LECTURES ON THE $SL(2,\mathbb{R})$ ACTION ON MODULI SPACE

1. Lecture 1

Suppose $g \geq 1$, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a partition of $2g-2$, and let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials, i.e. the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\alpha_1 \dots \alpha_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(\alpha)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [Zo].

The space $\mathcal{H}(\alpha)$ admits an action of the group $SL(2,\mathbb{R})$ which generalizes the action of $SL(2,\mathbb{R})$ on the space $GL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of flat tori.

Period Coordinates. Let $\Sigma \subset M$ denote the set of zeroes of ω. Let $\{\gamma_1, \dots, \gamma_k\}$ denote a symplectic $\mathbb{Z}$-basis for the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. We can define a map $\Phi : \mathcal{H}(\alpha) \to \mathbb{C}^k$ by

$$\Phi(M,\omega) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} w \right)$$

The map Φ (which depends on a choice of the basis $\{\gamma_1, \dots, \gamma_k\}$) is a local coordinate system on (M, ω). Alternatively, we may think of the cohomology class $[\omega] \in H^1(M, \Sigma, \mathbb{C})$ as a local coordinate on the stratum $\mathcal{H}(\alpha)$. We will call these coordinates *period coordinates*. The period coordinates give $\mathcal{H}(\alpha)$ the structure of an affine manifold.

The $SL(2,\mathbb{R})$-action and the Kontsevich-Zorich cocycle. We write $\Phi(M,\omega)$ as a 2 by n matrix x. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ in these coordinates is linear. We choose some fundamental domain for the action of the mapping class group, and think of the dynamics on the fundamental domain. Then, the $SL(2,\mathbb{R})$ action becomes

$$x = \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \to gx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} A(g,x),$$

where $A : SL(2,\mathbb{R}) \times \mathcal{H}_1(\alpha) \to Sp(2g,\mathbb{Z}) \ltimes \mathbb{R}^k$ is the *Kontsevich-Zorich cocycle*. Thus, $A(g,x)$ is change of basis one needs to perform to return the point $gx \in \mathcal{H}_1(\alpha)$ to the fundamental domain. It can be interpreted as the monodromy of the Gauss-Manin connection restricted to the orbit of $SL(2,\mathbb{R})$.

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Masur-Veech measures. We can consider the measure λ on $\mathcal{H}(\alpha)$ which is given by the pullback of the Lebesgue measure on $H^1(M,\Sigma,\mathbb{C}) \approx \mathbb{C}^k$. The measure λ is independent of the choice of basis $\{\gamma_1,\dots,\gamma_k\}$, and is easily seen to be $SL(2,\mathbb{R})$-invariant. We call λ the *Lebesgue* or the *Masur-Veech* measure on $\mathcal{H}(\alpha)$.

The area of a translation surface is given by

$$a(M,\omega) = \frac{i}{2}\int_M \omega \wedge \bar{\omega}.$$

A "unit hyperboloid" $\mathcal{H}_1(\alpha)$ is defined as a subset of translation surfaces in $\mathcal{H}(\alpha)$ of area one. The SL-invariant Lebesgue measure λ_1 on $\mathcal{H}_1(\alpha)$ is defined by disintegration of the Lebesgue measure λ on $\mathcal{H}_1(\alpha)$, namely

$$d\lambda = d\lambda_1\, da$$

A fundamental result of Masur [Mas1] and Veech [Ve1] is that $\lambda_1(\mathcal{H}_1(\alpha)) < \infty$. In this paper, we normalize λ_1 so that $\lambda_1(\mathcal{H}_1(\alpha)) = 1$ (and so λ_1 is a probability measure).

Affine measures and affine invariant submanifolds. For a subset $\mathcal{M}_1 \subset \mathcal{H}_1(\alpha)$ we write

$$\mathbb{R}\mathcal{M}_1 = \{(M,t\omega) \mid (M,\omega) \in \mathcal{M}_1, \quad t \in \mathbb{R}\} \subset \mathcal{H}(\alpha).$$

Definition 1.1. An ergodic $SL(2,\mathbb{R})$-invariant probability measure ν_1 on $\mathcal{H}_1(\alpha)$ is called *affine* if the following hold:

(i) The support $\mathcal{M}_1$ of ν_1 is an suborbitfold of $\mathcal{H}_1(\alpha)$. Locally $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ is given by a complex linear subspace defined over $\mathbb{R}$ in the period coordinates.
(ii) Let ν be the measure supported on $\mathcal{M}$ so that $d\nu = d\nu_1 da$. Then ν is an affine linear measure in the period coordinates on $\mathcal{M}$, i.e. it is (up to normalization) the restriction of the Lebesgue measure λ to the subspace $\mathcal{M}$.

Definition 1.2. We say that any suborbitfold $\mathcal{M}_1$ for which there exists a measure ν_1 such that the pair $(\mathcal{M}_1, \nu_1)$ satisfies (i) and (ii) an *affine invariant submanifold.*

We also consider the entire stratum $\mathcal{H}(\alpha)$ to be an (improper) affine invariant submanifold. Thus, in particular, an affine invariant submanifold is a closed $SL(2,\mathbb{R})$-invariant subset of $\mathcal{H}(\alpha)$ which in period coordinates looks like a linear subspace.

For many applications we need the following:

Proposition 1.3. *Any stratum $\mathcal{H}_1(\alpha)$ contains at most countably many affine invariant submanifolds.*

Proposition 1.3 is deduced as a consequence of some isolation theorems in [EMiMo]. This argument relies on adapting some ideas of G.A. Margulis to the Teichmüller space setting. Another proof is given by A. Wright in [Wr1], where it is proven that affine invariant submanifolds are always defined over a number field.

The classification of the affine invariant submanifolds is complete in genus 2 by the work of McMullen [Mc1], [Mc2], [Mc3], [Mc4], [Mc5], and Calta [Ca]. In genus 3 or greater it is an important open problem. See [Mö1] [Mö2] [Mö3] [Mö4] [BoM], [BaM], [HLM], and [Wr2] for some results in this direction.

1.1. **The main theorems.** Let

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R} \right\}.$$

Let $r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, and let $K = \{r_\theta \mid \theta \in [0, 2\pi)\}$. Then N, A and K are subgroups of $SL(2,\mathbb{R})$. Let $P = AN$ denote the upper triangular subgroup of $SL(2,\mathbb{R})$.

Theorem 1.4. *Let ν be any P-invariant probability measure on $\mathcal{H}_1(\alpha)$. Then ν is $SL(2,\mathbb{R})$-invariant and affine.*

The following (which uses Theorem 1.4) is joint work with A. Mohammadi and is proved in [EMiMo]:

Theorem 1.5. *Suppose $S \in \mathcal{H}_1(\alpha)$. Then, the orbit closure $\overline{PS} = \overline{SL(2,\mathbb{R})S}$ is an affine invariant submanifold of $\mathcal{H}_1(\alpha)$.*

For the case of strata in genus 2, the $SL(2,\mathbb{R})$ part of Theorem 1.4 and Theorem 1.5 were proved using a different method by Curt McMullen [Mc6].

The proof of Theorem 1.4 uses extensively entropy and conditional measure techniques developed in the context of homogeneous spaces (Margulis-Tomanov [MaT], Einsiedler-Katok-Lindenstrass [EKL]). Some of the ideas came from discussions with Amir Mohammadi.

But the main strategy is to replace polynomial divergence by the "exponential drift" idea of Benoist-Quint [BQ].

Stationary measures. Let μ be a K-invariant compactly supported measure on $SL(2,\mathbb{R})$ which is absolutely continuous with respect to Lebesgue measure. A measure ν on $\mathcal{H}_1(\alpha)$ is called *μ-stationary* if $\mu * \nu = \nu$, where

$$\mu * \nu = \int_{SL(2,\mathbb{R})} (g_* \nu)\, d\mu(g).$$

Recall that by a theorem of Furstenberg [F1], [F2], restated as [NZ, Theorem 1.4], μ-stationary measures are in one-to-one correspondence with P-invariant measures. Therefore, Theorem 1.4 can be reformulated as the following:

Theorem 1.6. *Any μ-stationary measure on $\mathcal{H}_1(\alpha)$ is $SL(2,\mathbb{R})$ invariant and affine.*

Counting periodic trajectories in rational billiards. Let Q be a rational polygon, and let $N(Q,T)$ denote the number of cylinders of periodic trajectories of length at most T for the billiard flow on Q. By a theorem of H. Masur [Mas2] [Mas3], there exist c_1 and c_2 depending on Q such that for all $t > 1$,

$$c_1 e^{2t} \le N(Q, e^t) \le c_2 e^{2t}.$$

Theorem 1.4 and Proposition 1.3 together with some extra work (done in [EMiMo]) imply the following "weak asymptotic formula" (cf. [AEZ]):

Theorem 1.7. *For any rational polygon Q, the exists a constant $c = c(Q)$ such that*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t N(Q, e^s) e^{-2s}\, ds = c.$$

The constant c in Theorem 1.7 is the Siegel-Veech constant (see [Ve2], [EMZ]) associated to the affine invariant submanifold $\mathcal{M} = \overline{SL(2,\mathbb{R})S}$ where S is the flat surface obtained by unfolding Q.

It is natural to conjecture that the extra averaging on Theorem 1.7 is not necessary, and one has $\lim_{t\to\infty} N(Q, e^t)e^{-2t} = c$. This can be shown if one obtains a classification of the measures invariant under the subgroup N of $SL(2,\mathbb{R})$. Such a result is in general beyond the reach of the current methods. However it is known in a few very special cases, see [EMS], [EMM], [CW] and [Ba].

1.2. **Some notes on the proofs.** The theorems of §1.1 are inspired by the results of several authors on unipotent flows on homogeneous spaces, and in particular by Ratner's seminal work. In particular, the analogues of Theorem 1.4 and Theorem 1.5 in homogeneous dynamics are due to Ratner [Ra4], [Ra5], [Ra6], [Ra7]. (For an introduction to these ideas, and also to the proof by Margulis and Tomanov [MaT] see the book [Mor].) The homogeneous analogue of the fact that P-invariant measures are $SL(2,\mathbb{R})$-invariant is due to Mozes [Moz] and is based on Ratner's work. All of these results are based in part on the "polynomial divergence" of the unipotent flow on homogeneous spaces.

However, in our setting, the dynamics of the unipotent flow (i.e. the action of N) on $\mathcal{H}_1(\alpha)$ is poorly understood, and plays no role in our proofs. The main strategy is to replace the "polynomial divergence" of unipotents by the "exponential drift" idea in the recent breakthrough paper by Benoist and Quint [BQ].

One major difficulty is that we have no apriori control over the Lyapunov spectrum of the geodesic flow (i.e. the action of A). By [AV] the Lyapunov spectrum is simple for the case of Lebesgue (i.e. Masur-Veech) measure, but for the case of an arbitrary P-invariant measure this is not always true, see e.g. [FoM].

In order to use the the Benoist-Quint exponential drift argument, we must show that the Zariski closure (or more precisely the algebraic hull, see §2.1) of the Kontsevich-Zorich cocycle is semisimple. The proof proceeds in the following steps:

Step 1. We use an entropy argument inspired by the "low entropy method" of [EKL] (using [MaT] together with some ideas from [BQ]) to show that any P-invariant measure ν on $\mathcal{H}_1(\alpha)$ is in fact $SL(2,\mathbb{R})$ invariant. We also prove Theorem 2.11 which gives control over the conditional measures of ν. This argument is outlined in Lecture 2.

Step 2. By some results of Forni and Forni-Matheus-Zorich (see §3.1), for an $SL(2,\mathbb{R})$-invariant measure ν, the absolute cohomology part of the Kontsevich-Zorich cocycle $A : SL(2,\mathbb{R}) \times X \to Sp(2g,\mathbb{Z})$ is semisimple, i.e. has semisimple algebraic hull. For an exact statement see Theorem 3.4.

Step 3. We pick an admissible measure μ on $SL(2,\mathbb{R})$ and work in the random walk setting as in [BQ]. Let B denote the space of infinite sequences $g_0, g_1, \dots$, where $g_i \in SL(2,\mathbb{R})$. We then have a skew product shift map $T : B \times X \to B \times X$ as in [BQ], so that $T(g_0, g_1, \dots; x) = (g_1, g_2, \dots; g_0^{-1}x)$. Then, we use (in §3.2) a modification of the arguments by Guivarc'h and Raugi [GR1], [GR2], as presented by Goldsheid and Margulis in [GM, §4-5], and an argument of Zimmer ([Zi1] or [Zi2]) to prove Theorem 3.5 which states that the Lyapunov spectrum of T is always "semisimple", which means that for each $SL(2,\mathbb{R})$-irreducible component of the cocycle, there is an T-equivariant non-degenerate inner product on the Lyapunov subspaces of T (or more precisely on the successive quotients of the Lyapunov flag of T). This statement is trivially true if the Lyapunov spectrum of T is simple.

Step 4. We can now use the Benoist-Quint exponential drift method to show that the measure ν is affine. This argument is outlined in Lecture 4.

At one point, to avoid a problem with relative homology, we need to use a result about the isometric (Forni) subspace of the cocycle, which is proved in joint work with A. Avila and M. Möller [AEM].

