Towards an n-category of cobordisms
extremely rough draft

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Abstract

We discuss an approach to constructing a weak n-category of cobordisms. First we present a generalisation of Trimble's definition of n-category which seems most appropriate for this construction; in this definition composition is parametrised by a contractible operad. Then we discuss the problem of defining an n-category **nCob**, whose k-cells are k-cobordisms, possibly with corners. We show how to make some preliminary constructions as "stepping stones" towards the desired goal. We follow Baez and Langford in using "manifolds embedded in cubes" rather than general manifolds. We make the construction for 1-manifolds embedded in 2- and 3-cubes.

Contents

Introduction

To be written.

1 Trimble's definition

We begin by presenting the theory of weak n-categories we will be using throughout the rest of the article. Our starting point is Trimble's definition [Tri99], so we begin by recalling this. Since none of our n-categories are strict we will omit the word "weak" throughout.

We will eventually include this definition for completeness; for now we refer the reader to [Lei02] and [CL04].

2 Generalised Trimble definition

In this section we present a generalisation of Trimble's definition which we will be using throughout the rest of the paper. The idea is to copy Trimble's definition but with a general operad E in some category $\mathcal{B}$, in place of the operad in **Top** used by Trimble. The delicate issue is to make the setting sufficiently general to include more examples of such operads, but to retain enough structure to enable us to make the induction step. The difficulty in the induction step is in knowing how to construct Π_n, the "fundamental n-category" functor. Our approach starts from the observation that, for the definition of n-category and n-functor, we only need the "fundamental n-category" of *certain* spaces, not all of them. We also observe that although Trimble's operad E is an operad in **Top**, the induction step of the definition of n-category is enabled by the associated operad in globular sets. This arises from considering the globular set associated with any space, with

0-cells	the points in the space
1-cells	paths between these points
2-cells	paths between paths i.e. maps $D^2 \longrightarrow X$
$\vdots$	
m-cells	maps $D^m \longrightarrow X$
$\vdots$	

So in fact we make the definition using an operad defined directly in globular sets. Among other things, this also enables us to construct more variants of the original definition by restricting the operad at each dimension as desired, giving us finer control over the situation.

In using globular sets we frequently make use of the fact that the cells $x \longrightarrow y$ in a globular set X form a globular set themselves. That is, $\forall x, y \in X_k$ we have a globular set $X(x, y)$ whose 0-cells are $k+1$-cells $x \longrightarrow y$, whose 1-cells are

$k+2$ cells between those, and so on. The globular set $X(x,y)$ can be thought of as analogous to a path space.

2.1 The definition

We begin with the following data

- a contractible operad E in **GSet**
- a subcategory $\mathcal{G}$ of **GSet** satisfying
  - i) $1 \in \mathcal{G}$
  - ii) $\forall k \geq 0$, $E(k) \in \mathcal{G}$
  - iii) $\forall X \in \mathcal{G}$ and $\forall x, y \in X_0$, $X(x,y) \in \mathcal{G}$
  - iv) $\forall X, Y \in \mathcal{G}$, $X \times Y \in \mathcal{G}$
- $\forall X \in \mathcal{G}$, an action of E on X such that every map $f : X \to Y$ in $\mathcal{G}$ is an E-action map. Note that in particular this means E must then act on products via the diagonal map.

We will define inductively for each $n \geq 0$ a category **nCat** with finite products and a functor $\Pi_n : \mathcal{G} \to \mathbf{nCat}$ preserving the above finite products. Π_n can be thought of as a "fundamental n-category" functor, but in fact we will only ever apply it to contractible globular sets, giving a fundamental n-groupoid.

Remark At this point we still need to work out exactly what $\mathcal{G}$ *needs*. So far we have only characterised some things that are true about the $\mathcal{G}$ used in our examples.

2.2 The case $n = 0$

For the case $n = 0$ we put $\mathbf{nCat} = \mathbf{Set}$ and define the functor Π_0 to send everything to 1.

N.B. Π_0 can be thought of as taking path components, but all objects of $\mathcal{G}$ in which we are interested are contractible. Π_0 clearly preserves products.

