed by Hida and Coleman–Mazur.

In Section 1 of this exposé we recall the basic theory of modular forms, Hecke operators, and the Galois representations associated to Hecke eigenforms. In Section 2, we outline the definitions and basic results and conjectures regarding the p-adic Hecke algebras $\mathbb{T}(N)$, and the families of Hida and Coleman–Mazur. We focus more on systems of Hecke eigenvalues attached to eigenforms, rather than on the eigenforms themselves. This is in keeping with our focus on the relationship with Galois representations (although it takes us somewhat far in spirit from the concrete viewpoint of [37]).

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0.1. Notation

As usual $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers, and $\mathbb{Z}$ denotes the ring of integers. For any prime p, we let $\mathbb{Z}_p$ denote the ring of p-adic integers, and $\mathbb{Q}_p$ denote the field of p-adic numbers.

We let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\overline{\mathbb{Z}}$ denote the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. For each prime p, we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and let $\overline{\mathbb{Z}}_p$ denote the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$. We also fix an embedding $\imath_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. This restricts to an embedding $\overline{\mathbb{Z}} \hookrightarrow \overline{\mathbb{Z}}_p$. We write $\overline{\mathbb{F}}_p$ to denote the residue field of $\overline{\mathbb{Z}}_p$. It is an algebraic closure of the field $\mathbb{F}_p$ of p elements. We let $\operatorname{ord}_p : \overline{\mathbb{Q}}_p \to \mathbb{Z} \cup \{\infty\}$ denote the p-adic valuation, normalized so that $\operatorname{ord}_p(p) = 1$. If $x \in \overline{\mathbb{Q}}_p$, then $\operatorname{ord}_p(x)$ is also called the *slope* of x. (Thus x has finite slope if and only if $x \neq 0$, while x has slope zero if and only if $x \in \overline{\mathbb{Z}}_p^{\times}$.)

1. MODULAR FORMS, HECKE ALGEBRAS, AND GALOIS REPRESENTATIONS

1.1. Modular forms

Let

$$\mathcal{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$$

denote the complex upper half-plane. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ in the usual way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Let $\mathcal{O}(\mathcal{H})$ denote the space of holomorphic functions on $\mathcal{H}$. If k is an integer, then we define the weight k-action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{O}(\mathcal{H})$ as follows:

$$(f\,|_k\,\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau),$$

for $f \in \mathcal{O}(\mathcal{H})$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$; as the notation indicates, this is a right action.

If $N \geq 1$, define

$$\Gamma_1(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \,|\, \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\right\}.$$

Definition 1.1. — *A modular form (resp. cuspform) of weight k and level N is a holomorphic function $f \in \mathcal{O}(\mathcal{H})$ that is invariant under the weight k-action of $\Gamma_1(N)$, and for which*

$$\lim_{y\to\infty} (f\,|_k\,\gamma)(iy) \tag{1}$$

exists and is finite (resp. vanishes) for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We let $\mathcal{M}_k(N)$ (resp. $\mathcal{S}_k(N)$) denote the space of modular forms (resp. cuspforms) of weight k and level N.

Remark 1.2. — If $f \in \mathcal{O}(H)$ is invariant under the weight k-action of $\Gamma_1(N)$, then, in order to check if f is a modular form or a cuspform, it suffices to study the limit (1) for finitely many $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (namely, for a set of coset representatives for $\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbb{Z})$).

Remark 1.3. — If f is a modular form of weight k and level N, then, applying the invariance property of f to the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1(N)$, one finds that $f(\tau+1) = f(\tau)$. We may thus expand the function $f(\tau)$ as a Fourier series

$$f(\tau) := \sum_{n=-\infty}^{\infty} c_n(f) q^n,$$

where $q := \exp(2\pi i\tau)$. Condition (1), with $\gamma = 1$, then shows that $c_n(f) = 0$ for $n < 0$ (resp. for $n \leq 0$ if f is a cuspform). We refer to this Fourier series as the q-expansion of f.

Clearly $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$ are vector subspaces of $\mathcal{O}(\mathcal{H})$. In fact they are also finite dimensional. (See [39] for a discussion of this and other basic facts concerning modular forms.)

Example 1.4. — If $k < 0$, then $\mathcal{M}_k(N) = 0$. When $k = 0$, the space $\mathcal{M}_0(N)$ consists simply of the constant functions on $\mathcal{H}$ (and so $\mathcal{S}_0(N) = 0$). To avoid these trivial cases, we will typically assume that $k \geq 1$ in all that follows. As k increases, the dimensions of both $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$ grow essentially linearly in k (with the exception that $\mathcal{M}_k(N) = 0$ if $N = 1$ or 2 and k is odd).

Example 1.5. — The simplest examples of modular forms of positive weight are the Eisenstein series $E_k \in \mathcal{M}_k(1)$. These are defined for even $k \geq 4$. (It is easily shown that $\mathcal{M}_k(1)$ vanishes if k is odd or $0 < k < 4$.) The q-expansion of E_k is given by the following formula:

$$E_k(\tau) = \frac{-B_k}{2k} + \sum_{n=1}^{\infty}(\sum_{d|n} d^{k-1})q^n,$$

where B_k is the kth Bernoulli number.

There is a direct sum decomposition

$$\mathcal{M}_k(1) = \mathbb{C}E_k \oplus \mathcal{S}_k(1).$$

More generally, for any N, we may decompose $\mathcal{M}_k(N)$ into the direct sum of a space of Eisenstein series (typically of dimension greater than one when $N > 1$) and the space of cuspforms. (See Example 1.18 below.)

1.2. Hecke operators

Fix integers $k \geq 1$ and $N \geq 1$. Write

$$\Gamma_0(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \,|\, \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod N\right\}.$$

Note that $\Gamma_0(N)$ contains $\Gamma_1(N)$ as a normal subgroup, and that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \bmod N$$

induces an isomorphism

$$\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times. \tag{2}$$

A simple computation, using the normality of $\Gamma_1(N)$ in $\Gamma_0(N)$, shows that the weight k-action of $\Gamma_0(N)$ preserves $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$. When restricted to these spaces, this action obviously factors through the quotient $\Gamma_0(N)/\Gamma_1(N)$, and hence, via the isomorphism (2), we obtain an action of the group $(\mathbb{Z}/N\mathbb{Z})^\times$ on $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$. If $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, then we denote the corresponding automorphism of $\mathcal{M}_k(N)$ by $\langle d \rangle$. (These operators are sometimes referred to as the diamond operators.)

Remark 1.6. — We note a simple but important identity for the action of the diamond operator $\langle -1 \rangle$, namely

$$\langle -1 \rangle f = (-1)^k f, \tag{3}$$

for any $f \in \mathcal{M}_k(N)$. This is easily verified by considering the weight k-action of the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma_0(N)$ on f.

Definition 1.7. — *If ℓ is a prime not dividing N, then we define the automorphism S_ℓ of $\mathcal{M}_k(N)$ via the formula*

$$S_\ell = \langle \ell \rangle \ell^{k-2}.$$

Since the diamond operators leave $\mathcal{S}_k(N)$ invariant, so do the operators S_ℓ. In fact, although it is traditional to single out the operators S_ℓ as defined above, it is the operators $\ell S_\ell = \langle \ell \rangle \ell^{k-1}$ that will be more important for us, as we see already in the next definition.

Definition 1.8. — *If ℓ is a prime not dividing N, then we define the endomorphism T_ℓ of $\mathcal{M}_k(N)$ via the formula*

$$(T_\ell f)(\tau) = \sum_{n=0}^{\infty} c_{n\ell}(f) q^n + \sum_{n=0}^{\infty} \ell c_n(S_\ell f) q^{n\ell}. \tag{4}$$

Remark 1.9. — It is not immediately obvious that T_ℓ, which we have defined simply by its effect on q-expansions, actually preserves the space $\mathcal{M}_k(N)$. In fact T_ℓ can be thought of as a certain double coset operator, corresponding to the double coset $\mathrm{GL}_2(\mathbb{Z}_\ell) \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathrm{GL}_2(\mathbb{Z}_\ell)$ (see e.g. [39, Ch. 3]). From this point of view, it is easy to verify that it preserves the space $\mathcal{M}_k(N)$, as well as the subspace $\mathcal{S}_k(N)$ of cuspforms.

The operator S_ℓ also has a double coset interpretation; it corresponds to the double coset $\mathrm{GL}_2(\mathbb{Z}_\ell) \left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right) \mathrm{GL}_2(\mathbb{Z}_\ell)$. This is one reason to consider S_ℓ as a primary object, rather than the diamond operator $\langle \ell \rangle$.

Definition 1.10. — *We let $\mathbb{T}_k(N)$, or simply $\mathbb{T}_k$ when the level N is understood, denote the $\mathbb{Z}$-subalgebra of $\mathrm{End}\big(\mathcal{M}_k(N)\big)$ generated by the operators ℓS_ℓ and T_ℓ as ℓ ranges over the primes not dividing N. The algebra $\mathbb{T}_k(N)$ is called the Hecke algebra (for the given weight k and level N).*

Remark 1.11. — Following [39, Ch. 3], one can extend Definition 1.8 and define Hecke operators T_m acting on $\mathcal{M}_k(N)$ for any positive integer m prime to N. The algebra $\mathbb{T}_k(N)$ defined above then coincides with the $\mathbb{Z}$-algebra of endomorphisms of $\mathcal{M}_k(N)$ generated by the collection of these operators T_m.

The following result encapsulates the basic properties of the algebra $\mathbb{T}_k$, and of its action on $\mathcal{M}_k(N)$.

Proposition 1.12. — *The algebra $\mathbb{T}_k$ is commutative, reduced, and free of finite rank over $\mathbb{Z}$. Furthermore, the tensor product $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{T}_k$ acts faithfully on $\mathcal{M}_k(N)$.*

Remark 1.13. — The commutativity part of the statement is not difficult to verify; for example, it is easily checked using the description of the Hecke operators in terms of double cosets. The additional properties of $\mathbb{T}_k$ are then equivalent to the following statements about the eigenspaces and eigenvalues of the Hecke operators:

1. Every eigenvalue of any of the Hecke operators is an algebraic integer. (Here one sees the importance, when $k = 1$, of taking ℓS_ℓ rather than S_ℓ in the definition of $\mathbb{T}_k$, so as to avoid introducing denominators.)
2. The systems of simultaneous eigenvalues for the action of the Hecke operators on $\mathcal{M}_k(N)$ (which are collections of algebraic integers, by 1) are closed under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

3. The space $\mathcal{M}_k(N)$ decomposes as a direct sum of simultaneous eigenspaces for the Hecke operators.

Definition 1.14. — *We say that $f \in \mathcal{M}_k(N)$ is a Hecke eigenform if it is a simultaneous eigenvector for the Hecke operators ℓS_ℓ and T_ℓ (where ℓ ranges over all primes not dividing N), or equivalently, if there is a ring homomorphism $\lambda : \mathbb{T}_k \to \mathbb{C}$ such that $Tf = \lambda(T)f$ for all $T \in \mathbb{T}_k$.*

We refer to a homomorphism $\lambda : \mathbb{T}_k \to \mathbb{C}$ as a system of Hecke eigenvalues. (Any such λ is the system of Hecke eigenvalues attached to some Hecke eigenform. Also, according to the preceding remark, any such λ factors through the ring of algebraic integers $\overline{\mathbb{Z}}$ in $\overline{\mathbb{Q}}$.) If λ is a system of Hecke eigenvalues, then we write $\mathcal{M}_k(N)[\lambda]$ to denote the corresponding subspace of Hecke eigenforms.