Finally, we note that the proof relies heavily on various recurrence to compact sets results for the $SL(2,\mathbb{R})$ action, such as those of [EMa] and [Ath]. All of these results originate in the ideas of Margulis and Dani, [Mar1], [Dan1], [EMM1].

In the rest of the lectures, we will give an outline of the proof of Theorem 1.4. Various shortcuts are taken, and many details are hidden. We refer the reader to [EMi] for the details.

2. Lecture 2

2.1. General statements about cocycles.

Cocycles. Suppose (X,ν) is a measure space, and G acts on X preserving ν. An ($SL(m,\mathbb{R})$-valued) cocycle is a Haar$\times\nu$-measurable map

$$\alpha : G \times X \to SL(m,\mathbb{R})$$

such that

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 x)\alpha(g_2 x).$$

Suppose $C : X \to SL(m,\mathbb{R})$ is a ν-measurable map. (We should think of C as "change of basis at x"). Then,

$$\beta(g,x) = C(gx)^{-1}\alpha(g,x)C(x)$$

is also a cocycle and we say that β is *homologous* to α. We write $\beta \sim \alpha$.

Algebraic Hull. The algebraic hull of a cocycle α is the smallest algebraic subgroup H of $SL(m,\mathbb{R})$ such that for some $\beta \sim \alpha$, β takes values in H. Zimmer proves that H exists and is unique up to conjugation.

Cocycles over $\mathbb{R}$ and $\mathbb{Z}$. Even though the Kontsevich-Zorich cocycle is naturally a cocycle over an action of $SL(2,\mathbb{R})$, we will sometimes restrict to the action of the subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, i.e. the geodesic flow, or consider the random walk context defined below. In the first case, we get a cocycle over a flow (i.e. an action of $\mathbb{R}$, and in the second a cocycle over a shift map (i.e. an action of $\mathbb{Z}$).

There is a general structure theory for cocycles over $\mathbb{R}$ and $\mathbb{Z}$. In fact there are two fundamental theorems:

Theorem 2.1 (Osceledets Multiplicative Ergodic Theorem)**.** *Suppose T acts on X, preserving an ergodic measure ν. Suppose T is invertible, and $\alpha : \mathbb{Z}\times X \to SL(m,\mathbb{R})$ is a cocycle, i.e.*

$$\alpha(n+k,x) = \alpha(n,T^k x)\alpha(k,x).$$

Then, there exists a T-equivariant splitting

$$\mathbb{R}^m = \bigoplus_i E_i(x),$$

such that for $v \in E_i(x)$, and $t \to \pm\infty$,

$$\lim_{t\to\pm\infty} \frac{1}{t}\log\|\alpha(n,x)v\| = \lambda_i. \tag{2.1}$$

The numbers λ_i (which are independent of x) are called the Lyapunov exponents of α.

In particular, this theorem says that every cocycle is homologous to one in "block diagonal" form. (For each x pick a basis for $\mathbb{R}^m$ which is compatible with the $E_i(x)$, and let $C(x)$ be the matrix which changes the standard basis for $\mathbb{R}^m$ to the new one).

If all the E_i are one-dimensional (i.e. the Lyapunov spectrum is simple), then Oseledets implies that the cocycle is homologous to a diagonal cocycle. (Of course we might have only one E_i of dimension n in which case we get nothing from this).

Theorem 2.2 (Zimmer's Amenable Reduction)**.** *Every cocycle over (an ergodic action of) $\mathbb{R}$ or $\mathbb{Z}$ is homologous to a cocycle taking values in an amenable algebraic subgroup of $SL(m,\mathbb{R})$.*

Every amenable subgroup of a linear group is conjugate to a subgroup of a group of the form

$$\begin{pmatrix} \mathrm{Conf}(n_1) & * & \dots & * \\ 0 & \mathrm{Conf}(n_2) & \dots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \dots & \mathrm{Conf}(n_k) \end{pmatrix}, \tag{2.2}$$

$\mathrm{Conf}(n)$ is the "conformal group of $\mathbb{R}^n$", i.e.

$$\mathrm{Conf}(n) \cong \mathbb{R} \times O(n)$$

where $O(n)$ is the orthogonal group of $\mathbb{R}^n$ and the $\mathbb{R}$ is a scaling factor. (So an element of $\mathrm{Conf}(n)$ is a scalar multiple of an orthogonal matrix).

Thus, Theorem 2.2 says that every cocycle is homologous to a cocycle taking values in the group given by (2.2).

Remark 1. One gets the best information by first applying the Osceledets theorem, and then applying Zimmer's amenable reduction within each Osceledets block.

Remark 2. In order to apply the Benoist-Quint argument, we need to get rid of the $*$'s in (2.2) [for the Kontsevich-Zorich cocycle]. (Preview: if we do that, we can use the scaling factors of the conformal blocks to define time changes. Otherwise, it is not clear how to make the right time change).

It turns out that some work, and a series of miracles (i.e. some results of Forni and Guivarc'h-Raugi) allows us to do that. [If not, we would have had to make some unverifiable assumption like "simple Lyapunov spectrum"].

Definition 2.3. We say that a cocycle over $\mathbb{R}$ or $\mathbb{Z}$ has *semisimple Lyapunov spectrum* if in each Lyapunov block it has the form

$$\begin{pmatrix} \mathrm{Conf}(n_1) & 0 & \dots & 0 \\ 0 & \mathrm{Conf}(n_2) & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \mathrm{Conf}(n_k) \end{pmatrix} \tag{2.3}$$

2.2. **Review of Lecture 1. The affine linear structure on $\mathcal{H}(\alpha)$.** Let Σ denote the set of marked points on the surface M. (These include all the cone points). Fix a symplectic basis $\{\gamma_1, \dots, \gamma_n\}$ for $H_1(M, \Sigma, \mathbb{Z})$. The local coordinates are

$$\Psi\left((M,\omega)\right) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_n} \omega\right) \in \mathbb{C}^n \approx (\mathbb{R}^2)^n$$

which we write as a 2 by n matrix x. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ in these coordinates is:

$$x = \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \to gx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} A(g,x),$$

where $A(g,x) \in Sp(2g,\mathbb{Z}) \ltimes \mathbb{R}^k$ is the Kontsevich-Zorich cocycle. $A(g,x)$ is the change of basis one needs to perform to return the point gx to the fundamental domain.

We can write our local coordinates as

$$H^1(M,\Sigma,\mathbb{R}^2) \cong \mathbb{R}^2 \otimes H^1(M,\Sigma,\mathbb{R}).$$

Then, $SL(2,\mathbb{R})$ acts on the $\mathbb{R}^2$ part, and the cocycle $A(g,x)$ acts on the $H^1(M,\Sigma,\mathbb{R})$ part.

Notational Conventions. For $t \in \mathbb{R}$, let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Let $A = \{g_t \ : \ t \in \mathbb{R}\}$, $N = \{u_t \ : \ t \in \mathbb{R}\}$. Let $P = AN$.

Let X denote the stratum $\mathcal{H}_1(\alpha)$ or a finite cover. Let $\tilde{X}$ denote the universal cover of X. Let $\pi : \tilde{X} \to X$ denote the natural map. A point of X is a pair (M,ω), where M is a compact Riemann surface, and ω is a holomorphic 1-form on M. Let Σ denote the set of zeroes of ω. The cohomology class of ω in the relative cohomology group $H^1(M,\Sigma,\mathbb{C}) \cong H^1(M,\Sigma,\mathbb{R}^2)$ is a local coordinate on X.

Stable and Unstable foliations. We also use the notation,

$$\hat{W}^+(x) = (1,0) \otimes H^1(M,\Sigma,\mathbb{R}), \qquad \hat{W}^-(x) = (0,1) \otimes H^1(M,\Sigma,\mathbb{R}).$$

and let $W^+(x)$ and (resp. $W^-(x)$) denote the parts of $\hat{W}^+(x)$ (resp. $\hat{W}^-(x)$) which are orthogonal to the g_t orbit of x. Then $W^+[x]$ and $W^-[x]$ are the unstable and stable foliations for the action of g_t on X for $t > 0$.

The notation $V(x)$ and $V[x]$. Let $V(x)$ denote a subspace of $H^1(M,\Sigma,\mathbb{R}^2)$. Then we denote

$$V[x] = \{y \in X \ : \ y - x \in V(x)\}.$$

This makes sense in a neighborhood of x.

The notation $V^+(x)$ and $V^-[x]$. For a subspace $V(x) \in H^1(M,\Sigma,\mathbb{R})$, we write

$$V^+(x) = (1,0) \otimes V(x), \qquad V^-(x) = (0,1) \otimes V(x).$$

Then $V^+(x) \subset \hat{W}^+(x)$, $V^-(x) \subset \hat{W}^-(x)$. Thus, $V^+[x] \subseteq \hat{W}^+[x]$ and $V^-[x] \subseteq \hat{W}^-[x]$.

The forward and backward Lyapunov flags. Recall that $E_j(x)$ denotes the Lyapunov subspace corresponding to the Lyapunov exponent λ_j, as in (2.1). Let

$$V_i(x) = \bigoplus_{j=1}^{i} E_j(x), \qquad \hat{V}_i(x) = \bigoplus_{j=k+1-i}^{k} E_j(x).$$

Then we have the Lyapunov flags

$$\{0\} = V_0(x) \subset V_1(x) \subset \cdots \subset V_k(x) = H^1_\perp \tag{2.4}$$

and

$$\{0\} = \hat{V}_0(x) \subset \hat{V}_1(x) \subset \cdots \subset \hat{V}_k(x) = H^1_\perp \tag{2.5}$$

Lemma 2.4. *The subspaces $V_i(x)$ are locally constant along $W^+[x]$, i.e. for almost all $y \in W^+[x]$ close to x we have $V_i(y) = V_i(x)$.*

Proof. Note that

$$V_i(x) = \left\{ v \ : \ \lim_{t\to\infty} \frac{1}{t} \log \frac{\|(g_{-t})_* v\|}{\|v\|} \le -(1+\lambda_i) \right\}$$

Therefore, the subspace $V_i(x)$ depends only on the trajectory $g_{-t}x$ as $t \to \infty$. However, if $y \in W^+[x]$ then $g_{-t}y$ will for large t be close to $g_{-t}x$, and so in view of the affine structure, $(g_{-s})_*$ will be the same linear map for large s. □

A measurable flat connection on $W^+[x]$. The flag (2.4) is locally constant on a leaf of W^+ but the individual Lyapunov subspaces $E_i(x)$ need not be. Still, for a.e. x, we have the canonical map

$$P_{i,x} : E_i(x) \to V_i(x)/V_{i-1}(x).$$

Suppose $y \in W^+[x]$. Then,

$$V_i(x)/V_{i-1}(x) \cong V_i(y)/V_{i-1}(y)$$

(since all the subspaces involved are locally constant). Therefore

$$\begin{array}{ccc} E_i(x) & \xrightarrow{P_i^+(x,y)} & E_i(y) \\ {\scriptstyle P_{i,x}}\downarrow & & {\scriptstyle P_{i,y}}\downarrow \\ V_i(x)/V_{i-1}(x) & \xrightarrow{\cong} & V_i(y)/V_{i-1}(y) \end{array}$$

we get a map $P_i^+(x,y) : E_i(x) \to E_i(y)$ so the diagram commutes. Putting the $P_i^+(x,y)$ together, we get a linear map:

$$P^+(x,y) : W^+(x) \to W^+(y). \tag{2.6}$$

This map can be viewed as a flat connection on the leaves of W^+ which is equivariant undet the action of g_t. [The dependence on x, y is only measurable]. If we identify

$W^+(x)$ with $W^+(y)$ (using the Gauss-Manin connection), then the matrix of $P^+(x,y)$ is unipotent, preserving the flag (2.4).

By construction, the Lyapunov subspaces E_i are P^+-covariantly constant. Furthermore, we have the following:

Proposition 2.5. *If L is a g_t-invariant subbundle of W^+, then for a.e. $y \in W^+[x]$, $L(y) = P^+(x,y)L(x)$.*

Proof. See [EMi, Proposition 4.4]. □

Invariant linear foliations. Suppose L^+ is an g_t-invariant linear foliation of W^+ (i.e. if we write $L^+[x]$ for the leaf of L^+ passing through x, then $L^+[x] \subseteq W^+[x]$ is an affine subspace.) For example L^+ could be one of the V_i.

If L^+ is a g_t-invariant linear foliation, then for a.e. $x \in X$,

$$L^+(x) = \bigoplus_i (L^+(x) \cap E_i(x))$$

2.3. **Conditional Measures.** References for this section are [EL] and [CK].

The construction should be done much more generally, but here we only consider conditional measures along invariant linear foliations. Suppose L^+ is an invariant linear foliation. (L^+ could be W^+).

Consider a (reasonably nice) fundamental domain for the action of the mapping class group, and consider the partition $\mathcal{P}_{L^+}$ of the stratum X whose atoms are connected components of the intersection of the leaves of L^+ with the fundamental domain. We now define the conditional measure on each atom of the partition $\mathcal{P}_{L^+}$ by disintegrating the measure ν along the atoms.

