Category Theory Examples 1

Winter 2006 Eugenia Cheng

Please report any mistakes.

1. (i) Show that identities in a category are unique.
   (ii) Show that a morphism with both a right inverse and a left inverse is necessarily an isomorphism.
   (iii) Show that functors preserve isomorphisms.
   (iv) Show that a natural transformation is a natural isomorphism if and only if each of its components is an isomorphism.

2. (i) Show that there is a functor $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ taking a category to its set of objects. Is this functor faithful? Is it full?
   (ii) Show that there is a functor $A : \mathbf{Cat} \longrightarrow \mathbf{Set}$ taking a category to its set of morphisms. Is this functor faithful? Is it full?
   (iii) Is there a functor $Z : \mathbf{Grp} \longrightarrow \mathbf{Grp}$ taking a group to its centre?

3. A *pointed set* is a set X equipped with an element ('basepoint') $x \in X$. Let $\mathbf{Set}_*$ be the category of pointed sets and basepoint-preserving functions. Show that $\mathbf{Set}_*$ is equivalent to **Par**, the category of sets and partial functions.

4. Fix a field k. Let $\mathbf{Mat}_k$ be the category whose objects are the natural numbers, and with

$$\mathbf{Mat}_k(m, n) = \{n \times m \text{ matrices over } k\}.$$

   Prove that $\mathbf{Mat}_k$ is equivalent to $\mathbf{FDVect}_k$, the category of finite-dimensional vector spaces over k.

5. (i) Show that an equalizer is monic. (Such a monic is called a *regular* monic.)
   (ii) Suppose that $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in a category $\mathcal{C}$. Show that
      a. f and g monic imply gf monic;
      b. gf monic implies f monic;
      c. gf regular monic and g monic implies f regular monic.
   (iii) A coequalizer is a *regular epi*. Show that if $f : A \longrightarrow B$ is both a monic and a regular epi, then f is an isomorphism.
   (iv) Identify the monics, the regular monics, the epis and the regular epis in the category **Top** of topological spaces and continuous functions.

6. A map $m : A \longrightarrow B$ in a category $\mathcal{C}$ is called *split monic* if there is a map $e : B \longrightarrow A$ with $em = 1_A$; *split epics* are defined dually.

   (i) Show that split monic $\Rightarrow$ regular monic $\Rightarrow$ monic.

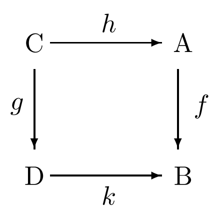
   (ii) In **Ab**, show that all monics are regular, but that not all monics are split.

   (iii) In **Top**, identify the regular monics, and find a monic which is not regular.

   (iv) Prove that in any category, a map is an isomorphism if and only if it is both epic and regular monic. Show by example that a map which is epic and monic need not be an isomorphism.

   (v) Show that epics, regular epics and split epics are all the same in **Set**. (For this you need the axiom of choice, which says *exactly* that epics split in **Set**; of course, this does not hold in every category.)

7. A monomorphism $f : A \longrightarrow B$ in a category is said to be *extremal* just when, for every commutative square



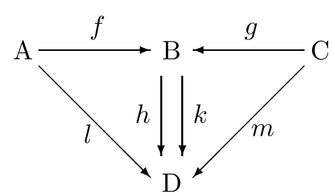
   with g epic, there exists a (necessarily unique) $t : D \longrightarrow A$ such that $ft = k$ and $tg = h$.

   (i) Show that

      a. f and g extremal monic imply gf extremal monic;

      b. gf extremal monic implies f extremal monic.

   (ii) Show that every regular monomorphism is extremal, but that in the finite category represented by the diagram



   the morphism f is extremal monic but not regular monic. *(Everything in the diagram commutes but $h \neq k$.)*

8. Suppose that

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

is a commutative diagram.

(i) Show that if both small squares are pullbacks then so is the large rectangle.

(ii) Show that if the large rectangle and the right hand square are pullbacks, then so is the left hand square.

(iii) Deduce from the above (or prove directly) that the pullback of a pullback square is a pullback square, stating clearly what you take this to mean.

9. (i) Let $\mathcal{D}$ be a locally small category. What does it mean for a functor $X : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ to be *representable*? What is a *representation* of such a functor?

(ii) Suppose that $F, G : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ are representable functors in a category $\mathcal{D}$ with binary products. Show that the functor $F(-) \times G(-) : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable.

(iii) Given a set I, an *I-fold power* A^I of $A \in \mathcal{D}$ is a product of I copies of A, that is

$$A^I = \prod_{i \in I} A.$$

Similarly an *I-fold copower* $A \times I$ is a coproduct of I copies of A. Suppose that $F : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable and $\mathcal{D}$ has I-fold powers and copowers. Show that the functor $F(- \times I) : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable.

10. Let $\mathbf{C}$ be a locally small category. Define the *Yoneda embedding* $H_\bullet : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. State and prove the Yoneda Lemma and deduce that the Yoneda embedding is full and faithful.

11. (i) What is a limit in a category $\mathcal{C}$?

(ii) Define products and equalizers.

(iii) Show that a category with all small products and equalizers has all small limits.