As already noted in the preceding remark, the space $\mathcal{M}_k(N)$ admits the direct sum decomposition

$$\mathcal{M}_k(N) = \bigoplus_{\lambda} \mathcal{M}_k(N)[\lambda],$$

where the direct sum is taken over all systems of Hecke eigenvalues.

Remark 1.15. — The formula (4) shows that if λ is a system of Hecke eigenvalues, then the q-expansion of a Hecke eigenform $f \in \mathcal{M}_k(N)[\lambda]$, and hence the eigenform f itself, is to a large extent determined by the system of Hecke eigenvalues λ. For example, if $N = 1$, then the group of diamond operators is trivial, and so $\ell S_\ell f = \ell^{k-1} f$. Formula (4) then shows that

$$\lambda(T_\ell)c_n(f) = c_{n\ell}(f) + c_{n/\ell}(f)\ell^{k-1}$$

for every prime number ℓ (where we set $c_{n/\ell} = 0$ if $\ell \nmid n$). Thus the Fourier coefficients $c_n(f)$ $(n \geq 1)$ are determined recursively by the single coefficient $c_1(f)$, and so f is determined up to a scalar by its associated system of Hecke eigenvalues. In particular, the λ-eigenspace in $\mathcal{M}_k(1)$ is one-dimensional.(1) If $N > 1$, then we find that f is determined by λ, together with the Fourier coefficients $c_m(f)$, for those positive integers m divisible only by primes dividing N. Thus f need not be uniquely determined (up to a scalar) by λ, and the λ-eigenspace in $\mathcal{M}_k(N)$ can be of dimension greater than one. However, the structure of this eigenspace is well-understood, either using the theory of so-called oldforms and newforms as in [1], or in terms of the action of $\mathrm{GL}_2(\mathbb{A})$ on the space of modular forms of weight k [25, 6]. We do not recall the details here, since they will not be important for us. As we will explain in the following subsection, our

(1)A slight amount of caution is required here, because $c_0(f)$ is not directly determined by the $c_n(f)$ for $n \geq 1$. However, since $k \geq 1$, then in fact $c_0(f)$ is so determined, as one easily sees, since a constant function cannot be modular of weight $k > 0$. As Serre notes [37, Rem. 2), p. 221], one can directly determine $c_0(f)$ from the $c_n(f)$ for $n \geq 1$ as follows: $-c_0(f)$ is the value at $s = 0$ of the meromorphic function defined by analytic continuation of the Dirichlet series $\sum_{n=1}^{\infty} c_n(f)n^{-s}$.

attention will be focussed on the systems of eigenvalues λ themselves, rather than on the associated Hecke eigenforms.

Remark 1.16. — Given a system of Hecke eigenvalues λ appearing in $\mathcal{M}_k(N)$, it follows from the definition of the operators S_ℓ that there is a $\overline{\mathbb{Q}}^\times$-valued character ε of $(\mathbb{Z}/N\mathbb{Z})^\times$ such that $\lambda(\ell S_\ell) = \varepsilon(\ell)\ell^{k-1}$. Thus we may recover the value of the weight k from the system of eigenvalues λ. Indeed, if ℓ is any prime not dividing N, then $k = (\log_\ell |\lambda(\ell S_\ell)|) + 1$.

Example 1.17. — If $k \geq 4$ is even, then the Eisenstein series $E_k \in \mathcal{M}_k(1)$ is a Hecke eigenform. The corresponding system of Hecke eigenvalues λ is given by

$$\lambda(\ell S_\ell) = \ell^{k-1}, \quad \lambda(T_\ell) = 1 + \ell^{k-1}.$$

(Here ℓ is an arbitrary prime, since we are in the case $N = 1$).

Example 1.18. — Let $\psi_1 : (\mathbb{Z}/M_1\mathbb{Z})^\times \to \mathbb{C}^\times$ and $\psi_2 : (\mathbb{Z}/M_2\mathbb{Z})^\times \to \mathbb{C}^\times$ be characters, and let $k \geq 1$ (unless $M_1 = M_2 = 1$, in which case we require that $k \geq 4$) be chosen so that $\psi_1(-1)\psi_2(-1)(-1)^k = 1$. Then the following system of Hecke eigenvalues, which we denote by $\lambda_{\psi_1,\psi_2,k}$, appears in $\mathcal{M}_k(M_1M_2)$:

$$\lambda_{\psi_1,\psi_2,k}(\ell S_\ell) = \psi_1(\ell)\psi_2(\ell)\ell^{k-1}, \quad \lambda_{\psi_1,\psi_2,k}(T_\ell) = \psi_1(\ell) + \psi_2(\ell)\ell^{k-1}.$$

In the case when $M_1 = M_2 = 1$, we obtain the systems of Hecke eigenvalues associated to the Eisenstein series E_k, as considered in the preceding example. In general, we refer to such a system of Hecke eigenvalues as an Eisenstein system of eigenvalues.

If we write

$$\mathcal{E}_k(N) = \bigoplus_{\lambda \text{ Eisenstein}} \mathcal{M}_k(N)[\lambda],$$

where the sum ranges over all Eisenstein systems of Hecke eigenvalues for which $M_1M_2 = N$, then we refer to modular forms $f \in \mathcal{E}_k(N)$ as Eisenstein series. There is a direct sum decomposition

$$\mathcal{M}_k(N) = \mathcal{E}_k(N) \oplus \mathcal{S}_k(N).$$

Unlike the Eisenstein systems of eigenvalues considered in Example 1.18, the systems of eigenvalues appearing in the spaces of cuspforms do not admit an elementary description. As we will see in the following subsection, they correspond to certain Galois representations.

Example 1.19. — We close this subsection with a careful presentation of the preceding concepts in the case $N = 1$ and $k = 12$. In this case

$$\mathcal{M}_{12}(1) = \mathcal{E}_{12}(1) \oplus \mathcal{S}_{12}(1),$$

where $\mathcal{E}_{12}(1)$ is one-dimensional, spanned by

$$E_{12} = \frac{691}{32760} + \sum_{n=1}^{\infty}(\sum_{d|n} d^{11})q^n = \frac{691}{32760} + q + 2049q^2 + 177148q^3 + \cdots,$$

and $\mathcal{S}_{12}(1)$ is also one-dimensional, spanned by Ramanujan's famous cuspform

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n = q - 24q^2 + 252q^3 + \cdots .$$

(Here $\tau(n) = c_n(\Delta)$ is by definition the nth Fourier coefficient of Δ.) Each of these modular forms is a Hecke eigenform, and correspondingly $\mathbb{T}_{12}$ admits two systems of Hecke eigenvalues.

If we write λ_1 (resp. λ_2) to denote the system of Hecke eigenvalues attached to E_{12} (resp. Δ), then

$$\lambda_1 \times \lambda_2 : \mathbb{T}_{12} \hookrightarrow \mathbb{Z} \times \mathbb{Z}. \tag{5}$$

Note that since each of these eigenforms has been normalized so that $c_1 = 1$, we may read off the corresponding systems of Hecke eigenvalues from the Fourier coefficients, as in Remark 1.15. The product $\lambda_1 \times \lambda_2$ provides an embedding $\lambda_1 \times \lambda_2 : \mathbb{T}_{12} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$. It was first observed by Ramanujan that this embedding is not an isomorphism. Indeed, Ramanujan showed that

$$\tau(n) \equiv \sum_{d|n} d^{11} \bmod 691$$

for every natural number n, or equivalently,

$$\lambda_1(T_\ell) \equiv \lambda_2(T_\ell) \bmod 691$$

for each prime ℓ. On the other hand, it is easily verified (just by considering the cases when $\ell = 2$ and 3) that no such congruence holds modulo any higher power of 691, nor modulo any other prime. Thus (5) induces an isomorphism

$$\mathbb{T}_{12} \xrightarrow{\sim} \{(u, v) \in \mathbb{Z} \times \mathbb{Z} \,|\, u \equiv v \bmod 691\}.$$

If p is a prime and $p \neq 691$, then $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_{12} \xrightarrow{\sim} \mathbb{Z}_p \times \mathbb{Z}_p$; this reflects the fact that the distinct systems of eigenvalues λ_1 and λ_2 remain distinct when reduced modulo p. On the other hand, the tensor product $\mathbb{Z}_{691} \otimes_{\mathbb{Z}} \mathbb{T}_{12}$ does not factor as a product in any non-trivial way; rather, it is a local ring, reflecting the congruence of λ_1 and λ_2 modulo 691.

1.3. Galois representations

As in the preceding section, fix integers $k \geq 1$ and $N \geq 1$. From a certain point of view, it is the systems of Hecke eigenvalues appearing in $\mathcal{M}_k(N)$ that are of the greatest interest, rather than the modular forms, or even the Hecke eigenforms, themselves. This is because they give rise to Galois representations, as we now recall.

Choose a prime number p. If λ is a system of Hecke eigenvalues appearing in $\mathcal{M}_k(N)$, then since λ takes values in the ring $\overline{\mathbb{Z}}$ of algebraic integers, we may compose it with our chosen embedding $\imath_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and so regard λ as taking values in $\overline{\mathbb{Z}}_p$. For the remainder of this subsection, we regard all systems of Hecke eigenvalues as being $\overline{\mathbb{Z}}_p$-valued. If $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$ is a system of Hecke eigenvalues, then we let $\overline{\lambda} : \mathbb{T}_k \to \overline{\mathbb{F}}_p$ be the homomorphism obtained by composing λ with the map $\overline{\mathbb{Z}}_p \to \overline{\mathbb{F}}_p$ given by reducing modulo the maximal ideal of $\overline{\mathbb{Z}}_p$.