Disintegration along leaves. We have

$$X = \bigsqcup_{y \in Y} L^+[y],$$

where for $x \in X$, $L^+[x]$ denotes the atom of the partition containing x (i.e. the connected component of the intersection of the leaf of L^+ through x with the fundamental domain).

In our setting, unlike [EKL] the space of atoms Y is Hausdorff. (This is because we consider only compact pieces of leaves. In [EKL] a much more elaborate construction is done, defining conditional measures on entire leaves.)

Let $\pi : X \to Y$ denote the map taking x to the piece of leaf (i.e. atom) containing x. We can then push the measure ν on X to a measure $\bar{\nu} \equiv \pi^*(\nu)$ on Y. Then, standard measure theory argument shows that for a.e. $y \in Y$ there exist measures $\hat{\nu}_y$ for each $y \in Y$ such that for continuous $h : X \to \mathbb{R}$,

$$\int_X h \, d\nu(x) = \int_Y \int_{L^+[y]} h \, d\hat{\nu}_y \, d\bar{\nu}(y).$$

Pullback to $\mathbb{R}^n$ using the affine structure. In our case, because of the affine structure, we have a (local) group, namely $\mathbb{R}^n$ where $n = \dim L$, acting on the sets $L^+[x]$. We can use this to identify $L^+[x]$ with $\mathbb{R}^n$, where the point x is identified with the origin. Then, we can pull back the measure $\hat{\nu}_x$ on $L^+[x]$ to a measure on $\mathbb{R}^n$ which we call $\nu_{L^+(x)}$. This is the *conditional measure* of ν along L^+ at x. (Sometimes the term "leafwise measure" is used).

From the construction, the following is clear:

Lemma 2.6. *Suppose $y \in L^+[x]$. Then,*

$$\nu_{L^+(y)} = (\mathrm{Trans}_{y-x})_* \nu_{L^+(x)}$$

where $\mathrm{Trans}_w : \mathbb{R}^n \to \mathbb{R}^n$ *is the map $v \to v + w$.*

We also have the following:

Lemma 2.7. *Suppose $D : X \to X$ is a linear map, which preserves the measure ν, and takes leaves of the foliation L^+ to leaves of the foliation L^+. Suppose $y = Dx$. Then,*

$$\nu_{L^+(y)} = D_* \nu_{L^+(x)}, \tag{2.7}$$

in the sense that the measures agree up to normalization on the set $DL^+(x) \cap L^+(y)$ where both are defined.

Remark. The reason we need the D_* in (2.7) is that the even though the measure ν (and thus its disintegration) is the same, the identification with $\mathbb{R}^n$ used to define $\nu_{L^+(y)}$ changes.

Recall that we are assuming that the measure ν is $P = AN$ invariant. Let

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \;:\; t \in \mathbb{R} \right\}, \quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

Lemma 2.8. *The measure ν is $\bar{N}$ invariant if and only if the conditional measure along the orbit foliation of $\bar{N}$ is Lebesgue.*

We have the following elementary:

Lemma 2.9. *If the conditional measure ν_{W^-} is non-trivial, then for any $\delta > 0$ and any compact $K \subset \mathcal{H}(\alpha)$ with $\nu(K) > 1 - \delta$ there exists a compact subset $K' \subset K$ with $\nu(K') > 1 - 2\delta'$ and $\rho = \rho(\delta) > 0$ so that: for any $q \in K'$ there exists $q' \in K \cap W^-[q]$ with*

$$\rho(\delta) < d(q, q') < 1.$$

Furthermore, $\delta' \to 0$ and $\rho(\delta) \to 0$ as $\delta \to 0$.

In other words, under the assumption that ν_{W-} is non-trivial, there is a set $K' \subset K$ of almost full measure such that every point $q \in K'$ has a "friend" in $q' \in W^-[q]$, with q' also in the "nice" set K, such that

$$d(q, q') \approx 1.$$

Thus, q can be chosen essentially anywhere in X.

2.4. **Entropy.** This subsection corresponds to [EMi, Appendix B].

Here we summarize the properties of entropy we will use. The references for this section are [EL], [MaT, §9]; see also [LY] and [BG]. We will state only the properties we need. All of this is true in much more generality.

Suppose $X = \mathcal{H}_1(\alpha)$, and let $T = g_1 : X \to X$ denote the time 1 map of the geodesic flow. Let ν be a measure invariant under T.

Let $h_\nu(T)$ denote the (measure-theoretic) entropy of T relative to the measure ν.

Theorem 2.10. *Suppose $L^+ \subseteq W^+$ is an invariant linear foliation. We can define a quantity $h_\nu(T, L^+)$, called "the contribution of L^+ to the entropy", so that the following properties hold:*

- (E0) $h_\nu(T, L^+)$ *depends only on ν only via the conditional measure ν_{L^+}.*
- (E1) $h_\nu(T) = h_\nu(T, W^+)$.
- (E2) $h_\nu(T, W^+) = 0$ *if and only if ν_{W^+} is trivial (i.e. the supported at the origin).*
- (E3) $h_\nu(T, L^+)$ *is maximized exactly when the conditional measure ν_{L^+} is Lebesgue. (In other words the Lebesque measure is the unique measure of maximal entropy).*
- (E4) *If ν_{L^+} is Lebesgue, then*

$$h_\nu(T, L^+) = \int_X J(x, L^+)\, d\nu(x)$$

where $J(x, L^+) = \det(T\ |_{L^+})(x)$ (relative to some choice of inner product at x.).

- (E5)

$$\int_X J(x, L^+)\, d\nu(x) = \sum_i \lambda_i \dim(L^+ \cap E_i)$$

where J is as in (E4).

- (E6) *If $L^+ \subset W^+$, then $h_\nu(T, L^+) \le h_\nu(T, W^+)$.*
- (E7) *If $\nu_{W^+(x)}$ is supported on $L^+(x)$, then $h_\nu(T, L^+) = h_\nu(T, W^+)$.*
- (E8) $h_\nu(T) = h_\nu(T^{-1})$.
- (E9) *All previous properties hold if we replace T by T^{-1}, W^+ by W^- and L^+ by L^-, where $L^- \subseteq W^-$ is an invariant linear foliation.*

2.5. **The proof of Step 1.** The aim of Step 1 is to prove the following:

Theorem 2.11. *Let ν be a P-invariant measure on the stratum X. Then ν is $SL(2,\mathbb{R})$-invariant. In addition, there exists an $SL(2,\mathbb{R})$-equivariant system of subspaces $\mathcal{L}(x) \subset H^1(M,\Sigma,\mathbb{R})$ such that for almost all x, the conditional measures of ν along $W^+[x]$ are the Lebesgue measures along $\mathcal{L}^+[x]$, and the conditional measures of ν along $W^-[x]$ are the Lebesgue measures along $\mathcal{L}^-[x]$.*

The general strategy is based on the idea of additional invariance which was used in the proofs of Ratner [Ra4], [Ra5], [Ra6], [Ra7] and Margulis-Tomanov [MaT].

In the sequel, we will often refer to a subspace $U^+(x) \subset W^+(x)$ on which we already proved that the conditional measure of ν is Lebesgue. The proof of Theorem 2.11 will be by induction, and in the beginning of the induction, $U^+[x] = Nx$.

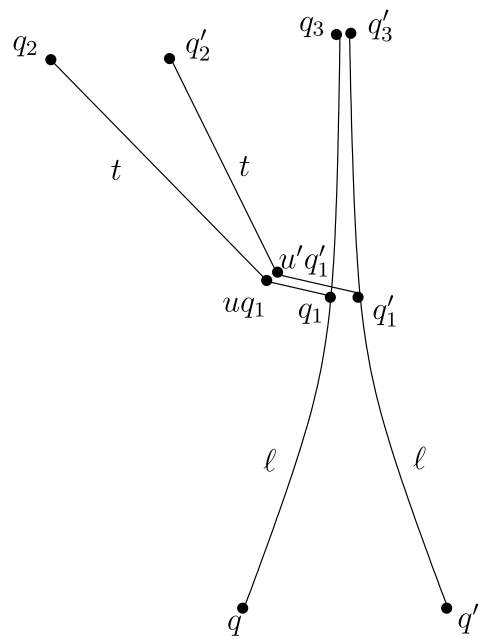

Figure 1. Outline of the proof of Theorem 2.11

Outline of the proof Theorem 2.11. Let $T = g_1$ be the time one map of the geodesic flow. Let ν be a P-invariant probability measure on X. Since ν is N-invariant, the conditional measure $\nu_{W^+(x)}$ is Lebesgue along the orbit $Nx \subset W^+[x]$. Then, by (E2), $h_\nu(T, W^+) > 0$. Then, by (E1), $h_\nu(T) > 0$. Hence, by (E8),

$$h_\nu(T^{-1}) > 0,$$

and then by (E9) and (E2), the conditional measure ν_{W^-} of ν along W^- is non-trivial. Therefore, by Lemma 2.9, on a set of almost full measure, we can pick points q and

q' in the support of ν such that q and q' are in the same leaf of W^- and $d(q,q') \approx 1$, see Figure 1.

Let $\ell > 0$ be a large parameter. Let $q_1 = g_\ell q$ and let $q_1' = g_\ell q'$. Then q_1 and q_1' are very close together. Pick $u \in U^+(q_1)$ and $u' \in U^+(q_1')$ with $\|u\| \approx \|u'\| \approx 1$, and consider the points uq_1 and $u'q_1'$. These are no longer in the same leaf of W^-, and we expect them to diverge under the action of g_t as $t \to +\infty$. Let t be chosen so that $q_2 = g_t u q_1$ and $q_2' = g_t u' q_1'$ be such that $d(q_2, q_2') \approx \epsilon$, where $\epsilon > 0$ is fixed.

Let

$$1 = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_n$$

denote the Lyapunov spectrum of the cocycle, and as in (2.1) let $E_i(x)$ denote the Lyapunov subspace corresponding to λ_i. Note that $E_0^+(x)$ corresponds to the unipotent direction inside the $SL(2,\mathbb{R})$ orbit of x. In the first step of the induction, $U^+(x) = E_0^+(x)$.

In general, we have, for $y \in U^+[x]$, in view of Lemma 2.4,

$$E_i^+(y) \subset \bigoplus_{j \leq i} E_j^+(x). \tag{2.8}$$

We say that the Lyapunov exponent λ_i is U^+-inert if for a.e x, $E_i^+(x) \not\in U^+(x)$ and also, for a.e $y \in U^+[x]$,

$$E_i^+(y) \subset U^+(x) + E_i(x).$$

(In other words, $E_i^+(x)$ is constant (modulo U^+) along $U^+[x]$.) Note that in view of (2.8), λ_1 is always U^+-inert. We now assume for simplicity that λ_1 is the only U^+-inert exponent. In this case, after possibly making a small change to u and u'. we may assume that $q_2' - q_2$ will be approximately in the direction of $E_1^+(q_2)$, see [EMi, §8] for the details.

For $x \in \mathcal{H}_1(\alpha)$, let $f_1(x)$ denote the conditional measure of ν along $E_1^+[x]$. We will also make the simplifying assumption that either E_1^+ is one dimensional, or else the restriction of the cocycle to E_1^+ consists of a single conformal block.

Let $q_3 = g_s q_1$ and $q_3' = g_s q_1'$ where $s > 0$ will be chosen later. By Lemma 2.7, as in [BQ]

$$f_1(q_2) = (D)_* f_1(q_3), \text{ and } f_1(q_2') = (D')_* f_1(q_3'), \tag{2.9}$$

where the linear transformation $D : E_1^+(q_3) \to E_1^+(q_2)$ is the restriction of the linear transformation $(g_t u g_s^{-1})_* : W^+(q_3) \to W^+(q_2)$ to the subspace $E_1^+(q_3)$.

In the case where E_1^+ is one-dimensional, the map D (like any linear transformation from $\mathbb{R}$ to $\mathbb{R}$) is given by multiplication by a scaling factor $\mathcal{D}$. In the other case we are considering, where the restriction of the cocycle to E_1^+ consists of a single conformal block, the map D is (essentially) conformal, so it is a (essentially) a product of the scaling factor $\mathcal{D}$ and an orthogonal matrix O.

We now choose $s > 0$ so that $\mathcal{D} \approx 1$, i.e. s is such that the amount of expansion along E_1^+ from q_1 to q_3 is equal to the amount of expansion along E_1^+ from uq_1 to q_2.

Note that D and D' are essentially the same bounded linear map. But q_3 and q_3' approach each other, so that

$$f_1(q_3) \approx f_1(q_3').$$

Hence, in in view of (2.9) and since $D \approx D'$,

$$f_1(q_2) \approx f_1(q_2').$$

Taking a limit as $\ell \to \infty$ of the points q_2 and q_2' we obtain points $\tilde{q}_2$ and $\tilde{q}_2'$ in the same leaf of E_1^+ and distance ϵ apart such that

$$f_1(\tilde{q}_2) = f_1(\tilde{q}_2'). \tag{2.10}$$

By Lemma 2.6, this means that the conditional measure $f_1(\tilde{q}_2)$ is invariant under a shift of size approximately ϵ. Repeating this argument with $\epsilon \to 0$ we obtain a point p such that $f_1(p)$ is invariant under arbitrarily small shifts. This implies that the conditional measure $f_1(p)$ restricts to Lebesgue measure on some subspace $U_{new}[p]$ of $E_1^+[p]$, which distinct from the orbit of N. Thus, we can enlarge $U^+(p)$ to be $U^+(p) \oplus U_{new}(p)$.

