Category Theory Examples 3

Winter 2006 Eugenia Cheng

Please report any mistakes.

1. A monad (T, η, μ) on a category $\mathcal{C}$ is *idempotent* if and only if the multiplication $\mu : T^2 \longrightarrow T$ is an isomorphism. Show that if (T, η, μ) is a idempotent monad on $\mathcal{C}$ then every T-algebra is (isomorphic to) a free T-algebra.

2. What does it mean for a category to be *well-powered*? Show that if $\mathcal{C}$ is a well-powered category with all pullbacks then there is a functor $\text{Sub} : \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Set}$ assigning to each object its set of subobjects. (*A representation of this functor is called a* subobject classifier, *and in fact it is possible to rephrase the definition of subobject classifier so that it makes sense in categories which are not necessarily well-powered. A* topos *is a cartesian closed category with finite limits and a subobject classifier; typical examples are* **Set** *and* $[\mathbb{C}^{op}, \mathbf{Set}]$.)

3. Let $G : \mathcal{D} \longrightarrow \mathcal{C}$. Show that the following are equivalent:

   i. G creates coequalizers for all G-absolute-coequalizer pairs.

   ii. $\mathcal{D}$ has coequalizers of all G-split-coequalizer pairs, and G preserves and reflects them.

4. A pair of maps $g, h : Y \longrightarrow Z$ in a category $\mathcal{D}$ is a *reflexive pair* just when there is a map $k : Z \longrightarrow Y$ such that $gk = 1_Z$ and $hk = 1_Z$. Suppose the adjunction $F \dashv G : \mathcal{D} \longrightarrow \mathcal{C}$ is such that

   (a) G reflects isomorphisms,

   (b) $\mathcal{D}$ has coequalizers of reflexive pairs, and

   (c) G preserves such coequalizers.

   Show that the adjunction is monadic. (*A result of this form is often called a Crude Monadicity Theorem. We just have sufficient conditions to ensure monadicity.*)