2.3 Objects of $(n+1)$-Cat

Inductively, an $(n+1)$-category A is given by

- a set A_0 of 0-cells
- $\forall a, a' \in A_0$, a hom-n-category $A(a,a') \in \mathbf{nCat}$
- $\forall k \geq 0$ and $a_0, \ldots, a_k$, a composition functor (i.e., a morphism of **nCat**)

$$\gamma = \gamma^A_{a_0,\ldots,a_k} : \Pi_n\big(E(k)\big) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \to A(a_0, a_k)$$

satisfying axioms ensuring compatibility with the composition and identity of the operad E. (This makes sense because Π_n preserves finite products and **nCat** has them.)

2.4 Maps in $(n+1)$-Cat

A map $F : A \to B$ in $(n+1)$**-Cat** is called an $(n+1)$-functor and is given by

- a function $F = F_0 : A_0 \to B_0$
- $\forall a, a' \in A_0$, an n-functor

$$F = F_{a,a'} : A(a, a') \to B(Fa, Fa')$$

such that $\forall k \geq 0$ and $\forall a_0, \ldots, a_k$, the following diagram commutes.

$$\begin{tikzcd}
\Pi_n(E(k)) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \arrow[r, "\gamma^A"] \arrow[d, "1 \times F \times \cdots F"] & A(a_0, a_k) \arrow[d, "F"] \\
\Pi_n(E(k)) \times B(Fa_{k-1}, Fa_k) \times \cdots \times B(Fa_0, Fa_1) \arrow[r, "\gamma^B"'] & B(Fa_0, Fa_k)
\end{tikzcd}$$

Composition for identities $(n+1)$-functors are obvious.

2.5 Π_{n+1} on objects

For $X \in G$, we define $\Pi_{n+1}(X) = A$, where A is given by

- $A_0 = X_0$
- $A(x, x') = \Pi_n\big(X(x, x')\big)$ (defined since $X(x, x') \in \mathcal{G}$ by hypothesis)
- for $x_0, \ldots, x_k$ the composition functor γ is given by

$$\begin{tikzcd}
\Pi_n(E(k)) \times \Pi_n\big(X(x_{k-1}, x_k)\big) \times \cdots \times \Pi_n\big(X(x_0, x_1)\big) \arrow[d, "\Pi_n \text{preserves products}"] \\
\Pi_n\Big(E(k) \times X(x_{k-1}, x_k) \times \cdots X(x_0, x_1)\Big) \arrow[d] \\
\Pi_n\Big(X(x_0, x_k)\Big)
\end{tikzcd}$$

where the second map is Π_n applied to the action of E on X.

2.6 Π_{n+1} on maps

Given $\varphi : X \to Y$ in $\mathcal{G}$, we define $F = \Pi_{n+1}\varphi : \Pi_{n+1}X \to \Pi_{n+1}Y$ as follows.

- On objects, $F_0 = \varphi_0 : X_0 \to Y_0$.
- The functor $F_{x,x'} : \Pi_{n+1}\big(X(x,x')\big) \to \Pi_{n+1}\big(Y(Fx,Fx')\big)$ is given by Π_n applied to the restriction of φ to $X(x,x') \to Y(\varphi x, \varphi x') = Y(Fx,Fx')$.

We check that the following diagram commutes.

$$\begin{tikzcd}
\Pi_n(E(k)) \times \Pi_n\big(X(x_{k-1},x_k)\big) \times \cdots \times \Pi_n\big(X(x_0,x_1)\big) \arrow[dd, "1\times\Pi_n\varphi\times\cdots\times\Pi_n\varphi"'] \arrow[dr] & \\
& \Pi_n\Big(E(k) \times X(x_{k-1},x_k) \times \cdots \times X(x_0,x_1)\Big) \arrow[d, "\Pi_n\alpha"] \\
\Pi_n(E(k)) \times \Pi_n\big(Y(\varphi x_{k-1},\varphi x_k)\big) \times \cdots \times \Pi_n\big(Y(\varphi x_0,\varphi x_1)\big) \arrow[d] & \Pi_n\Big(X(x_0,x_k)\Big) \arrow[d, "\Pi_n\varphi"] \\
\Pi_n\Big(E(k) \times Y(\varphi x_{k-1},\varphi x_k) \times \cdots \times Y(\varphi x_0,\varphi x_1)\Big) \arrow[r, "\Pi_n\alpha"'] & \Pi_n\Big(Y(\varphi x_0,\varphi x_k)\Big)
\end{tikzcd}$$

This can be seen by dividing the diagram into two squares via the map

$$\begin{tikzcd}
\Pi_n\Big(E(k) \times X(x_{k-1},x_k) \times \cdots \times X(x_0,x_1)\Big) \arrow[d, "\Pi_n(1\times\varphi\times\cdots\times\varphi)"] \\
\Pi_n\Big(E(k) \times Y(\varphi x_{k-1},\varphi x_k) \times \cdots \times Y(\varphi x_0,\varphi x_1)\Big)
\end{tikzcd}$$

(where here α is the action map of E on an object of $\mathcal{G}$).