2.6. **Completion of the proof of Theorem 2.11.** This subsection roughly corresponds to [EMi, §13].

Let $\mathcal{L}^-(x)$ be the smallest linear subspace which contains the support of $\nu_{W^-(x)}$. Note that in the argument in §2.5, we were forced to pick $q_1' \in \mathcal{L}^-(q_1)$. The subspaces $\mathcal{L}^-$ form an invariant linear foliation of W^-.

We now do an induction argument. At each step of the induction, we have an invariant linear foliation $U^+(x) \subset W^+(x)$ such that the conditional measures $\nu_{U^+(x)}$ are Lebesgue. (At the beginning of the induction, $U^+ = N$). We now use the argument of §2.5 as an inductive step to enlarge U^+.

The stopping condition. This process has to stop at some point. The "stopping condition" is roughly that $\mathcal{L}^- \subset U^+$; however this does not make sense since $\mathcal{L}^- \subset W^-$ and $U^+ \subset W^+$. A more correct condition can be expressed as follows: We can write $U^+ = (1,0) \otimes U$, where $U(x) \subset H^1(M,\Sigma,\mathbb{R})$. Also, we may write $\mathcal{L}^-(x) = (0,1) \otimes \mathcal{L}(x)$, where $\mathcal{L}(x) \subset H^1(M,\Sigma,\mathbb{R})$.

Let $p : H^1(M,\Sigma,\mathbb{R}) \to H^1(M,\mathbb{R})$ denote the natural map. For $x = (M,\omega)$ let $S = S(x) \subset H^1(M,\mathbb{R})$ denote the span of Re (ω) and Im (ω). Then $S(x)$ is an $SL(2,\mathbb{R})$-invariant subspace. Let let $H^1_\perp \subset H^1(M,\Sigma,\mathbb{R})$ denote the pullback under p of the symplectic complement of $S(x)$.

Then, the "stopping condition" is that for a.e. x, $\mathcal{L}(x) \cap H^1_\perp \subset U(x) \cap H^1_\perp$. In other words, if, on a set of positive measure, we have $\mathcal{L}(x) \cap H^1_\perp \not\subset U(x) \cap H^1_\perp$, then we can use the argument of §2.5 to enlarge U^+ further.

The situation at the end of the induction. When we cannot enlarge U^+ any more, the following hold:

(a) $\mathcal{L}(x) \cap H^1_\perp \subset U(x) \cap H^1_\perp$.
(b) The conditional measures $\nu_{U^+(x)}$ are Lebesgue.

(c) The subspaces $U(x)$ and $U^+(x)$ are $P = AN$-equivariant (or else one could enlarge U^+ further). Also the subspaces $\mathcal{L}(x)$ and $\mathcal{L}^-(x)$ are A-equivariant.
(d) The conditional measures $\nu_{W^-(x)}$ are supported on $\mathcal{L}^-[x]$.

An Entropy Calculation. (This is essentially the calculation in [MaT]). Let I denote an indexing set for the Lyapunov exponents of the cocycle in $U \cap H^1_\perp$, $J \subset I$ denote an indexing set for the Lyapunov exponents of the cocycle (taken with multiplicity) in $\mathcal{L} \cap H^1_\perp$. Since $U \cap H^1_\perp$ is AN-invariant, by Theorem 3.1 we have,

$$\sum_{i \in I} \lambda_i \geq 0. \tag{2.11}$$

Set $t = 1$. Note that, because of the tensor product strucure the Lyapunov exponents of g_t on $U^+ \subset W^+$ are

$$\{2\} \cup \{(1 + \lambda_i) \ : \ i \in I\},$$

where the 2 comes from the direction of $N \subset SL(2,\mathbb{R})$, and the Lyapunov exponents of g_t^{-1} on $\mathcal{L}^- \cap H^1_\perp \subset W^-$ are $\{-1 + \lambda_j \ : \ j \in J\}$.

We now compute the entropy of $T = g_1$. We have, by (E6), (E4) and (E5),

$$h_\nu(T, W^+) \geq 2 + \sum_{i \in I}(1 + \lambda_i) = 2 + |I| + \sum_{i \in I} \lambda_i \geq 2 + |I| \tag{2.12}$$

where the 2 comes from the direction of N, and for the last estimate we used (2.11). Also, by (E9), (E3), (E4) and (E5),

$$\begin{aligned} h_\nu(T^{-1}, W^-) &\leq 2 + \sum_{j \in J}(1 - \lambda_j), && \text{where the 2 is the potential contribution of } \bar{N} \\ &\leq 2 + \sum_{i \in I}(1 - \lambda_i) && \text{since } (1 - \lambda_i) \geq 0 \text{ for all } i \\ &\leq 2 + |I| && \text{by (2.11)} \qquad (2.13) \end{aligned}$$

However, by (E1), (E8), and again (E1),

$$h_\nu(T, W^+) = h_\nu(T) = h_\nu(T^{-1}) = h_\nu(T^{-1}, W^-).$$

Therefore, all the inequalities in (2.12) and (2.13) are in fact equalities. In particular, $I = J$ (i.e. $\mathcal{L} \cap H^1_\perp = U \cap H^1_\perp$) and the direction of $\bar{N}$ is contained in $\mathcal{L}^-$. Also by (E7),

$$h_\nu(T^{-1}, W^-) = h_\nu(T^{-1}, \mathcal{L}^-) = 2 + \sum_{i \in I}(1 - \lambda_i),$$

which coincides with the entropy of Lebesgue measure on $\mathcal{L}^-$. Then, by (E3) we get that the conditional measures along $\nu_{\mathcal{L}^-(x)}$ are Lebesgue. In particular, the conditional measure in the direction of $\bar{N}$ is Lebesgue, therefore by Lemma 2.8, ν is $\bar{N}$-invariant. Hence ν is $SL(2,\mathbb{R})$-invariant.

By the definition of $\mathcal{L}^-$, the conditional measures $\nu_{W^-[x]}$ are supported on $\mathcal{L}^-[x]$. Thus, the conditional measures $\nu_{W^-[x]}$ are (up to null sets) precisely the Lebesgue measures on $\mathcal{L}^-[x]$. Since ν is $SL(2,\mathbb{R})$-invariant, we can argue by symmetry that the conditional measures $\nu_{W^+[x]}$ are precisely the Lebesgue measures on the smallest subspace containing $U^+[x]$. Since $U \cap H^1_\perp = \mathcal{L} \cap H^1_\perp$, this completes the proof of Theorem 2.11. □

2.7. **Technical Problems.**

Technical Problem #1. The argument requires that all eight points q, q', q_1, q_1', q_2, q_2', q_3, q_3' belong to some "nice" set K of almost full measure.

Our solution to this problem is based in part on Lemma 2.9. We also note the following trivial:

Lemma 2.12. *Suppose ν is a measure on X invariant under the flow g_t. Let $\hat{\tau} : X \times \mathbb{R} \to \mathbb{R}$ be a function such that for $t > s$,*

$$\kappa^{-1}(t-s) \le \hat{\tau}(x,t) - \hat{\tau}(x,s) \le \kappa(t-s). \tag{2.14}$$

Let $\psi_t : X \to X$ be given by $\psi_t(x) = g_{\hat{\tau}(x,t)}x$. Then, for any $K^c \subset X$ and any $\delta > 0$, there exists a subset $E \subset \mathbb{R}$ of density at least $(1-\delta)$ such that for $t \in E$,

$$\nu(\psi_t^{-1}(K^c)) \le (\kappa^2/\delta)\nu(K^c).$$

(We remark that the maps ψ_t are not a flow, since ψ_{t+s} is not in general $\psi_t \circ \psi_s$. However, Lemma 2.12 still holds.)

In [EMi, §7] we show that roughly, $q_2 = \psi_t(q)$, where ψ_t is as in Lemma 2.12. (A more precise statement, and the strategy for dealing with this problem is given at the beginning of [EMi, §5]). Then, to make sure that q_2 avoids a "bad set" K^c of small measure, we make sure that $q \in \psi_t^{-1}(K)$ which by Lemma 2.12 has almost full measure. Combining this with Lemma 2.9, we can see that we can choose q, q' and q_2 all in an a priori prescribed subset K of almost full measure. A similar argument can be done for all eight points, see [EMi, §12], where the precise arguments are assembled.

Technical Problem #2. Beyond the first step of the induction, the subspace $U^+(x)$ may not be locally constant as x varies along $W^+(x)$. This complication has a ripple effect on the proof. In particular, instead of dealing with the divergence of the points g_tuq_1 and $g_tu'q_1'$ we need to deal with the divergence of the affine subspaces $U^+[g_tuq_1]$ and $U^+[g_tu'q_1']$. As a first step, we project $U^+[g_tu'q_1']$ to the leaf of W^+ containing $U^+[g_tuq_1]$, to get a new affine subspace $\mathcal{U}'$. One way to keep track of the relative location of $U^+ = U^+[g_tu'q_1']$ and $\mathcal{U}'$ is (besides keeping track of the linear parts of U^+ and $\mathcal{U}'$) to pick a transversal Z to U^+, and to keep track of the intersection of $\mathcal{U}'$ and Z, see Figure 2.

However, since we do not know at this point that the cocycle is semisimple, we cannot pick Z in a way which is invariant under the flow. Thus, we have no choice except to pick some transversal $Z(x)$ to $U^+(x)$ at ν-almost every point $x \in X$, and then deal with the need to change transversal.

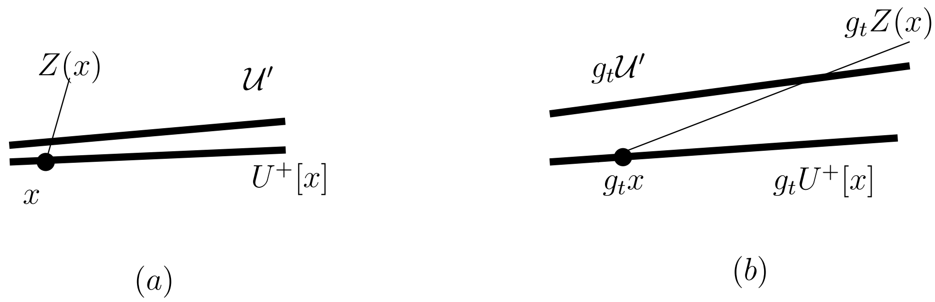

Figure 2.

(a) We keep track of the relative position of the subspaces $U^+[x]$ and $\mathcal{U}'$ in part by picking a transversal $Z(x)$ to $U^+[x]$, and noting the distance between $U^+[x]$ and $\mathcal{U}'$ along $Z[x]$.

(b) If we apply the flow g_t to the entire picture in (a), we see that the transversal $g_t Z[x]$ can get almost parallel to $g_t U^+[x]$. Then, the distance between $g_t U^+[x]$ and $g_t \mathcal{U}'$ along $g_t Z[x]$ may not be much larger then the distance between $g_t x \in g_t U^+[x]$ and the closest point in $g_t \mathcal{U}'$.

It turns out that the formula for computing how $\mathcal{U}' \cap Z$ changes when Z changes is non-linear (it involves inverting a certain matrix). However, we would really like to work with linear maps. This is done in two steps: first we show that we can choose the approximation $\mathcal{U}'$ and the transversals $Z(x)$ in such a way that changing transversals involves inverting a unipotent matrix. This makes the formula for changing transversals polynomial. In the second step, we embed the space of parametrers of affine subspaces near $U^+[x]$ into a certain tensor power space $\mathbf{H}(x)$ so that the on the level of $\mathbf{H}(x)$ the change of transversal map becomes linear. The details of this construction are in [EMi, §6].

Technical Problem #3. There may be more than one U^+-inert Lyapunov exponent. In that case, we do not have precise control over how q_2 and q_2' diverge. In particular the assumption that $q_2 - q_2'$ is nearly in the direction of $E_1^+(q_2)$ is not justified. Also we really need to work with $U^+[q_2]$ and $U^+[q_2']$. So let $\mathbf{v} \in \mathbf{H}(q_2)$ denote the vector corresponding to (the projection to $W^+(q_2)$ of) the affine subspace $U^+[q_2']$. (This vector $\mathbf{v}$ takes on the role of $q_2 - q_2'$). We have no a-priori control over the direction of $\mathbf{v}$ (even though we know that $\|\mathbf{v}\| \approx \epsilon$, and we know that $\mathbf{v}$ is almost contained in $\mathbf{E}(q_2) \subset \mathbf{H}(q_2)$, where $\mathbf{E}(x)$ is defined in [EMi, §8] as the union of the Lyapunov subspaces of $\mathbf{H}(x)$ corresponding to the U^+-inert Lyapunov exponents.)