Let Σ denote the (finite) set of primes dividing Np, let $\mathbb{Q}_\Sigma$ denote the maximal algebraic extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ that is unramified outside of the primes in Σ, and write $G_{\mathbb{Q},\Sigma} := \mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$. Recall that if ℓ is a prime not in Σ, then attached to ℓ is a Frobenius element $\mathrm{Frob}_\ell \in G_{\mathbb{Q},\Sigma}$, well-defined up to conjugacy, with the property that there is a prime ideal $\mathfrak{l}$ lying over ℓ in the ring of algebraic integers in $\mathbb{Q}_\Sigma$ that is preserved by Frob_ℓ, such that for any algebraic integer $x \in \mathbb{Q}_\Sigma$, $\mathrm{Frob}_\ell(x) \equiv x^\ell \bmod \mathfrak{l}$. The Čebotarev density theorem furthermore implies that the union of these conjugacy classes is dense in $G_{\mathbb{Q},\Sigma}$.

If M is any integer divisible only by primes dividing Np, and if ζ_M denotes a primitive Mth root of unity, then $\zeta_M \in \mathbb{Q}_\Sigma$, and so there is a group homomorphism

$$\chi_M : G_{\mathbb{Q},\Sigma} \to (\mathbb{Z}/M\mathbb{Z})^\times$$

describing the action of the elements of $G_{\mathbb{Q},\Sigma}$ on ζ_M, namely, for any $\sigma \in G_{\mathbb{Q},\Sigma}$, we have

$$\sigma(\zeta_M) = \zeta_M^{\chi_M(\sigma)}.$$

We refer to χ_M as the mod M cyclotomic character. It can also be characterized by the formula

$$\chi_M(\mathrm{Frob}_\ell) \equiv \ell \bmod M,$$

for any prime ℓ not dividing Np. Also, if $c \in G_{\mathbb{Q},\Sigma}$ denotes complex conjugation, then $\chi_M(c) = -1$.

We also define the p-adic cyclotomic character $\chi : G_{\mathbb{Q},\Sigma} \to \mathbb{Z}_p^\times$ to be the projective limit over n of the mod p^n-cyclotomic characters χ_{p^n}. Again, the character χ is characterized by the formula $\chi(\mathrm{Frob}_\ell) = \ell$ for any ℓ not dividing Np, and we also have that $\chi(c) = -1$.

The various cyclotomic characters give the basic examples of characters (i.e. one-dimensional representations) of the group $G_{\mathbb{Q},\Sigma}$. The following theorem shows that Hecke eigenforms are a source of two-dimensional representations of this group.

Theorem 1.20. — *There is a continuous, semi-simple representation*

$$\rho_\lambda : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p),$$

uniquely determined (*up to equivalence*) *by the condition that for each prime* $\ell \nmid Np$, *the matrix* $\rho_\lambda(\mathrm{Frob}_\ell)$ *has characteristic polynomial equal to* $X^2 - \lambda(T_\ell)X + \lambda(\ell S_\ell)$.

Remarks on the proof. — The uniqueness statement of the theorem is easily proved. Indeed, if ρ_1 and ρ_2 are two representations both satisfying the conditions of the theorem, then by assumption their characteristic polynomials agree on the set of elements Frob_ℓ, which by Čebotarev density are dense in $G_{\mathbb{Q},\Sigma}$. Since they are continuous, their characteristic polynomials then agree on all elements of $G_{\mathbb{Q},\Sigma}$. It follows that ρ_1 and ρ_2 are equivalent, as claimed, since a semi-simple finite-dimensional representation of a group is uniquely determined, up to equivalence, by its characteristic polynomials.

In the case when λ is an Eisenstein system of Hecke eigenvalues, the existence of ρ_λ is also easily proved; see Example 1.24 below. On the other hand, if λ is a system of eigenvalues attached to a cuspform, then the construction of ρ_λ is much less trivial. Its construction is due to Eichler, Shimura, and Igusa [13, 38, 24] (in the case $k = 2$), to Deligne [11] (for $k > 2$), and to Deligne and Serre [12] (for $k = 1$). □

It is useful to give a name to the characteristic polynomials appearing in Theorem 1.20.

DEFINITION 1.21. — *If $\lambda : \mathbb{T}_k(N) \to \mathbb{C}$ is a system of Hecke eigenvalues, then for each prime $\ell \nmid N$, we define the ℓth Hecke polynomial of λ to be the polynomial*

$$X^2 - \lambda(T_\ell)X + \lambda(\ell S_\ell).$$

Remark 1.22. — Since $G_{\mathbb{Q},\Sigma}$ is profinite, the representation ρ_λ may be conjugated so as to take values in $\mathrm{GL}_2(\overline{\mathbb{Z}}_p)$, and we let ρ_λ° denote such a $\mathrm{GL}_2(\overline{\mathbb{Z}}_p)$-valued representation underlying ρ_λ. The $\mathrm{GL}_2(\overline{\mathbb{Z}}_p)$-valued representation ρ_λ° is not always uniquely determined up to equivalence by λ. However, if we let $\overline{\rho}_{\overline{\lambda}}$ denote the semi-simplification of the representation $G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ obtained by reducing ρ_λ° modulo the maximal ideal of $\overline{\mathbb{Z}}_p$, then $\overline{\rho}_\lambda$ is uniquely determined, up to equivalence, by λ, and in fact, even by $\overline{\lambda}$ (as the notation suggests). Indeed, $\overline{\rho}_{\overline{\lambda}}$ is uniquely characterized, up to equivalence, by the condition that for each prime $\ell \nmid Np$, the matrix $\overline{\rho}_{\overline{\lambda}}(\mathrm{Frob}_\ell)$ has characteristic polynomial equal to $X^2 - \overline{\lambda}(T_\ell)X + \overline{\lambda}(\ell S_\ell)$. (The proof of the uniqueness is identical to that given in the proof of Theorem 1.20.)

Remark 1.23. — As in Remark 1.16, write $\lambda(\ell S_\ell) = \varepsilon(\ell)\ell^{k-1}$ for some $\overline{\mathbb{Q}}^\times$-valued character ε of $(\mathbb{Z}/N\mathbb{Z})^\times$. We may compose ε with the mod N cyclotomic character χ_N to obtain a $\overline{\mathbb{Q}}^\times$-valued character of $G_{\mathbb{Q},\Sigma}$, which we regard as being $\overline{\mathbb{Q}}_p^\times$-valued via our chosen embedding $\imath_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. It then follows from the condition on the determinant of $\rho(\mathrm{Frob}_\ell)$ in the statement of Theorem 1.20, together with Čebotarev density and the given relationship between $\lambda(\ell S_\ell)$ and $\varepsilon(\ell)$, that

$$\det \rho_\lambda := (\varepsilon \circ \chi_N)\chi^{k-1},$$

where as above χ denotes the p-adic cyclotomic character.

In particular, if $c \in G_{\mathbb{Q},\Sigma}$ denotes complex conjugation, then one computes that

$$\det \rho_\lambda(c) = \varepsilon(-1)(-1)^{k-1} = -1$$

(the last equality following from (3)). One says that ρ_λ is *odd*. Similarly, the representation $\overline{\rho}_{\overline{\lambda}}$ is odd.

Example 1.24. — If $\lambda_{\psi_1,\psi_2,k}$ is an Eisenstein system of Hecke eigenvalues attached to characters $\psi_i : (\mathbb{Z}/M_i\mathbb{Z})^\times \to \mathbb{C}^\times$ and the weight k, as in Example 1.18, then it is easy to write down a corresponding Galois representation $\rho_{\lambda_{\psi_1,\psi_2,k}}$ satisfying the conditions of Theorem 1.20; namely, we can take

$$\rho_{\lambda_{\psi_1,\psi_2,k}} = (\psi_1 \circ \chi_{M_1}) \oplus (\psi_2 \circ \chi_{M_2})\chi^{k-1}.$$

On the other hand, if λ arises from a cuspform, then ρ_λ does not admit a description in terms of characters. Indeed, one has the following result [35, Thm. 2.3].

PROPOSITION 1.25. — *If the system of Hecke eigenvalues λ is attached to a cuspform, then the representation ρ_λ associated to λ by Theorem 1.20 is irreducible.*

For any prime p, write $\mathbb{T}_k^{(p)}$ to denote the subalgebra of $\mathbb{T}_k$ generated by the elements ℓS_ℓ and T_ℓ for ℓ not dividing Np (i.e. we omit the Hecke operators at p). If $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$ is a system of Hecke eigenvalues, we write $\lambda^{(p)}$ to denote the restriction of λ to $\mathbb{T}_k^{(p)}$, and refer to $\lambda^{(p)}$ as the *p-deprived* system of Hecke eigenvalues associated to λ. Similarly, we let $\overline{\lambda}^{(p)} : \mathbb{T}_k^{(p)} \to \overline{\mathbb{F}}_p$ denote the restriction of $\overline{\lambda}$. The conditions on the Galois representation ρ_λ given in Theorem 1.20 evidently depend only on $\lambda^{(p)}$, and in fact we can use the existence of the Galois representations attached to λ to show that $\lambda^{(p)}$ already determines λ. Indeed, we have the following more general result.

PROPOSITION 1.26. — *If λ_1 and λ_2 are two systems of Hecke eigenvalues such that $\lambda_1(T_\ell) = \lambda_2(T_\ell)$ for all but finitely many primes ℓ not dividing N, then λ_1 and λ_2 coincide.*

Proof. — This is proved by the same argument used to establish the uniqueness claim of Theorem 1.20. Let q be some fixed prime not dividing N, and choose p to be distinct from q. Let ρ_{λ_1} and ρ_{λ_2} denote the Galois representations associated to λ_1 and λ_2 as in Theorem 1.20, regarded as representations over $\overline{\mathbb{Q}}_p$. Then ρ_{λ_1} and ρ_{λ_2} have the same traces on the elements Frob_ℓ, for all but finitely many ℓ. Čebotarev density implies that the set of elements Frob_ℓ (where ℓ ranges over all but finitely many primes not dividing Np) is dense in $G_{\mathbb{Q},\Sigma}$, and so, since ρ_{λ_1} and ρ_{λ_2} are continuous, we see that their traces coincide. Thus they have isomorphic semi-simplifications (since we are working over the field $\overline{\mathbb{Q}}_p$ of characteristic zero), and so their characteristic polynomials coincide on any element of $G_{\mathbb{Q},\Sigma}$. Applying this to Frob_q, we find that $\lambda_1(S_q) = \lambda_2(S_q)$ and that $\lambda_1(T_q) = \lambda_2(T_q)$. Since q was an arbitrary prime not dividing N, the proposition follows. □

The preceding proposition has the following technical corollary.