2.7 Finite products

A product of $(n+1)$-categories A and B is given as follows:

- $(A \times B)_0 = A_0 \times B_0$
- $(A \times B)\big((a,b),(a',b')\big) = A(a,a') \times B(b,b')$

- composition is given by

$$\begin{tikzcd}
\Pi_n(E(k)) \times (A \times B)\big((a_{k-1}, b_{k-1}), (a_k, b_k)\big) \times \cdots \times (A \times B)\big((a_0, b_0), (a_1, b_1)\big) \arrow[d, equal] \\
\Pi_n(E(k)) \times A(a_{k-1}, a_k) \times B(b_{k-1}, b_k) \times \cdots \times A(a_0, a_1) \times B(b_0, b_1) \arrow[d, "\cong"] \\
\Pi_n(E(k)) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \times B(b_{k-1}, b_k) \times \cdots \times B(b_0, b_1) \arrow[d, "\Delta \times 1"] \\
\Pi_n(E(k)) \times \Pi_n(E(k)) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \times B(b_{k-1}, b_k) \times \cdots \times B(b_0, b_1) \arrow[d, "\cong"] \\
\Pi_n(E(k)) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \times \Pi_n(E(k)) \times B(b_{k-1}, b_k) \times \cdots \times B(b_0, b_1) \arrow[d, "\gamma^A \times \gamma^B"] \\
A(a_0, a_k) \times B(b_0, b_k) \arrow[d, equal] \\
(A \times B)\big((a_0, b_0), (a_k, b_k)\big),
\end{tikzcd}$$

  where Δ is the diagonal map in **GSet**.

It is easy to check that Π_{n+1} preserves products (it follows from the definition of the action of E on products via the diagonal), so the inductive definition goes through.

Remark 2.1. Given an operad E we may be able to construct the smallest $\mathcal{G}$ satisfying the required conditions.

3 Examples of operads in this framework

In this section, we will give two examples of operads that fit into our framework. The first example is to show that our framework really is just a generalisation of Trimble's; the second example is the motivating one and might be thought of as the entire reason for making the generalisation at all.

3.1 Trimble's definition in the new framework

In this section we discuss how Trimble's original definition arises in the generalised setting. The key point is the "fundamental globular set" functor

$$D : \mathbf{Top} \to \mathbf{GSet}$$

which takes the original topological operad to an operad in globular sets.

The rest of this section is to be written.

3.2 A smooth version of the original operad

We will be using a suboperad E_s of Trimble's operad E as follows. For each $k \geq 0$, $E_s(k)$ has

- 0-cells which are endpoint-preserving diffeomorphisms $f : [1] \to [k]$ such that there exists $\delta > 0$ for which $\forall x \in [0, \delta)$ $f(x) = x$ and $\forall x \in (1-\delta, 1]$ $f(x) = x + k - 1$
- m-cells, $m > 0$, from α to β are smooth homotopies

  $$\Theta : I^m \times [1] \to [k]$$

  from α to β such that

  i) each $\Theta(t, -, -, \cdots, -) : I^{m-1} \times [1] \to [k]$ satisfies the conditions for being an $(m-1)$-cell and

  ii) there exists a $\delta > 0$ such that $\forall t \in [0, \delta)$

  $$\Theta(t, -, -, \cdots, -) = \alpha$$

  and $\forall t \in (1-\delta, 1]$

  $$\Theta(t, -, -, \cdots, -) = \beta.$$

Note that in effect we have demanded that "the 0-cells have derivative 1 on a neighbourhood of the boundary" to ensure that smoothness is preserved by operadic composition; E_s inherits its unit and composition from E so we only have to check closure to see that E_s is indeed an operad. The other conditions are to ensure the desired behavior of the n-category of manifolds when we eventually construct it. Furthermore we can check that E_s is contractible.