The idea is to vary u (while keeping q_1, q_1', ℓ fixed). To make this work, we need to define a *finite* collection of subspaces $\mathbf{E}_{[ij],bdd}(x)$ of $\mathbf{H}(x)$ such that

(a) By varying u (while keeping q_1, q_1', ℓ fixed) we can make sure that the vector $\mathbf{v}$ becomes close to one of the subspaces $\mathbf{E}_{[ij],bdd}$, and

(b) For a suitable choice of point $q_3 = q_{3,ij} = g_{s_{ij}} q_1$, the map

$$(g_t u g_{-s_{ij}})_* \mathbf{E}_{[ij],bdd}(q_3) \to \mathbf{E}_{[ij],bdd}(q_2)$$

is a bounded linear map.

(c) Also, for a suitable choice of point $q_3' = q_{3,ij}' = g_{s_{ij}'} q_1$, the map

$$(g_t u g_{-s_{ij}'})_* \mathbf{E}_{[ij],bdd}(q_3') \to \mathbf{E}_{[ij],bdd}(q_2')$$

is a bounded linear map.

For the precise conditions see [EMi, Proposition 10.1] and [EMi, Proposition 10.2]. This construction is done in detail in [EMi, §10]. (The general idea is as follows: Suppose $\mathbf{v} \in \mathbf{E}_i(x) \oplus \mathbf{E}_j(x)$ where $\mathbf{E}_i(x)$ and $\mathbf{E}_j(x)$ are the Lyapunov subspaces corresponding to the U^+-inert (simple) Lyapunov exponents λ_i and λ_j. Then, if while varying u, the vector $\mathbf{v}$ does not swing towards either $\mathbf{E}_i$ or $\mathbf{E}_j$, we say that λ_i and λ_j are "synchronized". In that case, we consider the subspace $\mathbf{E}_{[i]}(x) = \mathbf{E}_i(x) \oplus \mathbf{E}_j(x)$ and show that (b) and (c) hold.)

The conditions (b) and (c) allow us to define in [EMi, §11] conditional measures f_{ij} on $W^+(x)$ which are associated to each subspace $\mathbf{E}_{[ij],bdd}$. In fact the measures are supported on the points $y \in W^+[x]$ such that the affine subspace $U^+[y]$ maps to a vector in $\mathbf{E}_{[ij],bdd}(x) \subset \mathbf{H}(x)$.

Technical Problem #4. More careful analysis (see the discussion following the statement of [EMi, Proposition 11.4]) shows that the maps D and D' of (2.9) are not exactly the same. Then, when one passes to the limit $\ell \to \infty$ one gets, instead of (2.10),

$$f_{ij}(\tilde{q}_2) = P^+(\tilde{q}_2, \tilde{q}_2')_* f_{ij}(\tilde{q}_2')$$

where $P^+ : W^+(\tilde{q}_2) \to W^+(\tilde{q}_2')$ the unipotent map given by (2.6). Thus the conditional measure $f_{ij}(\tilde{q}_2)$ is invariant under the composition of a translation of size ϵ and a unipotent map. Repeating the argument with $\epsilon \to 0$ we obtain a point p such that the conditional measure at p is invariant under arbitrarily small combinations of (translation + unipotent map). Thus does *not* imply that the conditional measure $f_{ij}(p)$ restricts to Lebesgue measure on some subspace of W^+, but it does imply that it is in the Lebesgue measure class along some polynomial curve in W^+. More precisely, for ν-a.e $x \in X$ there is a subgroup $U_{new} = U_{new}(x)$ of the affine group of $W^+(x)$ such that the conditional measure of $f_{ij}(x)$ on the polynomial curve $U_{new}[x] \subset W^+[x]$ is induced from the Haar measure on U_{new}. (We call such a set a "generalized subspace"). The exact definition is given in [EMi, §6].

Thus, during the induction steps, we need to deal with generalized subspaces. This is not a very serious complication since the general machinery developed in [EMi, §6] can deal with generalized subspaces as well as with ordinary affine subspaces. Also, this makes the "stopping condition" for the induction more complicated, see [EMi, §6.2].

3. Lecture 3

3.1. The Kontsevich-Zorich cocycle over the $SL(2,\mathbb{R})$ action. This subsection corresponds to [EMi, Appendix A].

Here we summarize some of the results we use from the fundamental work of Forni [Fo]. The recent preprint [FoMZ] contains an excellent presentation of these ideas and also some additional results which we use as well.

In the sequel, a subbundle L of the Hodge bundle is called *isometric* if the action of the Kontsevich-Zorich cocycle restricted to L is by isometries in the Hodge metric. We say that a subbundle is *isotropic* if the symplectic form vanishes identically on the sections, and *symplectic* if the symplectic form is non-degenerate on the sections. A subbundle is *irreducible* if it cannot be decomposed as a direct sum, and *strongly irreducible* if it cannot be decomposed as a direct sum on any finite cover of X.

Theorem 3.1. *Let ν be a P-invariant measure, and suppose L is a P-invariant ν-measurable subbundle of the Hodge bundle. Let $\lambda_1, \dots, \lambda_n$ be the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L. Then,*

$$\sum_{i=1}^{n} \lambda_i \geq 0.$$

Proof. Let $\{c_1, \dots, c_n\}$ be a Hodge-orthonormal basis for the bundle L at the point $S = (M, \omega)$, where M is a Riemann surface and ω is a holomorphic 1-form on M. For $g \in SL(2,\mathbb{R})$, let $V_S(g)$ denote the Hodge norm of the polyvector $c_1 \wedge \cdots \wedge c_n$ at the point gS, where the vectors c_i are transported using the Gauss-Manin connection. Since $V_S(kg) = V_S(g)$ for $k \in SO(2)$, we can think of V_S as a function on the upper half plane $\mathbb{H}$. From the definition of V_S and the multiplicative ergodic theorem, we see that for ν-almost all $S \in X$,

$$\lim_{t\to\infty} \frac{\log V_S(g_{-t})}{t} = -\sum_{i=1}^{n} \lambda_i, \tag{3.1}$$

where the λ_i are as in the statement of Theorem 3.1.

Let Δ_{hyp} denote the hyperbolic Laplacian operator (along the Teichmüller disk). By [FoMZ, Lemma 2.8] (see also [Fo, Lemma 5.2 and Lemma 5.2']) there exists a non-negative function $\Phi : X \to \mathbb{R}$ such that for all $S \in X$ and all $g \in SL(2,\mathbb{R})$,

$$(\Delta_{hyp} \log V_S)(g) = \Phi(gS).$$

We now claim that the Kontsevich-Zorich type formula

$$\sum_{i=1}^{n} \lambda_i = \int_X \Phi(S)\, d\nu(S) \tag{3.2}$$

holds, which clearly implies the theorem. The formula (3.2) is proved in [FoMZ] (and for the case of the entire stratum in [Fo]) under the assumption that the measure ν is invariant under $SL(2,\mathbb{R})$. However, in the proofs, only averages over "large circles" in $\mathbb{H} = SO(2)\backslash SL(2,\mathbb{R})$ are used. It is then easy to modify the proof so that it works for P-invariant measures instead. □

Theorem 3.2. *Let ν be an $SL(2,\mathbb{R})$-invariant measure, and suppose L is an $SL(2,\mathbb{R})$-invariant ν-measurable subbundle of the Hodge bundle. Suppose all the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L vanish. Then, the action of the Kontsevich-Zorich cocycle on L is isometric with respect to the Hodge inner product, and the orthogonal complement $L^{\perp}$ of L with respect to the Hodge inner product is also an $SL(2,\mathbb{R})$-invariant subbundle.*

Proof. The first assertion is the content of [FoMZ, Theorem 3]. The second assertion then follows from [FoMZ, Lemma 4.3] □

Theorem 3.3. *Let ν be an $SL(2,\mathbb{R})$-invariant measure, and suppose L is an $SL(2,\mathbb{R})$-invariant ν-measurable subbundle of the Hodge bundle. Suppose L is isotropic. Then all the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L vanish (and thus Theorem 3.2 applies to L).*

Proof. Let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

denote the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L. Since ν is $SL(2,\mathbb{R})$-invariant, it is in particular, P-invariant. Then by Theorem 3.1,

$$\lambda_1 + \cdots + \lambda_n \geq 0.$$

In particular $\lambda_1 \geq 0$.

By [FoMZ, Corollary 3.1] the following formula holds: for all $1 \leq j \leq n$,

$$\lambda_1 + \cdots + \lambda_j = \int_X \Phi_j(x)\, d\nu(x),$$

where $\Phi_1 \leq \Phi_2 \leq \cdots \leq \Phi_n$. (This formula is proved in [Fo] for the case where ν is Lebesgue measure and L is the entire Hodge bundle). Therefore $\lambda_1 + \cdots + \lambda_j$ increases as j increases, and hence $\lambda_j \geq 0$ for all $1 \leq j \leq n$.

Note that

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}.$$

Therefore the flow by g_t is conjugate in $SL(2,\mathbb{R})$ to the time reversed flow. However, time reversal changes the signs of the Lyapunov exponents.

Since ν is assumed to be $SL(2,\mathbb{R})$-invariant, this implies that e.g. the $-\lambda_j \geq 0$ for all j. Hence, all the Lyapunov exponents λ_j are 0. □

Theorem 3.4. *Let ν be an $SL(2,\mathbb{R})$-invariant measure. Then, the ν-algebraic hull of the Kontsevich-Zorich cocycle is semisimple.*

Proof. Suppose L is an invariant subbundle. It is enough to show that there exists an invariant complement to L. Let the symplectic complement $L^\dagger$ of L be defined by

$$L^\dagger(x) = \{v \ : \ v \wedge u = 0 \quad \text{for all } u \in L(x)\}.$$

Then, $L^\dagger$ is also an $SL(2,\mathbb{R})$-invariant subbundle, and $Q = L \cap L^\dagger$ is isotropic. By Theorem 3.3, Q is isometric, and $Q^\perp$ is also $SL(2,\mathbb{R})$-invariant. Then,

$$L = Q \oplus (L \cap Q^\perp), \qquad L^\dagger = Q \oplus (L^\dagger \cap Q^\perp),$$

and

$$H^1(M,\mathbb{R}) = Q \oplus (L \cap Q^\perp) \oplus (L^\dagger \cap Q^\perp)$$

Thus, $L^\dagger \cap Q^\perp$ is an $SL(2,\mathbb{R})$-invariant complement to L. ☐

3.2. **Semisimplicity of the Lyapunov spectrum.** In this subsection (corresponding to [EMi, Appendix C], we show that (in the random walk context) for an $SL(2,\mathbb{R})$-invariant measure ν, the Kontsevich-Zorich cocycle has semisimple Lyapunov spectrum, as defined in §2.1. This argument is very general, and is mostly due to Y. Guivarc'h and A. Raugi, [GR1] and [GR2]. Part of the argument is due to R. Zimmer.

Random Walks. We choose a compactly supported absolutely continuous measure μ on $SL(2,\mathbb{R})$. We also assume that μ is spherically symmetric. Let ν be any ergodic μ-invariant stationary measure on X. By Furstenberg's theorem [NZ, Theorem 1.4]

$$\nu = \int_0^{2\pi} (r_\theta)_* \nu_0 \, d\theta$$

where $r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and ν_0 is a measure invariant under $P = AN \subset SL(2,\mathbb{R})$. Then, by Theorem 2.11, ν_0 is $SL(2,\mathbb{R})$-invariant. Therefore the stationary measure ν is also in fact $SL(2,\mathbb{R})$-invariant.

The subbundle L. Let L be an $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle. Let $A : SL(2,\mathbb{R}) \times X \to GL(L)$ denote the restriction of the Kontsevich-Zorich cocycle to L.

The forward shift map. Let B be the space of (one-sided) infinite sequences of elements of $SL(2,\mathbb{R})$. (We think of B as giving the "future" trajectory of the random walk.) We denote elements of B by the letter a (following the convention that these refer to "future" trajectiories). If we write $a = (a_1, a_2, \dots)$ then Let $T : B \to B$ be the shift map. (In our interpretation, T takes us one step into the future). Thus, $T(a_1, a_2, \dots) = (a_2, a_3, \dots)$.

We now define the skew-product map $\hat{T} : B \times X \to B \times X$ by the formula

$$\hat{T}(a, x) = (Ta, a_1 x).$$

By the Osceledec multiplicative ergodic theorem, for $\beta\times\nu$ almost every $(a,x)\in B\times X$ there exists a Lyapunov flag

$$\{0\} = \hat{V}_0 \subset \hat{V}_1(a,x) \subset \hat{V}_2(a,x) \subset \hat{V}_k(a,x) = L. \tag{3.3}$$

It is easy to see that the $\beta\times\nu$ is a $\hat{T}$-invariant measure on $B\times X$.

Recall that the notion of semisimple Lyapunov spectrum was defined in §2.1. In this subsection we indicate the proof of the following theorem:

Theorem 3.5. *Suppose A is strongly irreducible. Then the Lyapunov spectrum of T on the subbundle L is semisimple. Furthermore, the restriction of A to the top Lyapunov subspace $\hat{V}_1$ is homologous to a cocycle consisting of a single conformal block.*

Remark. It is possible to define semisimplicity of the Lyapunov spectrum in the context of the action of $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix} \subset SL(2,\mathbb{R})$ (instead of the random walk). Then the analogue of Theorem 3.5 remains true.