COROLLARY 1.27. — *The ring $\mathbb{T}_k^{(p)}$ has finite index in $\mathbb{T}_k$.*

Proof. — Since $\mathbb{T}_k$ is finite over $\mathbb{Z}$, it suffices to show that $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{T}_k^{(p)} \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{T}_k$. Equivalently, we must show that distinct systems of Hecke eigenvalues remain distinct after omitting the eigenvalues corresponding to the Hecke operators at p. This follows from the proposition. □

Remark 1.28. — The finite index of Corollary 1.27 can be greater than 1. For example, if $N = 23$, $k = 2$, and $p = 2$, then the index of $\mathbb{T}_2^{(2)}$ in $\mathbb{T}_2$ is equal to 2. (More precisely, $\mathbb{T}_2 \cong \mathbb{Z}[(1 + \sqrt{5})/2]$, while $\mathbb{T}_2^{(2)} \cong \mathbb{Z}[\sqrt{5}]$.)

Remark 1.29. — Corollary 1.27 (or better, its proof) shows that $\lambda \mapsto \lambda^{(p)}$ induces a bijection between the set of homomorphisms $\mathbb{T}_k \to \mathbb{C}$ and the set of homomorphisms $\mathbb{T}_k^{(p)} \to \mathbb{C}$, and hence between the set of homomorphisms $\mathbb{T}_k \to \overline{\mathbb{Z}}_p$ and the set of homomorphisms $\mathbb{T}_k^{(p)} \to \overline{\mathbb{Z}}_p$.

2. p-ADIC FAMILIES OF SYSTEMS OF HECKE EIGENVALUES

2.1. The p-adic Hecke algebra

Let N be a positive integer, and fix a prime p not dividing N. If $k \geq 1$ is a positive integer, then for each prime ℓ not dividing N we define the operators S_ℓ and T_ℓ on the direct sum $\bigoplus_{i=1}^k \mathcal{M}_i(N)$ in the obvious way: S_ℓ and T_ℓ act on each summand via the Hecke operator with the same name.

Definition 2.1. — *We let $\mathbb{T}_{\leq k}^{(p)}(N)$, or simply $\mathbb{T}_{\leq k}^{(p)}$ if the level N is understood, denote the $\mathbb{Z}$-algebra of endomorphisms of $\bigoplus_{i=1}^k \mathcal{M}_i(N)$ generated by the operators ℓS_ℓ and T_ℓ, as ℓ ranges over all primes not dividing Np.*

Since each operator S_ℓ and T_ℓ is determined by its action on each of the direct summands, there is a natural injection

$$\mathbb{T}_{\leq k}^{(p)} \hookrightarrow \prod_{i=1}^{k} \mathbb{T}_i. \tag{6}$$

Remark 2.2. — We could consider the analogous algebra in which we included the operators pS_p and T_p. However, for our later purposes, it is important to omit these operators from the algebra under consideration.

Proposition 2.3. — *The image of* (6) *has finite index in $\prod_{i=0}^k \mathbb{T}_i$.*

Proof. — Given that the source and target of (6) are both finite $\mathbb{Z}$-algebras, it suffices to show that (6) becomes an isomorphism after tensoring with $\mathbb{C}$ over $\mathbb{Z}$. This follows from the fact that the p-deprived systems of eigenvalues appearing in $\mathcal{M}_k(N)$ are distinct for different values of k, by Remarks 1.16 and 1.29. □

Example 2.4. — Take $N = 1$, $p = 2$, and $k = 6$. The spaces $M_i(1)$ for $1 \leq i \leq 6$ vanish unless $i = 4$ or 6, in which case they are one-dimensional, spanned by E_4 and E_6 respectively. Thus $\mathbb{T}_4 \xrightarrow{\sim} \mathbb{Z}$ and $\mathbb{T}_6 \xrightarrow{\sim} \mathbb{Z}$, and so (6) becomes in this case an embedding

$$\mathbb{T}_{\leq 6}^{(2)} \hookrightarrow \mathbb{Z} \times \mathbb{Z}. \tag{7}$$

Now

$$E_4 = \frac{1}{240} + \sum_{n=1}^{\infty} (\sum_{d|n} d^3) q^n,$$

while

$$E_6 = \frac{-1}{504} + \sum_{n=1}^{\infty} (\sum_{d|n} d^5) q^n.$$

One immediately checks that

$$1 + \ell^3 \equiv 1 + \ell^5 \bmod 12,$$

for all $\ell \neq 2$. Furthermore, no analogous congruence holds modulo any larger modulus, and thus the embedding (7) induces an isomorphism

$$\mathbb{T}^{(2)}_{\leq 6} \xrightarrow{\sim} \{(u, v) \in \mathbb{Z} \times \mathbb{Z} \,|\, u \equiv v \bmod 12\}.$$

A similar calculation shows that

$$\mathbb{T}^{(3)}_{\leq 6} \xrightarrow{\sim} \{(u, v) \in \mathbb{Z} \times \mathbb{Z} \,|\, u \equiv v \bmod 6\}.$$

These examples exhibit congruences similar to those discussed in Example 1.19, but involving congruences between systems of Hecke eigenvalues in different weights.

If $k' \geq k$, then

$$\bigoplus_{i=0}^{k} \mathcal{M}_i(N) \subset \bigoplus_{i=0}^{k'} \mathcal{M}_i(N),$$

and so restriction induces a surjection

$$\mathbb{T}^{(p)}_{\leq k'} \to \mathbb{T}^{(p)}_{\leq k}.$$

Tensoring this with $\mathbb{Z}_p$ over $\mathbb{Z}$, we obtain a surjection

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_{\leq k'} \to \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_{\leq k}. \tag{8}$$

Definition 2.5. — *The p-adic Hecke algebra* $\mathbb{T}(N)$*, or simply* $\mathbb{T}$ *if the level* N *is understood, is defined to be the projective limit*

$$\mathbb{T} := \varprojlim_{k} \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_{\leq k}, \tag{9}$$

where the transition maps are the maps (8)*.*

Remark 2.6. — Note that since any prime $\ell \neq p$ is invertible in $\mathbb{Z}_p$, the operator $S_\ell = \ell^{-1}(\ell S_\ell)$ lies in each of the algebras $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_k$, for each $\ell \nmid Np$, and so we may regard each of these algebras as being generated by the elements S_ℓ and T_ℓ ($\ell \nmid Np$), just as well as by ℓS_ℓ. Also, since the transition maps (8) take the elements S_ℓ and T_ℓ in the source to the elements S_ℓ and T_ℓ in the target, these elements give rise to well-defined elements S_ℓ and T_ℓ in the projective limit $\mathbb{T}$, for any prime $\ell \nmid Np$.

From the various embeddings (6), we obtain an embedding

$$\mathbb{T} \hookrightarrow \prod_{k \geq 1} \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_k.$$

The target of this embedding is a countable product of non-zero rings; in particular, it is not Noetherian. On the other hand, we have the following result regarding the source.

THEOREM 2.7. — *The ring $\mathbb{T}$ is a p-adically complete, Noetherian $\mathbb{Z}_p$-algebra, and is in fact the product of finitely many complete Noetherian local $\mathbb{Z}_p$-algebras.*

Remarks on the proof. — Each $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_k^{(p)}$ is a finite $\mathbb{Z}_p$-algebra, and so is a product of finitely many complete local finite $\mathbb{Z}_p$-algebras. It follows that $\mathbb{T}$ is p-adically complete, and is a product of a countable collection of complete local $\mathbb{Z}_p$-algebras. The fact that this product involves only finitely many local algebras is not formal; it is equivalent to a statement about $\overline{\mathbb{F}}_p$-valued systems of Hecke eigenvalues that is the subject of Proposition 2.8 below. The fact that these local factors are Noetherian is also not formal; it is proved via a consideration of relation between the ring $\mathbb{T}$ and Galois representations, as discussed in the following subsection. (More precisely, each local component of $\mathbb{T}$ is canonically the quotient of a certain Galois pseudo-deformation ring, and hence is Noetherian; see [30, §1.4] for a discussion of the latter, and in particular Lemma 1.4.2 for a proof of the Noetherianness of pseudo-deformation rings.(2)) □

Informally speaking, this theorem can be thought of as showing that the phenomenon exhibited in Example 2.4 is typical: as k grows, the power of p dividing the index of the image of (6) in its target grows progressively larger, reflecting the existence of many congruences modulo powers of p between systems of eigenvalues appearing in various weights.

We present one concrete manifestation of this abundance of congruences in the following proposition (due to Jochnowitz [26], generalizing an argument of Serre in the case $N = 1$), which is an important ingredient in the proof of Theorem 2.7. Indeed, its statement is a straightforward reformulation of the claim that $\mathbb{T}$ has only finitely many maximal ideals. In order to state the proposition, we introduce additional notation. Suppose given a p-deprived system of Hecke eigenvalues $\lambda^{(p)} : \mathbb{T}_k^{(p)} \to \overline{\mathbb{Z}}_p$. We then write $\overline{\lambda}^{(p)} : \mathbb{T}_k^{(p)} \to \overline{\mathbb{F}}_p$ to denote the reduction of $\lambda^{(p)}$ modulo the maximal ideal of $\overline{\mathbb{Z}}_p$.

PROPOSITION 2.8. — *As $\lambda^{(p)}$ ranges over all p-deprived systems of eigenvalues of all weights $k \geq 0$, there are only finitely many possibilities for the collection of eigenvalues* $\big(\overline{\lambda}^{(p)}(\ell S_\ell), \overline{\lambda}^{(p)}(T_\ell)\big)_{\ell \nmid Np}$.

One has the following precise conjecture regarding the Krull dimension of the ring $\mathbb{T}$.

CONJECTURE 2.9. — *The ring $\mathbb{T}$ is equidimensional of Krull dimension 4, i.e. each irreducible component of* Spec $\mathbb{T}$ *is of dimension* 4.

(2) Technically, the results of [30, §1.4] only apply when p is odd; however, with the appropriate modifications, they should also apply in the case when $p = 2$.

Since $\mathbb{T}$ is a torsion free and p-adically complete $\mathbb{Z}_p$-algebra, this is equivalent to $\operatorname{Spec}\mathbb{T}$ having relative dimension 3 over $\operatorname{Spec}\mathbb{Z}_p$. This conjecture is motivated by the known and conjectured relations between the ring $\mathbb{T}$ and Galois representations. (See Remark 2.14 below.) We will prove in Corollary 2.28 below that each irreducible component of $\operatorname{Spec}\mathbb{T}$ has Krull dimension at least 4.

2.2. Galois representations again

As in the preceding subsections, we regard all systems of Hecke eigenvalues as taking values in $\overline{\mathbb{Z}}_p$.