To show that E_s has the required action, we need to construct a suitable category $\mathcal{G}$. We specify the objects by taking the smallest collection of objects satsifying the necessary conditions; essentially this amounts to taking each $E_s(k)$, all path spaces (of all dimensions), and all products of them. To specify the morphisms we take not *all* globular set maps between objects of $\mathcal{G}$, but only those that are restrictions of maps that come from maps of path spaces in **Top**. To show that E_s then acts as necessary, we check that the action of the original operad E preserves the properties used to define cells of $E(k)$ above.

N.B. The above characterisation is evidently only a sketch. Our precise characterisation is as yet unilluminating and it remains to be seen if a more illuminating characterisation can be found.

4 n-categories of manifolds in cubes: sketch

In this section we discuss how to construct operadic n-categories of "manifolds in cubes". The aim is to define an n-category **nCob** of cobordisms with corners as below.

0-cells	0-manifolds
1-cells	1-manifolds with appropriate boundary
$\vdots$	
k-cells	k-manifolds with corners,
$\vdots$	
n-cells	diffeomorphism classes of n-manifolds with corners.

(In this paper, all manifolds are smooth and compact.) We follow Baez and Langford [BL03] and consider manifolds embedded in cubes. We aim to construct, for each $n \geq 1$ and $0 \leq k < n$ an n-category of "k-manifolds in n-cubes". The n-cube is the space I^n, where $I = [0, 1]$, and we consider k-manifolds that are subsets of I^n, equipped with the smooth structure inherited from the standard smooth structure on I^n.

Ultimately, these should be degenerate n-categories with extra structure, since the m-cells are given as follows.

0-cells	trivial
1-cells	trivial
$\vdots$	$\vdots$
$(n-k)$-cells	0-manifolds in $(n-k)$-cubes
$(n-k+1)$-cells	1-manifolds in $(n-k+1)$-cubes
$\vdots$	$\vdots$
n-cells	k-manifolds in n-cubes

So this is an $(n-k)$-degenerate n-category or an $(n-k)$-tuply monoidal k-category. (Note that for $m < n-k$, the unique m-cell can be thought of as being the empty set as a subset of I^m.)

The Tangle Hypothesis of Baez and Dolan [BD95] says "framed oriented n-tangles in $(n+k)$-dimensions are the n-cells of the free weak k-tuply monoidal n-category with duals on one object". Note that by the following reindexing the dimensions involved do match up.

Current paper		Tangle Hypothesis
$n-k$	$\longrightarrow$	k
k	$\longrightarrow$	n

The Stabilisation Hypothesis [BD95] then says that the notion of k-tuply monoidal n-category should stabilise when $k \geq n+2$. With our indexing that means

$$n-k \geq k+2$$

i.e.

$$n \geq 2k+2.$$

So if we fix k, the dimension of our manifolds, and embed them in cubes of higher and higher dimension n, the situation should stabilise. Later in this paper we will sketch this process for $k = 1$, where we expect 1-manifolds in 4-cubes to give "the same structure" as 1-manifolds in n-cubes $\forall n \geq 4$. This corresponds to the fact that under- and over-crossings are the same in 4-space (and all higher dimensions), or the fact that all knots can be untied in space of dimension 4 and above.

Finally note that our construction differs from that of Baez and Langford as we are constructing a *weak* n-category where theirs is strict. Essentially this means that when we compose k-cells by "sticking cubes together" we need to keep track of how we reparametrise the resulting cuboid to become a unit cube again. By contrast, in the strict case the strictifying equivalence relation ensures that the different reparametrisations will not be detected by the n-category.

5 1-manifolds in 2-cubes

We now construct a (degenerate) 2-category of 1-manifolds with boundary as subsets of the 2-cube I^2.

0-cells There is only one 0-cell, which we consider to be $\varnothing \subset I^0$.

1-cells These are 0-manifolds in $(0, 1)$.

2-cells

Given 1-cells x, y, a 2-cell $\theta : x \Rightarrow y$ is a 1-manifold with boundary, in I^2, satisfying the following conditions, subject to the equivalence relation below.

i) The boundary of θ is contained in $\{0, 1\} \times I$ with $\partial\theta|_{\{0\}\times I} = x$ and $\partial\theta|_{\{1\}\times I} = y$.

ii) θ has a product structure near the top ($\{0\} \times I$) and the bottom ($\{1\} \times I$); we refer to this as a "collar" region.

The equivalence relation on these is defined as follows. Let θ_1, θ_2 be two 1-manifolds with the same boundary. We say $\theta_1 \sim \theta_2$ if there is a self-diffeomorphism of $I \times I$ taking θ_1 to θ_2 and preserving the boundary of $I \times I$.

