

Hodge-Tate comparison

(1)

Some base-change properties for $\Delta_{R/A}$

(1) If R is p -completely smooth over A/I ,

if $(A, I) \rightarrow (A', I' = IA')$ is a map of

bounded prisms with finite (p, I) -complete Tor-amplitude, and if we set

$$R' := R \hat{\otimes}_A^L A', \quad \text{then these}$$

are isomorphisms

$$\Delta_{R/A} \hat{\otimes}_A^L A' \xrightarrow{\sim} \Delta_{R'/A'}$$

$$\S \quad \overline{\Delta}_{R/A} \hat{\otimes}_A^L A' \xrightarrow{\sim} \overline{\Delta}_{R'/A'}$$

derived (p, I) -completed

Pf: Choose B a free S -alg. with e surjection

$$B \twoheadrightarrow R \quad \text{factor through } B/I$$

If J is the kernel, let $C = B \left\{ \frac{J}{I} \right\}^\wedge$
 \uparrow prismatic envelope.

Our results on prismatic envelopes (suitably augmented as in 3.14 & 4.16 of [BS]) show that

(C, IC) is flat (ie. (p, I) -comp. flat) over (A, I) ,

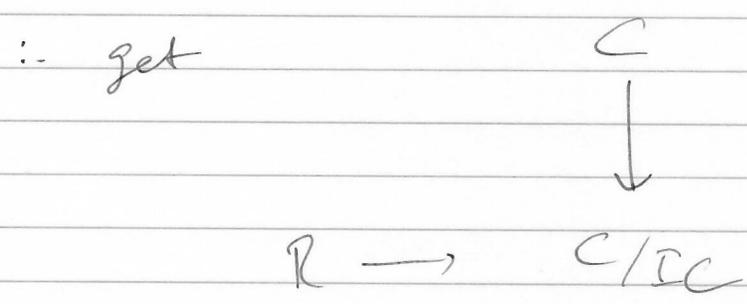
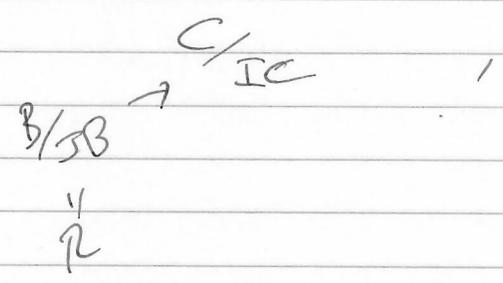
and that the formation of C is compatible with base-change,

ie. if $B' = B \hat{\otimes}_A^L A'$, and if $J' = \ker(B' \rightarrow R')$,

Then $C' := B' \left\{ \frac{J'}{I'} \right\}^\wedge = C \hat{\otimes}_A A'$

Since B is p -completely free, ~~and~~

Now by construction, $IC \supseteq IC$,



an object b the prismatic site of R over A . Since B is free, it is weakly hnd, \therefore

$$\Delta_{R/A} = \text{Tot.} (C^\bullet)$$

↑
 Completed ring ; $\text{Spec } C^\bullet = \check{C}ech \text{ nerve}$
 \downarrow
 $\text{Spec } C$
 over $\text{Spec } A$.
(p, I)-completed

Also $\Delta_{R'/A'} = \text{Tot.} ((C')^\bullet) = \text{Tot.} (C^\bullet \hat{\otimes}_A A')$

So the technical pt is to check that we can pass the \mathbb{Z} through the totalization.

The finite Tor-dim assumption lets us do this. \square

(Once we have the HT comparison, we can prove this with no finite Tor dim hypothesis $\sim (A, I) \rightarrow (A', I')$.)

② If $R \rightarrow S$ is a p -completely étale map of p -completely smooth A/I algs., then

$$\overline{\Delta}_{R/A} \begin{matrix} \mathbb{Z} \\ \otimes \\ \mathbb{Z} \end{matrix} \xrightarrow{\sim} \overline{\Delta}_{S/A}$$

Pf. The map $R \rightarrow S$ induces a "forgetful functor" from the prismatic site of S/A to that of R/A .