Definition 2.10. — *A p-adic system of Hecke eigenvalues is a homomorphism of $\mathbb{Z}_p$-algebras $\xi : \mathbb{T} \to \overline{\mathbb{Z}}_p$.*

Suppose that $\lambda^{(p)} : \mathbb{T}_k^{(p)} \to \overline{\mathbb{Z}}_p$ is a p-deprived system of Hecke eigenvalues. Since the target of $\lambda^{(p)}$ is a $\mathbb{Z}_p$-algebra, this homomorphism extends to a homomorphism $\lambda^{(p)} : \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_k^{(p)} \to \overline{\mathbb{Z}}_p$. Composing this homomorphism with the natural surjection $\mathbb{T} \to \mathbb{T}_k^{(p)}$, we obtain a homomorphism $\xi : \mathbb{T} \to \overline{\mathbb{Z}}_p$. We refer to p-adic systems of Hecke eigenvalues arising in this way as *classical*.

Theorem 2.11. — *If $\xi : \mathbb{T} \to \overline{\mathbb{Z}}_p$ is any p-adic system of Hecke eigenvalues, then there is a continuous, semi-simple representation*

$$\rho_\xi : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p),$$

uniquely determined (up to equivalence) by the condition that for each prime $\ell \nmid Np$, the matrix $\rho_\xi(\mathrm{Frob}_\ell)$ has characteristic polynomial equal to $X^2 - \xi(T_\ell)X + \xi(\ell S_\ell)$.

Sketch of proof. — The uniqueness proof is identical to that given in the proof of Theorem 1.20. As for existence, if ξ is a classical system, arising from the p-deprived system of eigenvalues $\lambda^{(p)}$, then we can clearly set $\rho_\xi := \rho_\lambda$. To construct ρ_ξ in general, one uses the fact that the classical ξ are dense in the set of all ξ (in a suitable sense), and then constructs ρ_ξ by an interpolation argument. □

Just as in the case of Theorem 1.20, one shows that if ξ is a p-adic system of Hecke eigenvalues, then ρ_ξ is odd, and so we see that $\mathbb{T}$ parametrizes a family of odd two-dimensional p-adic Galois representations. Furthermore, one has the following fundamental conjecture to the effect that all odd two-dimensional Galois representations should be of this form. (See e.g. the conjecture on p. 108 of [20].)

Conjecture 2.12. — *If Σ is any finite set of primes containing p, and if*

$$\rho : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

is continuous, semi-simple, and odd, then $\rho = \rho_\xi$ for some p-adic system of Hecke eigenvalues of some level N divisible only by primes in Σ distinct from p.

Remark 2.13. — In fact, one expects to be able to take N to be the tame (i.e. prime-to-p) Artin conductor of ρ.

Remark 2.14. — One can use techniques from Galois cohomology to show that if ρ is an odd, irreducible, continuous two-dimensional p-adic representation of $G_{\mathbb{Q},\Sigma}$ (for some fixed finite set of primes Σ), then the expected dimension of a neighbourhood of ρ in the space of all such representations is three. (See Corollary 3 on p. 405 of [31]. This reference treats the case of mod p Galois representations, but is easily adapted to the context of p-adic Galois representations.) Taken together with Conjecture 2.12, this motivates Conjecture 2.9.

Building on ideas of Gouvêa and Mazur [21] (in particular, the infinite fern, as considered in Subsection 2.5 below), together with the techniques of Wiles [45] and Taylor–Wiles [43], Böckle [3] has proved a strong result in the direction of Conjectures 2.12 and 2.9. Since the statement of his result is a little technical, we do not recall it here. However, by appealing to a result of Kisin, one can improve the part of Böckle's theorem that pertains to Conjecture 2.9, as follows.

Before stating it, we recall that $\mathbb{Q}(\zeta_{p^3})$ contains a unique quadratic extension of $\mathbb{Q}$ when p is odd, and three such extensions when $p = 2$. For any such quadratic $L \subset \mathbb{Q}(\zeta_{p^3})$, we write $G_{L,\Sigma} := \mathrm{Gal}(\mathbb{Q}_\Sigma/L)$.

THEOREM 2.15. — *Let $\xi : \mathbb{T} \to \overline{\mathbb{Z}}_p$ be classical, and suppose that $\overline{\rho}_{\xi\,|G_{L,\Sigma}}$ is irreducible, for each quadratic extension $L \subset \mathbb{Q}(\zeta_{p^3})$. Then* $\mathrm{Spec}\,\mathbb{T}$ *has dimension* 3 *in a neighbourhood of ξ.*

Proof. — It follows from [29, Thm., p. 277] that this dimension is at most 3, while Corollary 2.28 below establishes the opposite inequality. This proves the result. □

2.3. Families parametrized by weight: the Eisenstein family

Since $\mathrm{Spec}\,\mathbb{T}$ is (at least conjecturally) of relative dimension 3 over $\mathrm{Spec}\,\mathbb{Z}_p$, one can think of the set of all p-adic systems of Hecke eigenvalues ξ as depending on three parameters. Unfortunately, even in those cases when Conjecture 2.9 is known, there is no particularly canonical choice of these three parameters. A little more formally, if $\mathrm{Spec}\,\mathbb{T}$ has Krull dimension 4, then Noether normalization allows one to construct a finite map $\mathrm{Spec}\,\mathbb{T} \to \mathrm{Spec}\,\mathbb{Z}_p[[T_1, T_2, T_3]]$. However, there is no canonical choice for such a map.

On the other hand, there *is* a canonical map $\mathrm{Spec}\,\mathbb{T} \to \mathrm{Spec}\,\mathbb{Z}_p[[T]]$, as we now explain. Write $q = p$ if p is odd, and $q = 4$ if $p = 2$, and set $\Gamma = 1 + q\mathbb{Z}_p$. Let

$$\mathcal{L} := \{\ell \text{ prime} \mid \ell \equiv 1 \bmod Nq\}.$$

We regard $\mathcal{L}$ as a subset of Γ. Dirichlet's theorem on primes in arithmetic progression shows that $\mathcal{L}$ is in fact dense in Γ.

LEMMA 2.16. — *The map $\mathcal{L} \to \mathbb{T}$ given by $\ell \mapsto S_\ell$ extends uniquely to a continuous homomorphism of groups $\Gamma \to \mathbb{T}^\times$.*

Proof. — If $\ell \in \mathcal{L}$, and if λ is any system of Hecke eigenvalues of weight k, then $\lambda(S_\ell) = \ell^{k-2}$. (Because $\ell \equiv 1 \bmod N$, the diamond operator $\langle \ell \rangle$ is trivial.) The function $x \mapsto x^{k-2}$ is continuous on Γ, and so the map $\ell \mapsto S_\ell$ from $\mathcal{L} \to \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_k^{(p)}$ extends to a continuous homomorphism $\Gamma \to (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_k^{(p)})^\times$, for any weight k. The lemma now follows by an easy passage to the limit. □

Write $\mathbb{Z}_p[[\Gamma]] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$. This is the so-called completed group ring of Γ over $\mathbb{Z}_p$; there is an evident embedding of the usual group ring $\mathbb{Z}_p[\Gamma] \hookrightarrow \mathbb{Z}_p[[\Gamma]]$. If $x \in \Gamma$, we write $[x]$ to denote the corresponding element of $\mathbb{Z}_p[[\Gamma]]$ (so as to avoid confusion with the same element x regarded as belonging to the ring of coefficients $\mathbb{Z}_p$). There is an isomorphism of $\mathbb{Z}_p$-algebras $\mathbb{Z}_p[[T]] \xrightarrow{\sim} \mathbb{Z}_p[[\Gamma]$, determined by the condition $T \mapsto [1+q] - 1$.

The continuous map $\Gamma \to \mathbb{T}^\times$ of the preceding lemma extends uniquely to a homomorphism of $\mathbb{Z}_p$-algebras

$$w : \mathbb{Z}_p[[\Gamma]] \to \mathbb{T}, \tag{10}$$

which we may equally well regard as a map $\mathbb{Z}_p[[T]] \to \mathbb{T}$. Passing to Specs, we get the canonical map

$$\operatorname{Spec} \mathbb{T} \to \operatorname{Spec} \mathbb{Z}_p[[\Gamma]] \xrightarrow{\sim} \operatorname{Spec} \mathbb{Z}_p[[T]] \tag{11}$$

referred to above. What is the meaning of this map?

Well, giving a $\overline{\mathbb{Z}}_p$-valued point of $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$ is the same as giving a continuous character $\kappa : \Gamma \to \overline{\mathbb{Z}}_p^\times$. Thus $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$ is the space of characters of Γ. (The isomorphism $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]] \xrightarrow{\sim} \operatorname{Spec} \mathbb{Z}_p[[T]]$ is then given by mapping a character κ to the value $T = \kappa(1+q) - 1$; in this way, the space of continuous characters of Γ is identified with the maximal ideal of $\overline{\mathbb{Z}}_p$, or, in more geometric terms, the open unit disk around the origin of $\overline{\mathbb{Q}}_p$.) If k is an integer, then we may define a character $\kappa_k : \Gamma \to \overline{\mathbb{Z}}_p^\times$ via the formula $\kappa_k(x) = x^{k-2}$. These points are Zariski dense in $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$ (in fact, any infinite collection of them is Zariski dense), and so we regard $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$ as a certain kind of interpolation of the set of integers, and refer to it as *weight space*. In particular, the $\overline{\mathbb{Z}}_p$-valued point κ_k is said to be the point of weight k.

Now suppose that $\xi : \mathbb{T} \to \mathbb{Z}_p$ is classical, arising from the system of Hecke eigenvalues $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$. One computes that the composite $\xi \circ w$ is equal to κ_k, the point of weight k. Thus we may think of the w as mapping a system of Hecke eigenvalues to its corresponding weight (which explains our choice of notation). From this, we also see that w is injective (since there exist systems of Hecke eigenvalues of arbitrarily high weight), and hence that (11) is dominant.

Now the weight is a very natural parameter to consider, and so it is reasonable to ask whether we can find families of systems of Hecke eigenvalues, and hence families of Galois representations, that are parametrized by the weight. Somewhat more precisely, we can ask whether we can find a closed subscheme $Z \hookrightarrow \operatorname{Spec} \mathbb{T}$ such that the composite

$Z \hookrightarrow \operatorname{Spec} \mathbb{T} \to \operatorname{Spec} \mathbb{Z}_p[[T]]$ is dominant with finite fibres; such a subscheme Z could then be thought of as a family of Galois representations, parametrized by the weight.(3)

Of course, if we impose no other conditions on Z, then such subschemes Z exist for very general geometric reasons; on the other hand, a further natural condition to impose is that Z contain a Zariski dense set of points corresponding to classical systems of Hecke eigenvalues. The scheme Z could then be regarded as a one-dimensional family of systems of Hecke eigenvalues, parametrized by weight, and interpolating a collection of classical systems of Hecke eigenvalues.