To simplify the ideas, we outline the proof in the case where $\dim L = 2$. Then, without loss of generality, we may assume that A takes values in the special linear group $SL(L)\cong SL(2,\mathbb{R})$. (In this case simpler arguments are available, but we follow the argument which works in the general case).

Theorem 3.5 is becomes the assertion that either $\lambda_1 > -\lambda_1 = \lambda_2$, or else the cocycle A is homologous to a cocycle taking values in the orthogonal group $O(2)$.

Notation. We will use the notation

$$A_n(a,x) = A(a_n\dots a_1, x).$$

A trichotomy. We have three possibilities:

(A). For almost all $(a,x)\in B\times X$, $\|A_n(a,x)\|$ grows exponentially with n.

(B). For almost all $(a,x)\in B\times X$, $\|A_n(a,x)\|$ is bounded, i.e there exists a function $C: X\to\mathbb{R}$, such that

$$\|A_n(a,x)\| \le C(x)C(a_n\dots a_1 x). \tag{3.4}$$

(C). Neither (A) nor (B) holds.

The proof of Theorem 3.5 has three parts:

(i) In Case (A), we have $\lambda_1 > \lambda_2$.

(ii) In Case (B), the cocycle is homologous to a cocycle taking values in $O(2)$.

(iii) Case (C) never happens, so we always have either (A) or (B).

Parts (i) and (ii) seem quite reasonable, but part (iii) is very surprising. Why can't the $\|A_n(a,x)\|$ have some complicated subexponential behaviour?

Action on $\hat{X}$. Let ν be an ergodic $SL(2,R)$-invariant measure on X. Let the $\mathbb{P}^1(L)$ denote the projective space of L (i.e. the circle). Let $\hat{X} = X \times \mathbb{P}^1(L)$. We then have an action of $SL(2,\mathbb{R})$ on $\hat{X}$, by

$$g \cdot (x, W) = (gx, A(g,x)W).$$

Let $\hat{\nu}$ be an ergodic μ-stationary measure on $\hat{X}$ which projects to ν under the natural map $\hat{X} \to X$.

We may write

$$d\hat{\nu}(x, U) = d\nu(x)\, d\eta_x(U),$$

where η_x is a measure on $\mathbb{P}^1(L)$.

Lemma 3.6. *For a.e. x, the measure η_x on the circle $\mathbb{P}^1(L)$ has no atoms (i.e the η_x measure of any point is 0).*

Outline of proof. Essentially, the idea is that any atom of η_x will lead to an invariant 1-dimensional subspace of L, which contradicts the strong irreducibility of the cocycle. □

For $b \in B$, let $\hat{\nu}_a$ be as defined in [BQ, Lemma 3.2], i.e.

$$\hat{\nu}_a = \lim_{n\to\infty} (a_n \dots a_1)_*^{-1} \hat{\nu},$$

The limit exists by the martingale convergence theorem.

We disintegrate

$$d\hat{\nu}_a(x, v) = d\nu(x)\, d\eta_{a,x}(v).$$

where $\eta_{a,x}$ is a measure on the circle $\mathbb{P}^1(L)$.

We can now refine the strategy of the proof. We show:

(a) If $\eta_{a,x}$ has atoms, then $\lambda_1 > \lambda_2$.
(b) If $\eta_{a,x}$ does not have atoms, then (B) holds, and the cocycle is homologous to a cocycle taking values in $O(2)$.

Recall the following Lemma of Furstenberg (which is trivial in this 2-dimensional case):

Lemma 3.7. *Suppose μ_1 and μ_2 are two measures on the circle, and $g_n \in SL(2,\mathbb{R})$ a sequence such that $g_n\mu_1$ converges to μ_2. Then either the g_i are bounded (i.e. contained in a compact subset of $SL(2,\mathbb{R})$), or μ_2 is supported on at most two points.*

Proof of Lemma 3.7. This is immediate from the "north-south dynamics" of the action of $SL(2,\mathbb{R})$ on the circle (by Mobius transformations). □

Definition 3.8 (Uniformly (ϵ,δ)-regular). Suppose $\epsilon > 0$ and $\delta > 0$ are fixed. A sequence of measures η_j on the circle $\mathbb{P}^1(L)$ is *uniformly (ϵ,δ)-regular* if for any interval I in $\mathbb{P}^1(L)$ of length at most ϵ, and all j,

$$\eta_j(I) < \delta.$$

We will also need the following variant of Lemma 3.7.

Lemma 3.9. *Suppose $g_n \in SL(L) = SL(2,\mathbb{R})$ is a sequence of linear transformations, and η_n is a sequence of uniformly (ϵ,δ)-regular measures on $\mathbb{P}^1(L)$, and suppose also that $g_n\eta_n \to \lambda$. Then,*

(i) *If λ has no atoms, then the sequence g_n is bounded (in terms of ϵ, δ and λ).*

(ii) *Suppose the measure λ has an atom at the point $W \subset \mathbb{P}^1(L)$. Then, $\|g_n\| \to \infty$, and if we write $g_n = K(n)D(n)K'(n)$, where $K(n)$ and $K'(n)$ are orthogonal relative to the standard basis $\{e_1,\dots e_m\}$, and $D(n) = \begin{pmatrix} d_1(n) & 0 \\ 0 & d_2(n)\end{pmatrix}$ with $d_1(n) \ge d_2(n)$, we have*
$$K(n)e_1 \to W,$$
so in particular $K(n)$ converges as $n \to \infty$.

Proof of (b). We have for β-a.e. $a \in B$,
$$\lim_{n\to\infty} (a_n \dots a_1)_*^{-1}\hat{\nu} = \hat{\nu}_a.$$
Therefore, on a set of full measure,
$$\lim_{n\to\infty} A((a_n\dots a_1)^{-1}, y)\eta_y = \eta_{a,x}, \qquad \text{where } y = a_n\dots a_1 x.$$
Note that by the cocycle relation,
$$A((a_n\dots a_1)^{-1}, y) = A(a_n\dots a_1, x)^{-1} \equiv A_n(a,x)^{-1}$$
Hence, on a set of full measure,
$$\lim_{n\to\infty} A_n(a,x)^{-1}\eta_y = \eta_{a,x}, \qquad \text{where } y = a_n\dots a_1 x. \tag{3.5}$$

We apply Lemma 3.9 (i) with $g_n = A_n(a,x)^{-1}$ and $\lambda = \eta_{a,x}$. Then we immediately get (3.4). The rest of the argument is part of Zimmer's amenable reduction. Rougly, the idea is to construct an invariant inner product on L by taking a limit point of
$$\frac{1}{n}\sum_{k=1}^{n}\langle A_n(a,x)^{-1}u, A_n(a,x)^{-1}v\rangle.$$
□

For the proof of (a), we also need the following:

Lemma 3.10 (Furstenberg (special case))**.** *Let $\sigma_1 : G \times \hat{X} \to \mathbb{R}$ be given by*
$$\sigma_1(g,x,v) = \log\frac{\|A(g,x)v\|}{\|v\|}$$
Then, we have
$$\lambda_1 = \int_G\int_{\hat{X}} \sigma_1(g,x,v)\,d\hat{\nu}(x,v)\,d\mu(g).$$

Note that $\sigma_1 : SL(2,\mathbb{R}) \times \hat{X} \to \mathbb{R}$ is an additive cocycle, i.e.

$$\sigma_1(g_1 g_2, (x,v)) = \sigma_1(g_1, g_2(x,v)) + \sigma_2(g_2, x, v).$$

Proof of (a). Fix $\delta > 0$. Then, there exists $\epsilon > 0$ and a compact $\mathcal{K}_\delta \subset X$ such that the family of measures $\{\eta_x\}_{x \in \mathcal{K}_\delta}$ is uniformly (ϵ, δ)-regular. Let

$$\mathcal{N}_\delta(a,x) = \{n \in \mathbb{N} \;:\; a_n \dots a_1 x \in \mathcal{K}_\delta\}.$$

Write

$$A(a_n \dots a_1, x)^{-1} = K_n(a,x) D_n(a,x) K_n'(a,x)$$

where K_n and K_n' are orthogonal, and D_n is diagonal with decreasing entries. Then, by applying Lemma 3.9(ii) to (3.5), we see that for $\beta \times \nu$ almost all (a,x), as $n \to \infty$ along $\mathcal{N}_\delta(a,x)$, we have $\|A_n(a,x)\| \to \infty$, and $K_n(a,x)$ converges. We have

$$A(a_n \dots a_1, x) = K_n'(a,x)^{-1} D_n(a,x)^{-1} K_n(a,x)^{-1}.$$

Note that $K_n(a,x)^{-1}$ converges, and η_x is non-atomic. Therefore, for any $\epsilon_1 > 0$ there exists a subset H' of $B \times \hat{X}$ of measure at least $1 - \epsilon_1$, an integer $M > 0$ and a constant $C > 0$ such that for all $(a,x,v) \in H'$, all $n > M \in \mathcal{N}_\delta(a,x)$, we have

$$C > \frac{\|A(a_n \dots a_1, x)v\|}{\|A(a_n \dots a_1, x)\| \|v\|} > \frac{1}{C}, \tag{3.6}$$

Then, as $n \to \infty$ in $\mathcal{N}_\delta(b,x)$, for η_x-almost all v,

$$\log \frac{\|A(a_n \dots a_1, x)v\|}{\|v\|} \to \infty \tag{3.7}$$

Since ϵ_1 is arbitrary, (3.7) holds as $n \to \infty$ along $\mathcal{N}_\delta(a,x)$ for $\beta \times \hat{\nu}$ almost all $(b,x,v) \in B \times \hat{X}$. Then, the left hand side of (3.7) is exactly

$$\sigma_1(a_n \dots a_1, x, v) = \sum_{j=1}^{n-1} \sigma_1(\hat{T}^j(a,x,v)).$$

Also, we have $n \in \mathcal{N}_\delta(a,x)$ if and only if $\hat{T}^n(a,x) \in \mathcal{K}_\delta$. Then, by Lemma 3.12 below,

$$\int_{B \times \hat{X}} \sigma_1(a,x,v)\, d\beta(b) d\hat{\nu}(x,v) > 0.$$

But the left-hand-side of the above equation is λ_1 by Furstenberg's formula Lemma 3.10. Thus $\lambda_1 > 0$. This completes the proof of (a). □

3.3. **An Ergodic Lemma.** We recall the following well-known lemma:

Lemma 3.11. *Let $T : \Omega \to \Omega$ be a transformation preserving a probability measure β. Let $F : \Omega \to \mathbb{R}$ be an L^1 function. Suppose that for β-a.e. $x \in \Omega$,*

$$\liminf \sum_{i=1}^{n} F(T^i x) = +\infty.$$

Then $\int_\Omega F\, d\beta > 0$.

Proof. This lemma is due to Atkinson [At], and Kesten [Ke]. See also [GM, Lemma 5.3], and the references quoted there. □

We will need the following variant:

Lemma 3.12. *Let $T : \Omega \to \Omega$ be a transformation preserving a probability measure β. Let $F : \Omega \to \mathbb{R}$ be an L^1 function. Suppose for every $\epsilon > 0$ there exists $K_\epsilon \subset \Omega$ with $\beta(K_\epsilon) > 1 - \epsilon$ such that for β-a.e. $x \in \Omega$,*

$$\liminf \{ \sum_{i=1}^{n} F(T^i x) \quad : \quad T^n x \in K_\epsilon \} = +\infty.$$

Then $\int_\Omega F\, d\beta > 0$.

Outline of Proof. We just consider the first return map to K_ϵ, and apply Lemma 3.11. □

3.4. **The backwards shift.** In the next lecture, we need to consider the so-called backwards walk and bacwards shift map.

As above, let B be the space of (one-sided) infinite sequences of elements of $SL(2,\mathbb{R})$, but now we think of B as giving the "past" trajectory of the random walk.) Let $T : B \to B$ be the shift map. (In our interpretation, T takes us one step into the past). We define the skew-product map $\hat{T} : B \times X \to B \times X$ by

$$T(b, x) = (Tb, b_0^{-1} x), \qquad \text{where } b = (b_0, b_1, \dots)$$

We define the measure β on B to be $\mu \times \mu \dots$. Then let β^X denote the measure $\beta \times \nu$ on $B \times X$. This measure is $\hat{T}$-invariant.

By the Osceledec multiplicative ergodic theorem, for β^X almost every $(b, x) \in B \times X$ there exists a Lyapunov flag

$$\{0\} = V_0 \subset V_1(b, x) \subset V_2(b, x) \subset V_k(b, x) = L.$$

The following theorem is easily deduced from Theorem 3.5:

Theorem 3.13. *Suppose A is strongly irreducible. Then the Lyapunov spectrum of $\hat{T}$ on the subbundle L is semisimple. Furthermore, the restriction of A to the top Lyapunov subspace V_k/V_{k-1} is homologous to a cocycle consisting of a single conformal block.*

4. Lecture 4

This lecture corresponds to [EMi, §14-§16].