This has a right adjoint (right adjoint bc the prismatic site is defined via Spec's; it is left adjoint from the perspective of rings)

given as follows:

$$i_B \quad \begin{matrix} B \\ \downarrow \\ R \end{matrix} \longrightarrow B/I_B$$

then $B/I_B \rightarrow B/I_B \begin{matrix} \mathbb{Z} \\ \otimes \\ \mathbb{Z} \end{matrix}$, which is p -completely étale.

returns to a (p, I) -completely étale S -map

$$B \rightarrow B_{\mathbb{Z}} \quad (\text{by } S\text{-ring generality})$$

giving B_S

↓

$$S = R \overset{\oplus}{\underset{R}{\uparrow}} S \longrightarrow B_S / I B_S = (B/I) \overset{\oplus}{\underset{R}{\uparrow}} S$$

This right adjoint will preserve products and weakly final objects,

Let C and C' be as in ① ~~with \mathcal{C}~~

Then if D is the corresponding object over S , $D \cong C'_S$

D is weakly final, and D' ~~which arises from C' via the right adjoint~~ arises from C' via the right adjoint.

Then $D' / I D' = C' / I C' \overset{\oplus}{\underset{R}{\uparrow}} S$ by construction.

Also $\bar{\Delta}_{R/A} = \text{Tot}(C' / I C')$ by weak finality of C

$$\begin{aligned} \bar{\Delta}_{S/A} &= \text{Tot}(D' / I D') \quad \text{--- " --- of } D \\ &= \text{Tot}(C' / I C' \overset{\oplus}{\underset{R}{\uparrow}} S) \end{aligned}$$

Now ② follows by the same homological algebra as ①; the complete Nakayama of S over R lets us pass $\overset{\oplus}{\underset{R}{\uparrow}} S$ through the total complex. \square

We can now prove the crystalline case of the HT comparison: (A, \mathfrak{p}) a crystalline prism.
 R ~~smooth~~ smooth over $A/\mathfrak{p}A$.

Universal properties of $\Omega^i_{R/(A/\mathfrak{p})}$ give rise to

$$\Omega^i_{R/(A/\mathfrak{p})} \rightarrow H^i(\bar{\Delta}_{R/A})$$

compatible with wedge / cup product,
 exterior diff'l, and Bockstein.

Thm: This is an isomorphism.

Proof: By étale base change (② above) and naturality, we reduce to the case

$$\text{when } R = A/\mathfrak{p}A[x_1, \dots, x_n].$$

Then by ① above, applied to $\mathbb{Z}_p \rightarrow A$, we reduce to the case when $A = \mathbb{Z}_p$ and $R = \mathbb{F}_p[x_1, \dots, x_n]$.

In this case the primitive isom. reduces to crystalline isom., & the above map becomes the inverse Cartier isom., which is an isomorphism !! \square

⑥

We now consider the HT comparison Thm. in general.

~~Again~~ So now R is a p -completely smooth A/I -alg.

Again, universal property of $\Omega^i_{R/(A/I)}$ induces

$$\Omega^i_{R/(A/I)} \rightarrow H^i(\bar{\Delta}_{R/A}) \otimes_{A/I} I^i/I^{i+1}$$

wedge \longleftrightarrow cup product

exterior derivative \longleftrightarrow Bockstein

Thm: This is an isomorphism

Pf: We use p -complete étale base change to reduce to the case when $R = A/I \langle x_1, \dots, x_n \rangle$

(p -adic comp. of $A/I \langle x_1, \dots, x_n \rangle$).

Now the statement we want to check can be ~~checked~~ verified flat locally on (A, I) (here "flat" means in the prismatic sense, i.e. p - \mathbb{Z}_p -completely flat locally)

\therefore can assume A is oriented, i.e. that $I = (d)$.

Let $(A_0, (d)) =$ universal oriented prism

$$:= \mathbb{Z}_p \{d, s(d)^{-1}\}^\wedge$$

Then R is obtained by base-change from $R_0 := A_0/(d) \langle x_1, \dots, x_n \rangle$

We first verify the case when $A=A_0$.

The goal features of this case are that

• $\text{Emb}: A/p \rightarrow A/p$ is flat, and $A/(d)[p] = 0$,
i.e. d is a non-zero divisor on A/p .