Example 2.17. — The most basic example of a one-dimensional family of systems of Hecke eigenvalues, parametrized by weight, is the Eisenstein family. This is the original p-adic family of modular forms, introduced by Serre in [37]. We describe it here, in the language of systems of Hecke eigenvalues that we have introduced.

For simplicity we take $N = 1$, and we fix an even residue class $i \mod p-1$ if p is odd. Consider the p-deprived systems of Hecke eigenvalues $\lambda_k^{(p)}$ associated to the Eisenstein series E_k, for $k \geq 4$ and congruent to $i \bmod p-1$ if p is odd (resp. $k \geq 4$ and even if $p = 2$). Recall that these are given by

$$\lambda_k^{(p)}(\ell S_\ell) = \ell^{k-1}, \quad \lambda_k^{(p)}(T_\ell) = 1 + \ell^{k-1},$$

where ℓ ranges over all primes distinct from p.

We wish to rewrite these formulas slightly. Recall that $\mathbb{Z}_p^\times = \mu_{p-1} \times \Gamma$ (if p is odd) or $\mu_2 \times \Gamma$ (if $p = 2$). In either case, let μ denote the first factor, and write $\omega : \mathbb{Z}_p^\times \to \mu$ to denote the corresponding projection. Then we may rewrite the formulas for $\lambda_k^{(p)}$ as

$$\lambda_k^{(p)}(\ell S_\ell) = \ell\omega(\ell)^{i-2}\big(\ell\omega(\ell)^{-1}\big)^{k-2}, \quad \lambda_k^{(p)}(T_\ell) = 1 + \ell\omega(\ell)^{i-2}\big(\ell\omega(\ell)^{-1}\big)^{k-2},$$

where we set $i = 0$ if $p = 2$. We may evidently interpolate these formulas into a $\mathbb{Z}_p[[\Gamma]]$-valued point of $\operatorname{Spec} \mathbb{T}$. Namely, there is a homomorphism $E : \mathbb{T} \to \mathbb{Z}_p[[\Gamma]]$, defined by

$$S_\ell \mapsto \omega(\ell)^{i-2}[\ell\omega(\ell)^{-1}], \quad T_\ell \mapsto 1 + \ell\omega(\ell)^{i-2}[\ell\omega(\ell)^{-1}].$$

By construction, the composite $\kappa_k \circ E$ is equal to λ_k, for any $k \equiv i \bmod p-1$ (or any even k, if $p = 2$). Again by construction, $E \circ w$ is the identity on $\mathbb{Z}_p[[\Gamma]]$. Thus, in more geometric terms, we have constructed a map $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]] \to \operatorname{Spec} \mathbb{T}$ which is a section to the weight map $w : \operatorname{Spec} \mathbb{T} \to \operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$, namely a family of Eisenstein systems of eigenvalues, parametrized by their weight.

(3) We are using the word "parametrized" in a somewhat liberal sense, in that we are allowing our family to be a multi-valued function of the weight, i.e. we are asking that $Z \to \operatorname{Spec} \mathbb{Z}_p[[T]$ have finite fibres, but not that it necessarily be injective.

2.4. Families parametrized by weight: Hida families and the eigencurve

In our discussion of p-adic systems of Hecke eigenvalues, we have systematically ignored the Hecke operators S_p and T_p. This is important; for example, for the family λ_k of the Example 2.17, we have $\lambda_k(S_p) = p^{k-2}$ and $\lambda_k(T_p) = 1 + p^{k-1}$. These functions do not interpolate well as p-adic functions of k. However, if we consider the pth Hecke polynomial $X^2 - \lambda_k(T_p)X + p\lambda_k(S_p)$, we see that it has the form

$$X^2 - (1 + p^{k-1})X + p^{k-1} = (X - 1)(X - p^{k-1}).$$

One of the two roots of this polynomial is in fact constant in the family, and so interpolates without difficulty in the family. It is the second root which does not interpolate well. This motivates the idea of changing our context slightly, and considering points not just in $\operatorname{Spec}\mathbb{T}$, but in $\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m$ (here the fibre product is with $\operatorname{Spec}\mathbb{Z}[T, T^{-1}]$ over $\operatorname{Spec}\mathbb{Z}$, or equivalently, with $\operatorname{Spec}\mathbb{Z}_p[T, T^{-1}]$ over $\operatorname{Spec}\mathbb{Z}_p$). To any system of Hecke eigenvalues λ appearing in some $\mathcal{M}_k(N)$, we can plot a pair of $\overline{\mathbb{Q}}_p$-valued points in this fibre product, whose first coordinate (for either point) is the associated classical p-adic system of eigenvalues ξ, and whose second coordinates are the roots of the pth Hecke polynomial of λ.

Definition 2.18. — *Let $\mathcal{X}$ denote the set of $\overline{\mathbb{Q}}_p$-valued points of* $\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m$ *consisting of pairs (ξ, α), where $\xi : \operatorname{Spec}\mathbb{T} \to \overline{\mathbb{Z}}_p$ is classical, attached to some system of Hecke eigenvalues $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$ with $k \geq 1$, and α is a root of the pth Hecke polynomial*

$$X^2 - \lambda(T_p)X + p\lambda(S_p).$$

Let $\mathcal{X}^{\mathrm{ord}}$ denote the subset of $\mathcal{X}$ consisting of pairs (ξ, α) for which $\alpha \in \overline{\mathbb{Z}}_p^{\times}$. (The superscript ord *is for* ordinary.)

Remark 2.19. — The reason for singling out the subset $\mathcal{X}^{\mathrm{ord}}$ of $\mathcal{X}$ is that (since any system of Hecke eigenvalues is a $\overline{\mathbb{Z}}_p$-valued point of $\operatorname{Spec}\mathbb{T}$) these are precisely the points of $\mathcal{X}$ that consist of $\overline{\mathbb{Z}}_p$-valued points of $\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m$.

The following theorem, due to Hida [22, 23], describes the interpolation of the points in $\mathcal{X}^{\mathrm{ord}}$. (The map $\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m \to \operatorname{Spec}\mathbb{Z}_p[[\Gamma]]$ appearing in the statement of the theorem is the one obtained by first projecting onto the factor $\operatorname{Spec}\mathbb{T}$, and then applying the map w.)

Theorem 2.20. — *The Zariski closure $\mathcal{C}^{\mathrm{ord}}$ of $\mathcal{X}^{\mathrm{ord}}$ in $\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m$ is one-dimensional. The composite $\mathcal{C}^{\mathrm{ord}} \to \operatorname{Spec}\mathbb{T} \times \mathbb{G}_m \to \operatorname{Spec}\mathbb{Z}_p[[\Gamma]]$ is finite, and is furthermore étale in the neighbourhood of those points of $\mathcal{X}^{\mathrm{ord}}$ that are attached to systems of Hecke eigenvalues appearing in weight $k \geq 2$.*

Definition 2.21. — *We refer to $\mathcal{C}^{\mathrm{ord}}$ as the Hida family, or ordinary family, of tame level N.*

Remark 2.22. — We will see in Subsection 2.5 below that it is necessary to restrict to weights $k \geq 2$ in the final statement of the theorem.

The curve $\mathcal{C}^{\mathrm{ord}}$ is (almost) precisely a family of the type we envisaged in the previous subsection. (We say "almost" because it lies in $\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m$ rather than in $\operatorname{Spec}\mathbb{T}$ itself.) On the other hand, not every classical system of eigenvalues appears in $\mathcal{C}^{\mathrm{ord}}$; it is certainly possible that if $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$, then both roots of the pth Hecke polynomial may have positive slope.

We thus turn to our next result, due to Coleman and Mazur [10], which deals with the interpolation of the entire set $\mathcal{X}$. In this case, taking the algebraic Zariski closure of these points in $\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m$ turns out to be too coarse of an operation, and we cannot hope to construct an algebraic family of the type envisaged in the previous subsection that contains all the points of $\mathcal{X}$. Rather, we will construct a rigid analytic family, lying inside the associated rigid analytic space $(\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m)^{\mathrm{an}}$.(4)

THEOREM 2.23. — *The rigid analytic Zariski closure $\mathcal{C}$ of $\mathcal{X}$ in* $(\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m)^{\mathrm{an}}$ *is one-dimensional. More precisely, the composite*

$$\mathcal{C} \hookrightarrow (\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m)^{\mathrm{an}} \to (\operatorname{Spec}\mathbb{Z}_p[[\Gamma]])^{\mathrm{an}} \tag{12}$$

is flat, and has discrete fibres. Furthermore, for any positive constant C, there are only finitely many points (ξ, α) in any given fibre satisfying $\operatorname{ord}_p(\alpha) \le C$. (*In other words, the slopes of the $\mathbb{G}_m$-coordinates go to ∞ in each fibre.*)

DEFINITION 2.24. — *The curve $\mathcal{C}$ is called the eigencurve of tame level N. The analytification of $\mathcal{C}^{\mathrm{ord}}$ is called the slope zero part, or the ordinary part, of the eigencurve. It is a union of connected components of $\mathcal{C}$.*

Remark 2.25. — The map (12) is in fact étale in the neighbourhood of a point $(\xi, \alpha) \in \mathcal{X}$, unless α is a repeated root of the pth Hecke polynomial of the system of eigenvalues λ giving rise to ξ. (Compare the discussion of [10, p. 5].) It is conjectured that such repeated roots cannot occur when the weight $k \ge 2$ [9].

Remark 2.26. — By construction, each of $\mathcal{C}^{\mathrm{ord}}$ and $\mathcal{C}$ contains a(n algebraic or rigid analytic, as the case may be) Zariski dense set of points (ξ, α) for which ξ is a classical system of eigenvalues. It is natural to ask whether the converse holds, namely, if (ξ, α) is any $\overline{\mathbb{Q}}_p$-valued point of $\mathcal{C}^{\mathrm{ord}}$ of $\mathcal{C}$ lying over the weight $\kappa_k : \Gamma \to \overline{\mathbb{Z}}_p^{\times}$, for some positive integer k, then is ξ classical?