Examples go here.

Note that we can express a subset of I^m using a characteristic function $I^m \rightarrow \mathbf{2} = \{0, 1\}$. In this formalism a 1-cell x is expressed as a function $x : I \rightarrow \mathbf{2}$, and a 2-cell $\alpha : x \Rightarrow y$ is expressed as a function $I \times I \rightarrow \mathbf{2}$ where

$$\begin{aligned} \alpha(0, -) &= x, \text{and} \\ \alpha(1, -) &= y. \end{aligned}$$

The function $\alpha(t, -) : I \to \mathbf{2}$ gives the cross-section of the manifold at height t.

Vertical composition is vertical stacking of squares, followed by reparametrising to make the unit square again. The choice of reparametrisation does not matter as any two will give equivalent manifolds. Note that the collar regions ensure that such vertical composites are smooth.

Now we have to define the composition functor, i.e., horizontal composition. This is given by, for each $k \geq 0$, a functor

$$\Pi_1(E_s(k)) \times X^k \to X$$

where X is the (unique) vertical hom-category.

On objects (i.e., 1-cells of our 2-category), we have on the left a tuple

$$(f, x_1, x_2, \dots, x_k)$$

where f is a map $[1] \to [k]$ and each x_i is a subset of I, which we write as a function $x_i : I \to \mathbf{2}$. We can take a colimit to get

$$x_1 + x_2 + \cdots + x_k : [k] \to \mathbf{2}.$$

We now compose this with f to get

$$[1] \xrightarrow{f} [k] \xrightarrow{x_1 + \cdots + x_k} \mathbf{2},$$

i.e., a subset of I given by $(x_1 + \cdots + x_k) \circ f$; we check that the resulting subset is a manifold in $(0, 1)$. This is the required composite on 1-cells.

On morphisms (i.e. 2-cells of our 2-category) we must find a composite for

$$\begin{pmatrix} f, & x_1, & \dots, & x_k \\ \alpha \Downarrow, & \Downarrow \theta_1, & \dots, & \Downarrow \theta_k \\ g, & y_1, & \dots, & y_k \end{pmatrix}$$

whose source must be $(x_1 + \cdots + x_k) \circ f$ and target must be $(y_1 + \cdots + y_k) \circ g$. We take the equivalence class of the following composite.

$$I \times [1] \xrightarrow{\Delta \times 1} I \times I \times [1] \xrightarrow{1 \times \alpha} I \times [k] \xrightarrow{\theta_1 + \cdots + \theta_k} \mathbf{2}$$

It is straightforward to check that this has the correct source and target, and that it satisfies the conditions for being a 2-cell. This follows from the collar regions and the various "endpoint neighbourhood" conditions we demanded when defining the operad $E_s(k)$.

Finally we must check the remaining axioms for a 2-category, i.e., that this composition is functorial and interacts well with operad composition.

6 1-manifolds in 3-cubes

We now define a 3-category of 1-manifolds in 3-cubes. The idea is that the 0- and 1-cells are now trivial, and the 2- and 3-cells should be 0-manifolds and

1-manifolds with boundary respectively, now embedded in cubes one dimension higher than before. Composition of cubes can now occur in three ways as required – by stacking cubes in each of the three possible directions.

Note that this is a doubly degenerate 3-category, so should in fact be a braided monoidal category of some sort if we regard the 2-cells as objects and the 3-cells as morphisms. (See also [CG05].)

0-cells There is only one 0-cell, which we consider to be $\varnothing \subset I^0$.

1-cells There is only one 1-cell, which we consider to be $\varnothing \subset I$.

2-cells A 2-cell is a 0-manifold in $(0,1) \times (0,1)$.

3-cells

Given 1-cells x, y, a 3-cell $\theta : x \longrightarrow y$ is a 1-manifold-with-boundary in I^3, satisfying the following conditions, subject to the equivalence relation below.

i) The boundary of θ is contained in $\{0,1\} \times I^2$ with $\partial\theta|_{\{0\}\times I^2} = x$ and $\partial\theta|_{\{1\}\times I^2} = y$.

ii) As before θ has a product structure near the top ($\{0\} \times I^2$) and the bottom ($\{1\} \times I^2$).