In this context (i.e. if these properties hold), we set

$$B := A \left\{ \frac{\varphi(d)}{p} \right\}^{\uparrow} \begin{matrix} \text{derived } p\text{-completion} \\ \text{(and then } \varphi(d)) \end{matrix}$$

Since both d & p are distinguished in $\pi_0(B)$, we have $\varphi(d) = u \cdot p$ for $u \in \pi_0(B)^\times$, and so

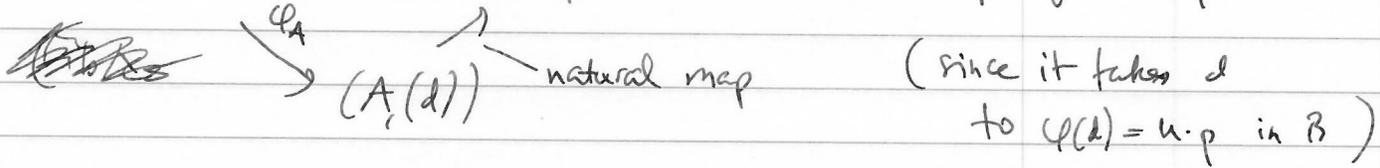
B is $\left\{ \begin{matrix} \text{derived} \\ (p, d) \end{matrix} \right\}$ -complete.

Also $A \left\{ \frac{\varphi(d)}{p} \right\} = D_A(d)$ by our p -envelope result,

\therefore is p -torsion free, $\therefore A \left\{ \frac{\varphi(d)}{p} \right\}^{\uparrow} =$ classical completion $B^{\#}$

So B is discrete, i.e. ~~complete~~ just a usual ring, and being $\left\{ \begin{matrix} \text{derived} \\ (p, d) \end{matrix} \right\}$ -complete it is a p -adically complete ring with primitive ideal (p) .

and $(A, (d)) \xrightarrow{\alpha} (B, (p))$ is a map of ~~prims~~ prims.



Now mod p , α becomes

$$A/p \xrightarrow{F_{\text{in}}} A/p \longrightarrow D_{A/pA}(d) \quad \textcircled{8}$$

Since $d^p = p \cdot \gamma_p(d) \underset{=0}{=}$ in $D_{A/pA}(d)$,

This factors as

$$\begin{array}{ccccc}
 A/p & \longrightarrow & A/(p,d) & \xrightarrow{F_{\text{in}}} & A/(p,d^p) \longrightarrow D_{A/pA}(d) \\
 \downarrow & & \downarrow & & \uparrow \\
 \text{finite} & & \text{base change} & & \text{inclusion of a} \\
 \text{Tor-amplitude} & & \text{of } A/p \xrightarrow{F_{\text{in}}} A/p & & \text{summand in a free} \\
 \text{b/c } d \text{ is} & & \text{over } A/(p,d), & & A/(p,d^p)\text{-module.} \\
 \text{a non-zero} & & \therefore \text{faithfully flat} & & \\
 \text{divisor} & & & &
 \end{array}$$

\therefore The composite has finite Tor-amplitude,

$\therefore \alpha: A \rightarrow B$ has p -completely finite Tor-amplitude,

so we get $\hat{\alpha}^* \triangleleft_{\mathbb{R}A} = \triangleleft_{\mathbb{R}B} B$

by base-change property ①.

Also, one finds that $\hat{\alpha}^*$ is conservative on

derived (p,d) -complete objects of $D(A)$.

(By p -derived Nakayama, it's enough to check that if $X \rightarrow Y$ is a morphism/in $D(A/p)$, of derived d -complete objects in

s.t. $(B/pB) \hat{\otimes}_{A/pA}^L X \rightarrow (B/pB) \hat{\otimes}_{A/pA}^L Y$ is an iso.

Then $X \rightarrow Y$ is an \cong .

Since $A/pA \rightarrow B/pB$ factors as $A/pA \rightarrow A/(p, d)$ followed by faithfully flat maps, it's enough to check the corresponding statement

for $(X \rightarrow Y) \rightsquigarrow (A/(p, d) \hat{\otimes}_{A/p}^L X \rightarrow A/(p, d) \hat{\otimes}_{A/p}^L Y)$.

But this is just derived Nakayama for (d)!!