The answer is *no* in general, for trivial reasons. One already sees this with the Eisenstein family of Example 2.17. Indeed, in the notation of that example (and assuming

(4) Concretely, if $\mathbb{T} = \prod_{i=1}^{m} \mathbb{Z}_p[[T_1, \ldots, T_{r_i}]]/(f_{i,1}, \ldots, f_{i,s_i})$, then

$$\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m = \coprod_{i=1}^{m} \operatorname{Spec}\mathbb{Z}_p[[T_1, \ldots, T_{r_i}]][T, T^{-1}]/(f_{i,1}, \ldots, f_{i,s_i}),$$

and $(\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m)^{\mathrm{an}}$ is the rigid analytic space

$$\coprod_{i=1}^{m} \{\, (T_1, \ldots, T_{r_i}, T) \;\mid\; |T_1|, \ldots, |T_{r_i}| < 1,\, T \ne 0,\, f_{i,1}(T_1, \ldots, T_{r_i}) = \cdots = f_{i,s_i}(T_1, \ldots, T_{r_i}) = 0 \,\}.$$

Also, $(\operatorname{Spec}\mathbb{Z}_p[[\Gamma]])^{\mathrm{an}} \cong (\operatorname{Spec}\mathbb{Z}_p[[T]])^{\mathrm{an}} = \{T \mid |T| < 1\}$, i.e. the open unit disk in $\overline{\mathbb{Q}}_p$.

p is odd for simplicity), if $k \not\equiv i \bmod p-1$ then the associated system of eigenvalues is not associated to a modular form of level 1; rather, its values on ℓS_ℓ and T_ℓ (for $\ell \nmid Np$) coincide with those of the system of eigenvalues $\lambda^{(p)}_{1,\omega^{i-k},k}$ (in the notation of Example 1.18), corresponding to an Eisenstein series of level p.

Hida showed in general that if (ξ, α) is a $\overline{\mathbb{Z}}_p$-valued (or equivalently, $\overline{\mathbb{Q}}_p$-valued) point of $\mathcal{C}^{\mathrm{ord}}$ lying over the character κ_k for $k \geq 2$ (or, more generally, a character of the form $\psi\kappa_k$, where ψ has finite order and $k \geq 2$), then there is a system of eigenvalues $\lambda : \mathbb{T}_k(Np) \to \overline{\mathbb{Z}}_p$ such that $\xi(\ell S_\ell) = \lambda(\ell S_\ell)$ and $\xi(T_\ell) = \lambda(T_\ell)$ for all $\ell \nmid Np$.

In the non-ordinary case, the situation is more complicated. The fibre of $\mathcal{C}$ over any κ_k (or over any character $\psi\kappa_k$, where ψ is of finite order) is typically infinite, and all but finitely many of the points do not arise from a classical eigenform (of any level). However, Coleman showed [7, 8] that if (ξ, α) is such a point, and if the slope of α is less than $k-1$, then just as in the ordinary case, there is a system of eigenvalues $\lambda : \mathbb{T}_k(Np) \to \overline{\mathbb{Z}}_p$ such that $\xi(\ell S_\ell) = \lambda(\ell S_\ell)$ and $\xi(T_\ell) = \lambda(T_\ell)$ for all $\ell \nmid Np$. (One can show, e.g. using Theorem 2.33 below, that, conversely, if such a λ exists, then the slope of α is at most $k-1$. Of course, all but finitely many of the points lying over κ_k have slope $> k-1$.)

Idea of proofs. — The first step in the proof of Theorems 2.20 and 2.23 is to define a space of p-adic modular forms on which the p-adic Hecke algebra $\mathbb{T}$ acts. In fact one can literally work with such a space, namely the space of *generalized p-adic modular functions* of Katz (as defined in [27], see also [19] and [22]) — this is the approach taken in [22] for the ordinary case and in [10] for the general case — or with a surrogate, constructed from the group cohomology of $\Gamma_1(N)$ and certain of its subgroups. The cohomological approach to the ordinary case is developed in [23], and for the general case is developed in [42, 2]. There is another approach, via the p-adically completed cohomology of modular curves [15, §4], which is somewhat different, and which we will say a little about below. To simplify the exposition, from now on we will speak simply of "the space of p-adic modular forms", meaning either the space of generalized p-adic modular functions, or one of the cohomological surrogates of [2, 23, 42].

The next step is to introduce an additional Hecke operator on this space, the so-called U_p-operator. In the context of p-adic modular forms, this operator has the following effect on q-expansions:

$$U_p f = \sum_{n=0}^{\infty} a_{np}(f) q^n.$$

We let $\mathbb{T}^*$ denote the quotient of $\mathbb{T}[U_p]$ that acts faithfully on the space of p-adic modular forms. Evidently, $\operatorname{Spec} \mathbb{T}^* \hookrightarrow \operatorname{Spec} \mathbb{T} \times \mathbb{A}^1$.

Suppose for a moment that f is a modular form of weight k and level N, with $p \nmid N$, and let α and β be the roots of the pth Hecke polynomial. Then $f(\tau) - \beta f(p\tau)$ is a modular form of level Np, which is a U_p-eigenform with eigenvalue α. (This can be checked directly on the level of q-expansions.) Thus $\operatorname{Spec} \mathbb{T}^*$ contains the set $\mathcal{X}$, and hence also the Zariski closure of this set. The technical difficulty that arises in establishing the theorems is that Spec $\mathbb{T}^*$ is much bigger than either $\mathcal{C}^{\mathrm{ord}}$ or $\mathcal{C}$, roughly speaking because U_p has a huge kernel on the space of the p-adic modular forms, while we are trying to construct curves lying in $\operatorname{Spec}\mathbb{T} \times \mathbb{G}_m$, i.e. systems of eigenvalues of $\mathbb{T}^*$ for which the associated U_p-eigenvalue is non-zero.

It is at this point that the proofs of the two theorems diverge somewhat, with the proof of Theorem 2.20 being technically simpler than that of Theorem 2.23. The points of $\mathcal{X}^{\mathrm{ord}}$ correspond to eigenforms whose U_p-eigenvalue is ordinary. If f is any eigenform for $\mathbb{T}^*$ whose U_p-eigenvalue α is of *positive* slope, then $\lim_n U_p^n f = \alpha^n f \to 0$ as $n \to 0$. Thus by iterating U_p on the space of p-adic modular forms and passing to a limit, we can cut out the *ordinary part* of the space of p-adic modular forms, on which U_p acts with only ordinary eigenvalues. (This process can be summarized by using the limits of powers of U_n to construct the so-called ordinary projector, which projects to the ordinary part.) The quotient of $\mathbb{T}^*$ acting faithfully on this ordinary part is denoted by $\mathbb{T}^{\mathrm{ord}}$, and $\mathcal{C}^{\mathrm{ord}} = \operatorname{Spec}\mathbb{T}^{\mathrm{ord}}$. The key fact, underlying the proof of Theorem 2.20, is that $\mathbb{T}^{\mathrm{ord}}$ is finite over $\mathbb{Z}_p[[\Gamma]]$. This can be proved in various ways; either using the theory of mod p modular forms, if one is working with generalized p-adic modular functions (this is the approach taken in [22]), or by arguments with group cohomology (this is the approach of [23]).

As already indicated, the proof of Theorem 2.23 is more technical. The reason is as follows: if $\mathfrak{m}^*$ is a maximal ideal of $\mathbb{T}^*$ lying over a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$, and if $\mathfrak{m}^*$ is not ordinary (i.e. if $U_p \in \mathfrak{m}^*$), then it follows from [19, Prop. II.3.14] that

$$\mathbb{T}^*_{\mathfrak{m}^*} \cong \mathbb{T}_{\mathfrak{m}}[[U_p]].$$

(Here $\mathbb{T}^*_{\mathfrak{m}^*}$ and $\mathbb{T}_{\mathfrak{m}}$ denote completions, and $\mathbb{T}_{\mathfrak{m}}[[U_p]]$ is the formal power series ring in $\mathbb{T}_{\mathfrak{m}}$ with variable U_p.) Thus if $\xi : \mathbb{T}_{\mathfrak{m}} \to \overline{\mathbb{Z}}_p$ is a system of eigenvalues, we can extend it to a system of eigenvalues of $\mathbb{T}^*_{\mathfrak{m}}$ by assigning any positive-slope value of U_p that we choose; even if ξ is classical, attached to some system of eigenvalues $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$, the algebra $\mathbb{T}^*$ has no way of distinguishing the positive slope roots of the pth Hecke polynomial of λ from any other positive slope elements of $\overline{\mathbb{Z}}_p$.

Thus one cannot reasonably interpolate the points $\mathcal{X}$ by algebra alone; it is necessary to use some analysis. In the generalized p-adic modular functions setting, the key step is to replace this space by a certain subspace of so-called overconvergent modular forms. (This is the approach of [10].) In the cohomological framework, this step can be taken at the beginning, by working with rigid analytic (rather than merely continuous) modular symbols (as is done in [42, 2]). In these settings, the operator U_p is a compact operator, and so has a reasonable spectral theory. One can then analyze, and obtain finiteness results for, all of its non-zero eigenspaces, rather than just the ordinary eigenspaces. The analysis of these eigenspaces is at the heart of the proof of Theorem 2.23.

As mentioned above, there is another approach to the proof of Theorem 2.23, via p-adically completed cohomology [15]. In this setting, one does not directly have an action of the U_p-operator, but rather has an action of the entire group $\mathrm{GL}_2(\mathbb{Q}_p)$, and the introduction of the U_p-operator and the passage to its non-zero eigenspaces is effected in a single step, by applying the locally analytic Jacquet module functor of [14].

We make some further technical remarks. In the paper [10], the authors prove Theorem 2.23 only in the case when $N = 1$. The generalization to arbitrary N can be found in [4, Part II], or [15, §4]. Also, in most of the papers cited, the authors work with Hecke algebras that contain the Hecke operators U_ℓ for $\ell | N, \ell \neq p$, as well as the operators S_ℓ and T_ℓ that we have considered. We have avoided any consideration of these operators, since they are not essential for the consideration of Galois representations. It is not difficult to deduce the results in the form that we have stated them from the corresponding results cited, which perhaps involve these additional operators. □

2.5. The infinite fern

The composition of the closed embedding $\mathcal{C}^{\text{ord}} \hookrightarrow \operatorname{Spec} \mathbb{T} \times \mathbb{G}_m$ with the projection onto the first factor gives a map

$$\mathcal{C}^{\text{ord}} \to \operatorname{Spec} \mathbb{T}, \tag{13}$$

which is very close to being injective on $\overline{\mathbb{Z}}_p$-points. Indeed, if (ξ_1, α) and (ξ_2, β) are two such points mapping to the same point of $\operatorname{Spec} \mathbb{T}$, then $\xi_1 = \xi_2 = \xi$ (say), and we see that $\rho_{\xi \,|\, G_{\mathbb{Q}_p}}$ admits unramified quotients on which Frob_p acts by α and β respectively. Thus if $\alpha \neq \beta$, we see that $\rho_{\xi \,|\, G_{\mathbb{Q}_p}}$ is unramified. It is then conjectured (as a special case of [18, Conj. 3c]), and is proved in most cases [5], that ξ is a classical system of Hecke eigenvalues, arising from a weight 1 Hecke eigenform of level N. Hence (13) has (or at least, is expected to have) at most finitely many double points, arising from classical systems of Hecke eigenvalues in weight one.

