Category Theory Examples 2

Winter 2006 Eugenia Cheng

Please report any mistakes.

1. Let $\mathbf{Set}_*$ be the category of pointed sets and basepoint-preserving functions. Let 1 be a one-element set. Show that $\mathbf{Set}_*$ is isomorphic to $1/\mathbf{Set}$.

2. What does it mean for a functor to reflect isomorphisms? Show that a full and faithful functor reflects isomorphisms.

3. (i) Show that if a category has a terminal object, all binary products and all equalizers, then it has all finite limits.

   (ii) Deduce that if a category has a terminal object and all pullbacks then it has all finite limits. (Hint: consider diagrams

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \longrightarrow & 1 \end{array} \qquad \begin{array}{ccc} & & B \\ & & \downarrow \\ A & \longrightarrow & B \times B \end{array} .)$$

   (iii) Let $\mathcal{C}$ be a category with all finite colimits and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor which preserves pushouts and the initial object. Show that F preserves all finite colimits.

   (iv) Show that a functor which preserves pushouts also preserves epics.

   (v) Characterise the monics and epics in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$.

4. (i) Suppose that $F : \mathcal{A}^{\mathrm{op}} \times \mathcal{B}$ is such that we have, for each $B \in \mathcal{B}$, a choice

$$\alpha_B : \mathcal{A}(-, A_B) \cong F(-, B)$$

   of representation for $F(-, B)$. Show that the assignment $B \longrightarrow A_B$ extends to a functor and explain in what sense this extension is uniquely determined.

   (ii) Give a definition of limits in terms of representability.

   (iii) Show that if a category $\mathcal{D}$ has limits of shape $\mathbb{I}$ then there is a functor $\int_I : [\mathbb{I}, \mathcal{D}] \longrightarrow \mathcal{D}$ which gives a (vertex of the) limit.

5. (i) Suppose that $F : \mathbb{I} \times \mathbb{J} \longrightarrow \mathcal{D}$ is such that the functors $F_J = F(-, J) : \mathbb{I} \longrightarrow \mathcal{D}$ have limits in $\mathcal{D}$ for all $J \in \mathbb{J}$. Show that there is a functor $\int_I F(I, -) : \mathbb{J} \longrightarrow \mathcal{D}$ with

$$\mathcal{D}(V, \int_I F(I, J)) \cong [\mathbb{I}, \mathcal{D}](\Delta V, F(-, J))$$

   naturally in V and J. (Hint: use the previous question.)

(ii) Now suppose that the functor $\int_I F(I, -)$ has a limit. Show that $F : \mathbb{I} \times \mathbb{J} \longrightarrow \mathcal{D}$ has a limit.

(iii) Deduce a result (which you should state precisely) to the effect that limits commute with limits.

6. Explain and prove the Density Formula

$$X(U) \cong \int^W \mathbb{C}(U, W) \times X(W)$$

for a presheaf $X \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$.

7. What does it mean for a category $\mathbb{C}$ to be *cartesian closed*?

(i) Show that the category of presheaves on a category $\mathbb{C}$ has products.

(ii) Show that the Yoneda embedding $H_\bullet : \mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ preserves products.

(iii) Show that the category of presheaves on a category $\mathbb{C}$ is cartesian closed.

(iv) Suppose that $\mathbb{C}$ is cartesian closed. Show that the Yoneda embedding $H_\bullet : \mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ preserves the cartesian closed structure.

8. Suppose that we have functors $G : \mathcal{D} \longrightarrow \mathcal{C}$ and $F : \mathcal{C} \longrightarrow \mathcal{D}$, together with natural transformations $\epsilon : FG \longrightarrow 1_{\mathcal{D}}$ and $\eta : 1_{\mathcal{C}} \longrightarrow GF$ such that the composite

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

is the identity. Show that the composite

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

is an idempotent (in the functor category $[\mathcal{C}, \mathcal{D}]$). Show that G has a left adjoint if and only if the idempotent splits. [An idempotent is a map $e : X \longrightarrow X$such that $e^2 = e$. An idempotent $e : X \longrightarrow X$ splits just when there are maps $i : Y \longrightarrow X$ and $r : X \longrightarrow Y$ such that $ri = 1_Y$ and $ir = e$.]

9. (i) Fix a topological space S and let $\mathcal{O}(S)$ be the poset of open subsets of S ordered by inclusion. Let

$$\Delta : \mathbf{Set} \longrightarrow [\mathcal{O}(S)^{\mathrm{op}}, \mathbf{Set}]$$

be the functor assigning to a set A the presheaf ΔA with constant value A. Exhibit a chain of adjoints

$$\Lambda \dashv \Pi \dashv \Delta \dashv \Gamma \dashv \nabla.$$

You are only required to describe the effect of these functors on objects, and when you show adjointness you are not required to carry out any formal checks of naturality. Does this chain of adjoints extend further in either direction?

(ii) Let $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor taking a small category to its set of objects. Exhibit a chain of adjoints

$$C \dashv D \dashv O \dashv I.$$

Does this chain of adjoints extend further in either direction?

10. Let $\mathcal{C}$ be a category with initial and terminal objects. Let $\mathbf{n}$ denote an n-element totally ordered set, so that a functor $\mathbf{n} \longrightarrow \mathcal{C}$ is a composable string of $n-1$ morphisms in $\mathcal{C}$.

(i) Show that there are functors

$$F_0, \ldots, F_{n+1} : [\mathbf{n}, \mathcal{C}] \longrightarrow [\mathbf{n+1}, \mathcal{C}]$$

and

$$G_0, \ldots, G_n : [\mathbf{n+1}, \mathcal{C}] \longrightarrow [\mathbf{n}, \mathcal{C}]$$

which form a chain of adjunctions

$$F_0 \dashv G_0 \dashv F_1 \dashv G_1 \dashv \cdots \dashv G_n \dashv F_{n+1}.$$

(ii) Now suppose that the initial and terminal objects are not isomorphic, and prove that the chain of adjunctions extends no further in either direction.

11. Let $F \dashv G : \mathcal{D} \longrightarrow \mathcal{C}$ be an adjunction with unit η and counit ϵ. Show that the following conditions are equivalent:

i $F\eta$ is a natural isomorphism.

ii ϵF is a natural isomorphism.

iii $G\epsilon F$ is a natural isomorphism.

iv $GF\eta = \eta GF$.

v $GF\eta G = \eta GFG$.

vi–x the duals of **i–v**

(An adjunction with the above properties is said to be *idempotent*.)

12. Let $A : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor taking a small category to its set of arrows.

(i) Show that A has a left adjoint T.

(ii) Show that T does not have a left adjoint and A does not have a right adjoint.