4.1. Random Walks. We choose a compactly supported absolutely continuous measure μ on $SL(2,\mathbb{R})$. We also assume that μ is spherically symmetric. Let ν be any ergodic μ-stationary measure on X. By Furstenberg's theorem [NZ, Theorem 1.4]

$$\nu = \int_0^{2\pi} (r_\theta)_* \nu_0 \, d\theta$$

where $r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and ν_0 is a measure invariant under $P = AN \subset SL(2,\mathbb{R})$. Then, by Theorem 2.11, ν_0 is $SL(2,\mathbb{R})$-invariant. Therefore the stationary measure ν is also in fact $SL(2,\mathbb{R})$-invariant.

By Theorem 2.11, there is a $SL(2,\mathbb{R})$-equivariant family of subspaces $U(x) \subset H^1(M,\Sigma,\mathbb{R})$, and that the conditional measures of ν along $U_{\mathbb{C}}(x)$ are Lebesgue.

In what follows we assume there is only one zero (so no relative homology). We also make the following simplifying assumption:

$$H^1(M,\mathbb{R}) = U(x) \oplus \mathcal{L}(x)$$

where the action of the cocycle on $\mathcal{L}(x)$ is *simple*, so no invariant subspaces inside $\mathcal{L}(x)$. (In the general case we have $\mathcal{L}(x) = \bigoplus_k \mathcal{L}_k(x)$ where the $\mathcal{L}_k$ are simple).

The backwards shift map. Let B be the space of (one-sided) infinite sequences of elements of $SL(2,\mathbb{R})$. (We think of B as giving the "past" trajectory of the random walk.) Let $T : B \to B$ be the shift map. (In our interpretation, T takes us one step into the past). We define the skew-product map $T : B \times X \to B \times X$ by

$$T(b,x) = (Tb, b_0^{-1}x), \qquad \text{where } b = (b_0, b_1, \dots)$$

We define the measure β on B to be $\mu \times \mu \dots$.

We have the Lyapunov flag for T

$$\{0\} = V_0 \subset V_1(b,x) \subset \dots V_n(b,x) = \mathcal{L}(x).$$

The two-sided shift space. Let $\tilde{B}$ denote the two-sided shift space. We define the measure $\tilde{\beta}$ on $\tilde{B}$ as $\dots \times \mu \times \mu \times \dots$.

Notation. For $a, b \in B$ let

$$a \vee b = (\dots, a_2, a_1, b_0, b_1, \dots) \in \tilde{B}. \tag{4.1}$$

If $\omega = a \vee b \in \tilde{B}$, we think of the sequence

$$\dots, \omega_{-2}, \omega_{-1} = \dots a_2, a_1$$

as the "future" of the random walk trajectory. (In general, following [BQ], we use the symbols b, b' etc. to refer to the "past" and the symbols a, a' etc. to refer to the "future").

The opposite Lyapunov flag. Note that on the two-sided shift space $\tilde{B} \times X$, the map T is invertible. Thus, for each $a \vee b \in B$, we have the Lyapunov flag for T^{-1}:

$$\{0\} = \hat{V}_0 \subset \hat{V}_1(a,x) \subset \dots \hat{V}_n(a,x) = \mathcal{L}(x),$$

(As reflected in the above notation, this flag depends only on the "future" i.e. "a" part of $a \vee b$).

The top Lyapunov exponent $\hat{\lambda}$. Let $\hat{\lambda} \geq 0$ denote the top Lyapunov exponent in $\mathcal{L}$.

The following lemma is a consequence of (the general version of) Lemma 3.6:

Lemma 4.1. *For every $\delta > 0$ and every $\delta' > 0$ there exists $E_{good} \subset X$ with $\nu(E_{good}) > 1 - \delta$ and $\sigma = \sigma(\delta) > 0$, such that for any $x \in E_{good}$, and any vector $w \in \mathcal{L}(x)$,*

$$\beta\left(\{a \in B \ : \ d(w, \hat{V}_{n-1}(a,x)) > \sigma\}\right) > 1 - \delta' \tag{4.2}$$

(In (4.2), $d(\cdot,\cdot)$ is some distance on the projective space $\mathbb{P}^1(H^1(M,\mathbb{R}))$).

The action on $H^1(M,\Sigma,\mathbb{C})$. By the multiplicative ergodic theorem applied to the action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$, for β-almost all $b \in B$,

$$\sigma_0 = \lim_{n\to\infty} \frac{1}{n} \log \|b_0 \dots b_n\|$$

where $\sigma_0 > 0$ is the Lyapunov exponent for the measure μ on $SL(2,\mathbb{R})$. Also, by the multiplicative ergodic theorem, for β-almost all $b \in B$ there exists a one-dimensional subspace $W_+(b) \subset \mathbb{R}^2$ such that $v \in W_+(b)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|b_n^{-1} \dots b_0^{-1} v\| = -\sigma_0.$$

For $x = (M,\omega)$ let $i_x : \mathbb{R}^2 \to H^1(M,\mathbb{R})$ denote the map $(a,b) \to a \operatorname{Re} x + b \operatorname{Im} x$. Let $W_+^{\perp}(b,x) \subset H^1(M,\Sigma,\mathbb{R})$ be defined by

$$W_+^{\perp}(b,x) = \{v \in H^1(M,\Sigma,\mathbb{R}) \ : \ p(v) \wedge w = 0 \text{ for all } w \in i_x(W_+(b)), \}$$

and let

$$W^+(b,x) = W_+(b) \otimes W_+^{\perp}(b,x).$$

Since we identify $\mathbb{R}^2 \otimes H^1(M,\Sigma,\mathbb{R})$ with $H^1(M,\Sigma,\mathbb{C})$, we may consider $W^+(b,x)$ as a subspace of $H^1(M,\Sigma,\mathbb{C})$. This is the "stable" subspace for T. (Recall that T moves into the past).

For a "future trajectory" $a \in B$, we can similarly define a 1-dimensional subspace $W_-(a) \subset \mathbb{R}^2$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|a_n \dots a_1 v\| = -\sigma_0 \quad \text{ for } v \in W_-(a).$$

Let $A : SL(2,\mathbb{R}) \times X \to Hom(H^1(M,\Sigma,\mathbb{R}), H^1(M,\Sigma,\mathbb{R}))$ denote the Kontsevich-Zorich cocycle. We then have the cocycle

$$\hat{A} : SL(2,\mathbb{R}) \times X \to Hom(H^1(M,\Sigma,\mathbb{C}), H^1(M,\Sigma,\mathbb{C}))$$

given by

$$\hat{A}(g,x)(v \otimes w) = gv \otimes A(g,x)w$$

and we have made the identification $H^1(M,\Sigma,\mathbb{C}) = \mathbb{R}^2 \otimes H^1(M,\Sigma,\mathbb{R})$.

From the definition we see that the Lyapunov exponents of $\hat{A}$ are of the form $\pm\sigma_0 + \lambda_i$, where the λ_i are the Lyapunov exponents of A.

4.2. **Time changes and suspensions.** There is a natural "forgetful" map $f : \tilde{B}^X \to B^X$ which carries $\tilde{\beta}^X$ to β^X. We extend functions on B^X to $\tilde{B}^X$ by making them constant along the fibers of f.

The cocycle τ. By Theorem 3.5, there is a inner product $\langle\, ,\rangle_{b,x}$ defined on $W_+(b) \otimes V_1(b,x)$ and a cocycle $\tau : B \times X \to \mathbb{R}$ such that for all $u, v \in W_+(b) \otimes V_1(b,x)$,

$$\langle \hat{A}(b_0^{-1},x)u, \hat{A}(b_0^{-1},x)v\rangle_{Tb,b_0^{-1}x} = e^{-\tau(b,x)}\langle u,v\rangle_{b,x}, \tag{4.3}$$

Recall that β denotes the measure on B which is given by $\mu \times \mu \dots$.

Suspension. Let $B^X = B \times X \times (0,1]$. Let β^X denote the measure on B^X given by $\beta \times \nu \times dt$, where dt is the Lebesgue measure on $(0,1]$. In B^X we identify $(b,x,0)$ with $(T(b,x),1)$, so that B^X is a suspension of T. We can then define a flow $T_t : B^X \to B^X$ in the natural way. Then T_t preserves a measure β^X.

The time change. Here we differ slightly from [BQ] since we would like to have several different time changes of the flow T_t on the same space. Hence, instead of changing the roof function, we keep the roof function constant, but change the speed in which one moves on the $[0,1]$ fibers.

Let $T_t^\tau : B^X \to B^X$ be the time change of T_t where on $(b,x) \times [0,1]$ one moves at the speed $\tau(b,x)$. More precisely, we set

$$T_t^\tau(b,x,s) = (b,x,s-\tau(b,x)t), \quad \text{if } 0 < s - \tau(b,x)t \le 1, \tag{4.4}$$

and extend using the identification $((b,x),0) = (T(b,x),1)$.

Then T_ℓ^τ is the operation of moving backwards in time far enough so that the cocycle multiplies the direction of the top Lyapunov exponent in $\mathcal{L}$ by $e^{-\ell}$. It is easy to see that the flow T_ℓ^τ preserves a measure $\beta^{\tau,X}$ on B^X which is in the same measure class as β^X. (In fact, $\beta^{\tau,X}$ differs from β^X by a multiplicative constant on each $[0,1]$ fiber).

The map T^τ and the two-sided shift space. On the space $\tilde{B}^X$, T^τ is invertible, and we denote the inverse of T_ℓ^τ by $T_{-\ell}^\tau$. We write

$$T_{-\ell}^\tau(a \vee b, x, s)_* \tag{4.5}$$

for the linear map on $H^1(M,\Sigma,\mathbb{R})$ induced by $T^\tau_{-\ell}(a\vee b,x,s)$. In view of the definition (4.3) of τ and the definition (4.4) of T^τ_ℓ, we have for $v \in W_+(b)\otimes V_1(b,x)$,

$$\|T^\tau_{-\ell}(a\vee b,x,s)_* v\| = \exp(\ell)\|v\|. \tag{4.6}$$

4.3. The Exponential Drift Argument.

Standing Assumptions. We assume that the conditional measures of ν along $W^\pm(b,x)$ are supported on $U^\pm(b,x)\equiv U_{\mathbb{C}}(x)\cap W^\pm(b,x)$, and also that the conditional measures of ν along $U_{\mathbb{C}}(x)$ are Lebesgue.

Lemma 4.2. *There exists a subset $\Psi\subset B^X$ with $\beta^X(\Psi)=1$ such that for all $(b,x)\in\Psi$,*

$$\Psi\cap W^+(b,x)\cap \text{ ball of radius } 1 \subset \Psi\cap U^+(b,x).$$

Proof. See [MaT] or [EL, 6.23]. □

The parameter δ. Let $\delta>0$ be a parameter which will eventually be chosen sufficiently small. We use the notation $c_i(\delta)$ and $c_i'(\delta)$ for functions which tend to 0 as $\delta\to 0$. In this section we use the notation $A\approx B$ to mean that the ratio A/B is bounded between two positive constants depending on δ.

The set K. We choose a compact subset $K\subset\Psi$ with $\beta^X(K)>1-\delta>0.999$, where the conull set Ψ is as in Lemma 4.2.

Warning. In the rest of this section, we will often identify K with its pullback $f^{-1}(K)\subset\tilde{B}^X$ where $f:\tilde{B}^X$ to B^X is the forgetful map.

The Martingale Convergence Theorem. Let $\mathcal{B}^{\tau,X}$ denote the σ-algebra of $\beta^{\tau,X}$ measurable functions on B^X. As in [BQ], let

$$Q^{\tau,X}_\ell = (T^\tau_\ell)^{-1}(\mathcal{B}^{\tau,X}),$$

(Thus if a function F is measurable with respect to $Q^{\tau,X}_\ell$, then F depends only on what happened at least ℓ time units in the past, where ℓ is measured using the time change τ.)

Let

$$Q^{\tau,X}_\infty = \bigcap_{\ell>0} Q^{\tau,X}_\ell.$$

The $Q^{\tau,X}_\ell$ are a decreasing family of σ-algebras, and then, by the Martingale Convergence Theorem, for $\beta^{\tau,X}$-almost all $(b,x,s)\in B^X$,

$$\lim_{\ell\to\infty}\mathbb{E}(1_K \mid Q^{\tau,X}_\ell)(b,x,s) = \mathbb{E}(1_K\mid Q^{\tau,X}_\infty)(b,x,s) \tag{4.7}$$

where $\mathbb{E}$ denotes expectation with respect to the measure $\beta^{\tau,X}$.