On the other hand, if we consider the analogous map

$$\mathcal{C} \to (\operatorname{Spec} \mathbb{T})^{\text{an}}, \tag{14}$$

then every system of eigenvalues $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$ gives rise to a pair of points (ξ, α) and (ξ, β), where ξ is a p-adic system of Hecke eigenvalues associated to λ, and

$$X^2 - \lambda(T_\ell)X + \lambda(\ell S_\ell) = (X - \alpha)(X - \beta).$$

Unless $\alpha = \beta$ (which, as we already noted, is expected to be impossible unless $k = 1$), we see that the image of (14) admits a double point at ξ. Thus the image of (14) is a very complicated curve, with an infinite number of double points. It is known as the *infinite fern* [32, 21]. The following theorem, due to Gouvêa and Mazur [21], shows that it is a kind of "space-filling curve" in $(\operatorname{Spec} \mathbb{T})^{\text{an}}$.

THEOREM 2.27. — *Each component of the Zariski closure of the infinite fern in* $(\operatorname{Spec} \mathbb{T})^{\text{an}}$ *is at least two-dimensional.*

Sketch of proof. — Since $\mathcal{C}$ is defined to be the Zariski closure of $\mathcal{X}$, we see that the Zariski closure of the image of (14) is equal to the Zariski closure of the set ξ of classical p-adic systems of Hecke eigenvalues. Suppose that this Zariski closure contains a component $\mathcal{Z}$ that is one-dimensional. Since the singular locus of $\mathcal{Z}$ is a Zariski closed proper subset of $\mathcal{Z}$, we may find a classical ξ lying in the smooth locus of $\mathcal{Z}$. Since $\mathcal{Z}$ is one-dimensional, the map $\mathcal{C} \to \mathcal{Z}$ must be surjective in a neighbourhood of ξ. Let $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$, for some $k \geq 1$, be the system of eigenvalues giving rise to ξ, and let α and β be the two roots of the pth Hecke polynomial of λ. Then (unless $\alpha = \beta$), the image of $\mathcal{C}$ is branched at ξ, contradicting the fact that $\mathcal{Z}$ is smooth at ξ.

If $\alpha = \beta$, then by appealing to the result of Coleman mentioned in Remark 2.26, it is easy to see that we may find arbitrarily small perturbations ξ' of ξ, for which ξ' is classical and such that the corresponding roots α' and β' of the pth Hecke polynomial *are* distinct. Then we may apply the above argument to ξ' instead, and again derive a contradiction. □

The previous result has the following corollary on the Krull dimension of $\operatorname{Spec}\mathbb{T}$, which is again due to Gouvêa and Mazur [21].

COROLLARY 2.28. — *The Krull dimension of each component of* $\operatorname{Spec}\mathbb{T}$ *is at least* 4.

Sketch of proof. — It is equivalent to show that each component of the associated rigid analytic space $(\operatorname{Spec}\mathbb{T})^{\mathrm{an}}$ is at least three-dimensional. Any such component $\mathcal{Y}$ contains a component $\mathcal{Z}$ of the image of (14), which is two-dimensional. Twisting by characters of p-power conductor then provides a one-dimensional deformation of $\mathcal{Z}$ inside $\mathcal{Y}$, showing that the $\mathcal{Y}$ is at least three-dimensional. □

2.6. Galois representations over Hida families and the eigencurve

The following result, due to Mazur–Wiles [34] and Wiles [44], gives a Galois-theoretic interpretation of the points of $\mathcal{C}^{\mathrm{ord}}$, and in particular, of the $\mathbb{G}_m$-coordinate. Before stating it, we note that the chosen embedding $\imath_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ induces a map $G_{\mathbb{Q}_p} \to G_{\mathbb{Q}} \to G_{\mathbb{Q},\Sigma}$ (where we have written $G_{\mathbb{Q}_p}$ and $G_{\mathbb{Q}}$ to denote $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ respectively, and where the second arrow is the natural surjection). For any representation ρ of $G_{\mathbb{Q},\Sigma}$, we write $\rho_{|\,G_{\mathbb{Q}_p}}$ to denote the restriction of ρ to a representation of $G_{\mathbb{Q}_p}$ via this map. Recall that $G_{\mathbb{Q}_p}$ contains a normal subgroup I_p (the inertia subgroup), such that $G_{\mathbb{Q}_p}/I_p \xrightarrow{\sim} G_{\mathbb{F}_p}$, the absolute Galois group of $\mathbb{F}_p$. This latter group is topologically generated by the Frobenius automorphism Frob_p. We say that a representation of $G_{\mathbb{Q}_p}$ is unramified if it is trivial when restricted to I_p; any such representation is then endowed with an action of Frob_p.

THEOREM 2.29. — *If (ξ, α) is $\overline{\mathbb{Z}}_p$-valued point of $\mathcal{C}^{\mathrm{ord}}$, then $\rho_{\xi\,|\,G_{\mathbb{Q}_p}}$ admits a one-dimensional unramified quotient on which* Frob_p *acts with eigenvalue α.*

One has the following conjecture, which is an analogue for $\mathcal{C}^{\mathrm{ord}}$ of Conjecture 2.12. It was first made by Mazur and Tilouine [33].

Conjecture 2.30. — *If Σ is any finite set of primes containing p, and if*

$$\rho : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

is a continuous, semi-simple, and odd representation whose restriction to $G_{\mathbb{Q}_p}$ admits a one-dimensional unramified quotient on which Frob_p *acts through the eigenvalue α, then there is a $\overline{\mathbb{Z}}_p$-valued point (ξ, α) in the Hida family for some level N divisible only by primes in Σ distinct from p such that $\rho = \rho_\xi$.*

In their papers [40, 41], Skinner and Wiles have established this conjecture in a large number of cases.

Theorem 2.31. — *Let ρ and α be as in the statement of Conjecture* 2.30, *and let $\overline{\rho} : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be the representation obtained by descending ρ to $\overline{\mathbb{Z}}_p$ and then reducing modulo the maximal ideal of $\overline{\mathbb{Z}}_p$. If $\det \rho_{|I_p} = \psi\chi^{k-1}$ for some finite order character ψ and some integer $k \geq 2$ (recall that χ denotes the p-adic cyclotomic character), and if the semi-simplification of $\overline{\rho}_{\,|\,G_{\mathbb{Q}_p}}$ (which is necessarily the direct sum of two characters, and which is well-defined independently of the choice of $\overline{\mathbb{Z}}_p$-model of ρ giving rise to $\overline{\rho}$) is the direct sum of* distinct *characters, then Conjecture* 2.30 *holds for ρ.*

Remark 2.32. — Suppose that ρ and α are as in the preceding result, and let (ξ, α) be the point on the Hida family (of an appropriately chosen level), whose existence is given by the theorem, for which $\rho = \rho_\xi$. The assumption on $\det \rho$ in the theorem (and in particular, the assumption that $k \geq 2$) implies, by the result of Hida recalled above, that ξ is obtained from a system of eigenvalues $\lambda : \mathbb{T}_k(Np) \to \overline{\mathbb{Z}}_p$. Thus, in the context of this result, one concludes that ρ actually arises from the system of Hecke eigenvalues attached to a classical modular form (of level possibly divisible by p).

The following result gives a Galois-theoretic interpretation of the points on $\mathcal{C}$, analogous to Theorem 2.29. It is due to Kisin [28]. The statement requires the language of Fontaine's theory [17]. Recall that Fontaine has defined a ring B^+_{cris}, equipped with commuting actions of the group $G_{\mathbb{Q}_p}$ and of a "Frobenius" operator φ. If V is any representation of $G_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_p$, then $D^+_{\mathrm{cris}}(V) := (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\mathbb{G}_{\mathbb{Q}_p}}$ is a $\overline{\mathbb{Q}}_p$-vector space of dimension at most that of V, equipped with an operator φ induced by the operator φ on B^+_{cris}.

Theorem 2.33. — *If (ξ, α) is a $\overline{\mathbb{Q}}_p$-valued point of $\mathcal{C}$, and if $\rho^\vee_{\xi\,|\,G_{\mathbb{Q}_p}}$ denotes the contragredient representation to $\rho_{\xi\,|\,G_{\mathbb{Q}_p}}$, then $D^+_{\mathrm{cris}}(\rho^\vee_{\xi\,|\,G_{\mathbb{Q}_p}})$ contains a one-dimensional subspace on which φ acts via α.*

Sketch of proof. — If the p-adic system of Hecke eigenvalues ξ is classical, arising from a system of Hecke eigenvalues $\lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$ attached to some modular form of weight $k \geq 1$, then the representation $\rho^\vee_{\xi\,|\,G_{\mathbb{Q}_p}}$ is in fact crystalline, with Hodge–Tate weights equal to 0 and $1-k$, and so $D^+_{\mathrm{cris}}(\rho^\vee_{\xi\,|\,G_{\mathbb{Q}_p}})$ is two-dimensional over $\overline{\mathbb{Q}}_p$. In this context, it is known that the characteristic polynomial of φ is equal to the pth Hecke polynomial of λ [36]. Thus, if α and β are the two roots of this polynomial, then we see that $D^+_{\mathrm{cris}}(\rho^\vee_{\xi\,|\,G_{\mathbb{Q}_p}})^{\varphi=\alpha}$ and $D^+_{\mathrm{cris}}(\rho^\vee_{\xi\,|\,G_{\mathbb{Q}_p}})^{\varphi=\beta}$ are both non-zero. The theorem is then proved by showing that these non-zero spaces interpolate over the curve $\mathcal{C}$. □

Remark 2.34. — In the context of Theorem 2.33, if $\alpha \in \overline{\mathbb{Z}}_p^\times$, then $D^+_{\mathrm{cris}}(\rho^\vee_{\xi\,|\,G_{\mathbb{Q}_p}})^{\varphi=\alpha}$ is non-zero if and only if $\rho_{\xi\,|\,G_{\mathbb{Q}_p}}$ contains an unramified quotient on which Frob_p acts via α. Thus Theorem 2.29 is a consequence of Theorem 2.33.

The following conjecture is analogous to Conjecture 2.30 in the ordinary case. (See the hope expressed in Remark (2) of [28, p. 450].)

Conjecture 2.35. — *If Σ is any finite set of primes containing p, and if*

$$\rho : \ G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

is a continuous, semi-simple, and odd representation such that $D^+_{\mathrm{cris}}(\rho^\vee_{\xi\,|\,G_{\mathbb{Q}_p}})^{\varphi=\alpha}$ is non-zero, then there is a $\overline{\mathbb{Q}}_p$-valued point (ξ,α) in $\mathcal{C}$ for some level N divisible only by primes in Σ distinct from p such that $\rho = \rho_\xi$.

There has been recent progress on this conjecture (see the corollary on p. 3 of [30] as well as the forthcoming paper [16]).

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