Now, since $\Omega_{R/A}^i$ are ~~free~~ (topologically) free over R , we may

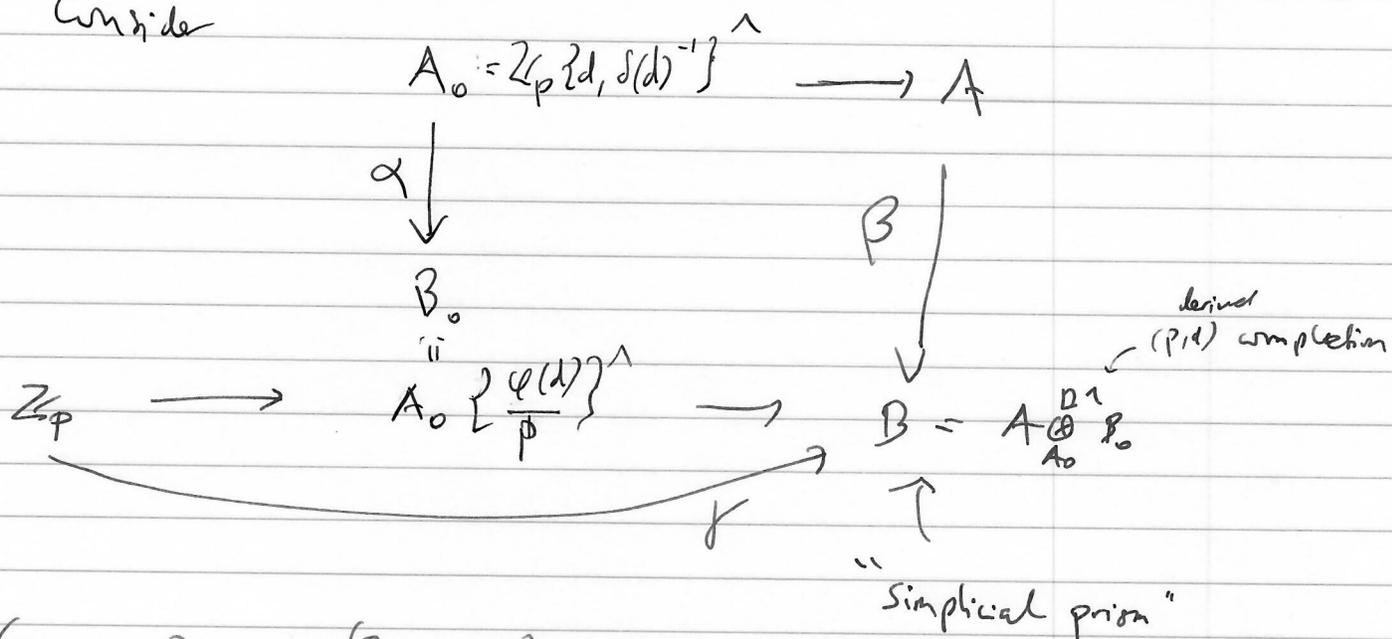
choose $\eta: \bigoplus \Omega_{R/A}^i[-i] \rightarrow \bar{\Omega}_{R/A}$

inducing the HT comparison maps on cohom.

Applying $\hat{\alpha}^*$, we know we get an iso., by the crystalline case. Since $\hat{\alpha}^*$ is conservative, we find that η is an iso., and we're done.

Finally, we want to base-change from this case ($A = A_0$) to the general case. If we knew general base-change results for $\bar{\Omega}_{R/A}$, we'd be done — but these will be deduced as a consequence of only the theorem we're trying to prove!!

Consider



$$(\mathbb{Z}_p, (p)) \longrightarrow (B_0, (p))$$

" (PID) completion" map of prisms.

Since $\hat{\alpha}^*$ is conservative, so is $\hat{\beta}^*$.

But now, using an explicit ω -simplicial model for $\overline{\Delta} R/A$, we ~~show~~ show that $\hat{\gamma}^* \overline{\Delta} \mathbb{F}_p[x_1, \dots, x_n] = \hat{\beta}^* \overline{\Delta} R/A$.

(We build an explicit ω -simplicial model for $\overline{\Delta} A_0 \langle x_1, \dots, x_n \rangle / A_0$ compatible which is manifestly base-change ~~invariant~~, and use that γ , and also α , and thus β , have finite p -^{complete}, resp. (PID)-complete Tor dim?, ~~so~~ so as to interchange completed pull-back with the formation of total complexes.)

Here it is :

$$B = A \{x_1, \dots, x_n\}$$

derived
← (p, I)-completion

$$x_i \mapsto x_i, \quad \delta^j(x_i) \mapsto 0 \text{ if } j > 0$$

$$A/I \langle x_1, \dots, x_n \rangle$$

and $C = B \{ \frac{\partial}{\partial x_i} \}^{\wedge}$

Then C is the model for $\langle A/I \langle x_1, \dots, x_n \rangle / A \rangle$ (p.d)-completed simplicial Cech nerve of C

is the model for $\langle A/I \langle x_1, \dots, x_n \rangle / A \rangle$

that we want (~~is~~ manifestly base-change compatible.)

So now we reduce to the (already proved) case of

$$(A, I) = (\mathbb{Z}_p, (p))$$

□