The equivalence relation on these is defined as follows. Let θ_1, θ_2 be two 1-manifolds with the same boundary. We say $\theta_1 \sim \theta_2$ if there is a self-diffeomorphism of I^3 taking θ_1 to θ_2 and preserving the boundary of I^3.

We must now define three kinds of composition.

Composition along a bounding 2-cell

$\forall k \geq 0$ we need a function

$$\Pi_0(E_s(k)) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1) \longrightarrow X(x_0, x_k).$$

As with vertical composition in the previous example, this composition is achieved by stacking boxes in the direction that identifies boundaries of 1-manifolds.

examples go here

As before, we then reparametrise to a unit cube and take equivalences classes, and observe that

- a different choice of reparametrisation gives the same equivalence class, and
- collar regions ensure that the composite is smooth.

Composition along a bounding 1-cell

Recall that there is only one 1-cell, so there is only one hom-category. We write this as X; then for all $k \geq 0$ we need a 1-functor

$$\Pi_1(E_s(k)) \times X^k \longrightarrow X.$$

On objects (i.e., 2-cells of our 3-category), we have on the left a tuple

$$(f, x_1, x_2, \ldots, x_k)$$

where f is a map $[1] \to [k]$ and each x_i is a subset of $I \times I$, which we write as a function $x_i : I \times I \to \mathbf{2}$. We form the required composite by the following map:

$$I \times [1] \xrightarrow{1 \times f} I \times [k] \xrightarrow{x_1 + \cdots + x_k} \mathbf{2},$$

and we check that the resulting subset is a manifold in $(0,1)$.

On morphisms (i.e. 3-cells of our 3-category) we must find a composite for

$$\begin{pmatrix} f, & x_1, & \ldots, & x_k \\ \alpha \Downarrow, & \Downarrow \theta_1, & \ldots, & \Downarrow \theta_k \\ g, & y_1, & \ldots, & y_k \end{pmatrix}$$

We take the equivalence class of the following composite.

$$I \times I \times [1] \xrightarrow{1 \times \Delta \times 1} I \times I \times I \times [1] \xrightarrow{1 \times 1 \times \alpha} I \times I \times [k] \xrightarrow{\theta_1 + \cdots + \theta_k} \mathbf{2}$$

As before, it is straightforward to check that this has the correct source and target, and that it satisfies the conditions for being a 3-cell, and that the composition is a functor satisfying the required properties.

Composition along a bounding 0-cell

Recall that there is only one 0-cell, so there is only one hom-2-category. We write this as X; then for all $k \geq 0$ we need a 2-functor

$$\Pi_2(E_s(k)) \times X^k \longrightarrow X.$$

On 0-cells (i.e. 1-cells of our 3-category) this is trivial.

On 1-cells (i.e., 2-cells of our 3-category), we have on the left a tuple

$$(f, x_1, x_2, \ldots, x_k)$$

where f is a map $[1] \to [k]$ and each x_i is a subset of I^3, which we write as a function $x_i : I^3 \to \mathbf{2}$. We form the required composite by the following map:

$$I \times [1] \xrightarrow{\Delta \times 1} I \times I \times [1] \xrightarrow{1 \times \alpha} I \times [k] \xrightarrow{\theta_1 + \cdots + \theta_k} \mathbf{2}$$

as for the 2-cells in the 2-category case, but now without the equivalence relation.

On 2-cells (i.e. 3-cells of our 3-category) we have on the left a tuple $(\phi, \gamma_1, \ldots, \gamma_k)$ where $\phi : \alpha \longrightarrow \alpha'$ is a map

$$I \times I \times [1] \longrightarrow [k]$$

and each $\gamma_i : \theta_i \longrightarrow {\theta_i}'$ is a map

$$I \times I \times I \longrightarrow \mathbf{2}.$$

We take the equivalence class of the following composite

$$I_1 \times I_2 \times [1] \xrightarrow{\Delta \times 1} I_1 \times I_2 \times I_1 \times I_2 \times [1] \xrightarrow{1 \times 1 \times \phi} I_1 \times I_2 \times [k] \xrightarrow{\gamma_1 + \cdots + \gamma_k} \mathbf{2}$$

which takes the triple (x, y, z) to $(\gamma_1 + \cdots + \gamma_k)(x, y, \phi(x, y, z))$. (The subscripts on the I's here are simply to show which component is which.)

As before, we must check that this has the correct source and target, and that it satisfies the conditions for being a 3-cell, and that the composition is a functor satisfying the required properties.

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