The set S'. In view of (4.7) and the condition (K2) we can choose $S' = S'(\delta) \subset B^X$ with

$$\beta^X(S') > 1 - c_2(\delta). \tag{4.8}$$

such that for all $\ell > \ell_0$ and all $(b,x,s) \in S'$,

$$\mathbb{E}(1_K \mid Q_\ell^{\tau,X})(b,x,s) > 1 - c_2(\delta). \tag{4.9}$$

The set E_{good}. By Lemma 4.1 we may choose a subset $E_{good} \subset \tilde{B}^X$ (which is actually of the form $\tilde{B} \times E'_{good}$ for some subset $E'_{good} \subset X \times [0,1]$), with $\beta^X(E_{good}) > 1 - c_3(\delta)$, and a number $\sigma(\delta) > 0$ such that for any $(b,x,s) \in E_{good}$, any j and any unit vector $w \in \mathcal{L}(x)$,

$$\beta\Big(\{a \in B \ : \ d(w, \hat{V}_{n-1}(a,x)) > \sigma(\delta)\}\Big) > 1 - c_3'(\delta). \tag{4.10}$$

We may assume that $E_{good} \subset K$. By the Osceledec multiplicative ergodic theorem we may also assume that there exists $\alpha > 0$ (depending only on the Lyapunov spectrum), and $\ell_0 = \ell_0(\delta)$ such that for $(b,x,s) \in E_{good}$, $\ell > \ell_0$, at least $1 - c_3''(\delta)$ measure of $a \in B$, and all $v \in \hat{V}_{n-1}(a,x)$,

$$\|T^\tau_{-\ell}(a \vee b, x, s)_* v\| \le e^{(1-\alpha)\ell}\|v\|. \tag{4.11}$$

The sets Ω_ρ. In view of (4.8) and the Birkhoff ergodic theorem, for every $\rho > 0$ there exists a set $\Omega_\rho = \Omega_\rho(\delta) \subset \tilde{B}^X$ such that

(Ω1) $\beta^X(\Omega_\rho) > 1 - \rho$.

(Ω2) There exists $\ell_0' = \ell_0'(\rho)$ such that for all $\ell > \ell_0'$, and all $(b,x,s) \in \Omega_\rho$,

$$|\{t \in [-\ell,\ell] \ : \ T_t(b,x,s) \in S' \cap E_{good}\}| \ge (1 - c_5(\delta))2\ell.$$

Lemma 4.3. *Suppose the measure ν is not affine. Then there exists $\rho > 0$ so that for every $\delta' > 0$ there exist $(b,x,s) \in \Omega_\rho$, $(b,y,s) \in \Omega_\rho$ with $\|y - x\| < \delta'$ such that*

$$p(y-x) \in p(U^\perp)_{\mathbb{C}}(x), \tag{4.12}$$

and

$$d(y-x, W^+(b,x)) > \frac{1}{3}\|y-x\| \tag{4.13}$$

(so $y-x$ is in general position with respect to $W^+(b,x)$.)

The proof of this lemma (which is omitted here) uses a theorem proved by S. Filip [Fi, Corollary 5.4].

Standing Assumption. We fix $\rho = \rho(\delta)$ so that Lemma 4.3 holds.

The main part of the proof is the following:

Proposition 4.4. *There exists $C(\delta) > 1$ such that the following holds: Suppose for every $\delta' > 0$ there exist $(b,x,s),(b,y,s) \in \Omega_\rho$ with $\|x-y\| \le \delta'$, $p(x-y) \in p(U_{\mathbb{C}}^{\perp})(x)$, and so that (4.12) and (4.13) hold. Then for every $\epsilon > 0$ there exist $(b'',x'',s'') \in K$, $(b'',y'',s'') \in K$, such that $y''-x'' \in U_{\mathbb{C}}^{\perp}(x'')$,*

$$\frac{\epsilon}{C(\delta)} \le \|y''-x''\| \le C(\delta)\epsilon,$$

$$d(y''-x'', U_{\mathbb{C}}(x'')) \ge \frac{1}{C(\delta)}\|y''-x''\|, \tag{4.14}$$

$$d(y''-x'', W^+(b'',x'')) < \delta'', \tag{4.15}$$

where δ'' depends only on δ', and $\delta'' \to 0$ as $\delta' \to 0$.

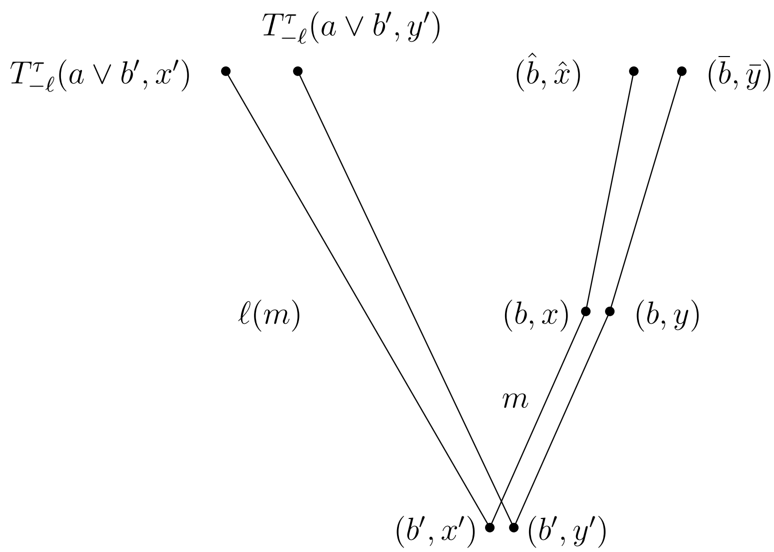

Figure 3. Proof of Proposition 4.4.

Proof. Write

$$(b',x',s') = T_m(b,x,s), \quad (b',y',s') = T_m(b,y,s).$$

and let

$$w(m) = x'-y'$$

(We will always have m small enough so that the above equation makes sense). For a positive integer m, let $\ell(m)$ be such that

$$e^{\ell(m)}\|w(m)\| = \epsilon,$$

In view of (4.13), (after some uniformly bounded time), $\|w(m)\|$ is an increasing function of m. Therefore, $\ell(m)$ is a decreasing function of m.

For a bi-infinite sequence $b \in \tilde{B}$ and $x \in X$, let

$$G(b,x,s) = \{m \in \mathbb{N} \ : \ T^{\tau}_{-\ell(m)} T_m(b,x,s) \in S'\}$$

Let $G_1(b,x,s) = G(b,x,s) \cap \{m \ : \ T_m(b,x,s) \in E_{good}\}$.

Lemma 4.5. *For $(b,x,s) \in \Omega_\rho$ and N sufficiently large,*

$$\frac{|G_1(b,x,s) \cap [1,N]|}{N} \geq 1 - c_6(\delta).$$

Proof. This follows from ($\Omega 2$) and the fact that $\ell(m)$ is a billipshitz function of m. ☐

Suppose $(b,x,s) \in \Omega_\rho$, $(b,y,s) \in \Omega_\rho$. By Lemma 4.5, we can fix $m \in G_1(x)$. Write $\ell = \ell(m)$. Let

$$(b',x',s') = T_m(b,x,s), \quad (b',y',s') = T_m(b,y,s).$$

Let

$$(\hat{b},\hat{x},\hat{s}) = T^{\tau}_{-\ell(m)}(b',x',s'), \quad (\bar{b},\bar{y},\bar{s}) = T^{\tau}_{-\ell(m)}(b',y',s').$$

Since $m \in G_1(b,x,s)$, we have $(\hat{b},\hat{x},\hat{s}) \in S'$, $(\bar{b},\bar{y},\bar{s}) \in S'$. Then, by (4.9),

$$\mathbb{E}(1_K \mid Q^{\tau,X}_\ell)(b,x,s) > (1 - c_2(\delta)), \quad \mathbb{E}(1_K \mid Q^{\tau,X}_\ell)(\bar{b},y,\bar{s}) > (1 - c_2(\delta)).$$

Since $T^{\tau}_\ell(b,x,s) = (b',x',s')$, by [BQ, (7.5)] we have

$$\mathbb{E}(1_K \mid Q^{\tau,X}_\ell)(b,x,s) = \int_B 1_K(T^{\tau}_{-\ell}(a \vee b',x',s'))\, d\beta(a),$$

where the notation $a \vee b'$ is as in (4.1). Thus,

$$\beta\left(\{a \ : \ T^{\tau}_{-\ell}(a \vee b',x',s') \in K\}\right) > 1 - c_2(\delta). \tag{4.16}$$

Similarly,

$$\beta\left(\{a \ : \ T^{\tau}_{-\ell}(a \vee b',y',s') \in K\}\right) > 1 - c_2(\delta).$$

Let $w = w(m) = x' - y'$. For any $a \in B$, we may write

$$w = \xi(a) + v(a),$$

where $\xi(a) \in W_+(b') \otimes V_1(b',x')$, and

$$v(a) \in W_+(b) \otimes \hat{V}_{n-1}(a,x') + W_-(a) \otimes \mathcal{L}(x').$$

(Then $v(a)$ will grow with a smaller Lyapunov exponent then $\xi(a)$ under $T^{\tau}_{-\ell}$).

Since $m \in G_1(b,x,s)$, we have $(b',x',s') \in E_{good}$. Then, by (4.10), for at least $1 - c'_3(\delta)$ fraction of $a \in B$,

$$\|v(a)\| \approx \|\xi(a)\| \approx \|w\| \approx \epsilon e^{-\ell}, \tag{4.17}$$

where the notation $A \approx B$ means that A/B is bounded between two constants depending only on δ. Also by (4.16), for at least $1 - c_2(\delta)$ fraction of $a \in B$, we have

$T^{\tau}_{-\ell}(a \vee b', x', s') \in K$. Thus, by (4.17), (4.6) and (4.11), we have, for all $j \in \tilde{\Lambda}$, and at least $1 - c_4(\delta)$ fraction of $a \in B$,

$$\|T^{\tau}_{-\ell}(a \vee b', x', s')_* \xi(a)\| \approx \epsilon, \text{ and } \|T^{\tau}_{-\ell}(a \vee b', x', s')_* v(a)\| = O(e^{-\alpha \ell}), \tag{4.18}$$

where $\alpha > 0$ depends only on the Lyapunov spectrum. (The notation in (4.18) is defined in (4.5)). Hence, for at least $1 - c_4(\delta)$ fraction of $a \in B$,

$$\|T^{\tau}_{-\ell}(a \vee b', x', s')_* w\| \approx \epsilon, \tag{4.19}$$

We now choose $\delta > 0$ so that $c_4(\delta) + 2c_2(\delta) < 1/2$, and using (4.16) we choose $a \in B$ so that (4.19) holds, and also

$$T^{\tau}_{-\ell}(a \vee b', x', s') \in K, \qquad T^{\tau}_{-\ell}(a \vee b', y', s') \in K.$$

One can check that (with some small modifications) (4.15) holds. □

Proof of Measure Classification Theorem. Suppose ν is an ergodic P-invariant measure. It was already proved in Theorem 2.11 that ν is $SL(2,\mathbb{R})$-invariant. Now suppose ν is not affine. We can apply Lemma 4.3, and then iterate Proposition 4.4 with $\delta' \to 0$ and fixed ϵ and δ. Taking a limit along a subsequence we get points $(b_\infty, x_\infty, s_\infty) \in K$ and $(b_\infty, y_\infty, s_\infty) \in K$ such that $\|x_\infty - y_\infty\| \approx \epsilon$, $y_\infty \in W^+(b_\infty, x_\infty)$ and $y_\infty \in (U^\perp)^+(b_\infty, x_\infty)$. This contradicts Lemma 4.2 since $K \subset \Psi$. Hence ν is affine. □

Some comments on the general case. In this lecture we made two simplifying assumptions, namely that $\mathcal{L}$ is simple and tat there is no relative homology. Removing the first assumption is easy: we have the $SL(2,\mathbb{R})$-invariant decomposition $\mathcal{L} = \bigoplus_k \mathcal{L}_k$, where the $\mathcal{L}_k$ are simple. We can then define a time change for each k, and the argument goes through modulo some extra bookkeeping and notation.

However, handling relative homology creates a susbstantial difficulty. Essentially the problem is that in the presence of relative homology, the cocycle may not be semisimple. However, relative homology is present only in (Kontsevich-Zorich cocycle) Lyapunov exponent 0, and due to the nature of the argument outlined in this lecture we see a problem only if the top Lyapunov exponent $\hat{\lambda}_k$ of some $\mathcal{L}_k$ is zero. This is somewhat unusual, but can happen in some examples, see e.g. [FoMZ] for a discussion. We define the direct sum of all $\mathcal{L}_k$ for which the top Lyapunov exponent is 0 to be the *Forni subspace*, denoted by F. The Forni subspace has several equivalent definitions, see e.g. the discussion in [EMi, Appendix A]. Remarkably $F(x)$ depends real-analytically on x. (Most Lyapunov subspaces are only measurable).

In the presense of the Forni subspace, the argument described in this section may fail in the case where the vector $y - x$ has a component along $F(x)$, because we have no way of defining a time change as in (4.3) for the bundle $p^{-1}(F)$, where $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ is the natural map.

However, the following is proved in [AEM]:

Theorem 4.6. *There exists a subset Φ of the stratum $\mathcal{H}_1(\alpha)$ with $\nu(\Phi) = 1$ such that for all $x \in \Phi$ there exists a neighborhood $\mathcal{U}(x)$ such that for all $y \in \mathcal{U}(x) \cap \Phi$ we have $p(y - x)$ is orthogonal to $F(x)$.*

In view of Theorem 4.6, in our setup, for the points x, y of Figure 3 we have $y - x \in F^{\perp}(x)$, and the proof goes through.

We remark that the main ingredient in the proof of Theorem 4.6 is the analyticity of $F(x)$.

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