The extremal symmetry of arithmetic simplicial complexes

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1 Introduction

Let K be a nonarchimedean local field, for example the p-adic numbers $\mathbf{Q}_p$ ($\mathrm{char}(K) = 0$) or the field of Laurent series over a finite field $\mathbf{F}_p((t))$ ($\mathrm{char}(p) > 0$) . Let $G = \mathrm{PGL}_n(K)$, or more generally the K-points of any absolutely simple, connected, algebraic K-group of adjoint form.

There is a natural way to associate to each cocompact lattice Γ in G a finite simplicial complex B_Γ, as follows. Bruhat-Tits theory (see below) provides a contractible, $\mathrm{rank}_K\, G$-dimensional simplicial complex X_G on which G acts by simplicial automorphisms. The lattice Γ acts properly discontinuously on X_G with quotient a simplicial complex B_Γ. [1]

Margulis proved (see, e.g., [Ma]) that $\mathrm{rank}_K\, G \geq 2$ implies that every lattice Γ in G is arithmetic. We also note that $\mathrm{char}(K) = 0$ implies every lattice in $G(K)$ is cocompact. In this paper we explore one aspect of the theme that, since the complex B_Γ is constructed using number theory, it should have remarkable properties. Here we concentrate on the extremal nature of the symmetry of B_Γ and all of its covers.

Our first theorem shows that the simplicial structure of B_Γ realizes all simplicial symmetries of any simplicial complex homeomorphic to B_Γ. For any simplicial complex C we denote by $\mathrm{Aut}(C)$ the group of simplicial automorphisms of C. We denote by $|C|$ the simplicial complex C thought of as a topological space, without remembering the simplicial structure.

Theorem 1.1. *Let K be a nonarchimedean local field, and let G be the K-points of an absolutely simple, connected algebraic K-group of adjoint form. Let Γ be a cocompact*

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[1] If Γ has torsion, one needs to barycentrically subdivide each simplex in X_G in order to make the quotient a true (not orbi) simplicial complex.

lattice in G, and let B_Γ be the quotient by Γ of the Bruhat-Tits building X_G associated to G. Suppose C is any simplicial complex homeomorphic to $|B_\Gamma|$. Then there is an injective homomorphism

$$\mathrm{Aut}(C) \longrightarrow \mathrm{Aut}(B_\Gamma).$$

Of course the simplicial structure on the space $|B_\Gamma|$ coming from the Bruhat-Tits building is not the unique simplicial structure satisfying Theorem 1.1. One can, for example, take all the top-dimensional simplices of B_Γ and subdivide them in the same way, so that the triangulation restricted to any maximal simplex gives a fixed simplicial isomorphism type. Each of these new triangulations of $|B_\Gamma|$ has automorphism group $\mathrm{Aut}(B_\Gamma)$. We call such a simplicial structure on B_Γ an *arithmetic simplicial structure.*

Our first main result is a rigidity theorem characterizing arithmetic simplicial structures among all simplicial structures on $|B_\Gamma|$. It gives a universal constraint on the symmetry of the universal covers of all other simplicial structures on $|B_\Gamma|$.

Theorem 1.2. *Let G and Γ as in Theorem 1.1 be given. Further assume that* $\mathrm{rank}_K G \geq 2$. *Fix a normalization of Haar measure μ on G. Then there exists a constant $N \geq 1$, depending only on $\mu(G/\Gamma)$, with the following property: Let C be any simplicial complex homeomorphic to $|B_\Gamma|$, and let Y be the universal cover of C (which therefore inherits a Γ-equivariant simplicial structure from C). Then either:*

1. $[\mathrm{Aut}(Y) : \Gamma] < N$, *so in particular* $\mathrm{Aut}(Y)$ *is finitely generated, or*
2. *C is an arithmetic simplicial structure, and so* $\mathrm{Aut}(Y)$ *is uncountable and acts transitively on the set of chambers of X_G.*

Remarks.

1. Theorem 1.2 is not true in the case that $\mathrm{rank}_K G = 1$, i.e. when X_G is a tree. An example is given in Section 5. The obstruction in this case is the fact that $\mathrm{Aut}(X_G)$ is "far" from G.
2. One is tempted to weaken the hypotheses of Theorem 1.1 and Theorem 1.2, for example to only require that C is homotopy equivalent to $|B_\Gamma|$ rather than homeomorphic to it. However the conclusion of each theorem is not true in this case, even for C of the same dimension as $|B_\Gamma|$. One can see this by taking, for any given $n \geq 2$,

a triangulation of the closed disk D^2 by dividing D^2 into n equal sectors based at the origin. This triangulation is invariant by the $2\pi/n$ rotation. Now let C be the complex obtained by attaching the central vertex of D^2 to some vertex of B_Γ. It is clear that $\mathrm{Aut}(C)$ contains $\mathbf{Z}/n\mathbf{Z}$. Since $n \geq 2$ was arbitrary, the conclusions both of Theorem 1.1 and of Theorem 1.2 do not hold.

3. Theorem 1.1 (resp. Theorem 1.2) is a simplicial analogue of a theorem of Farb-Weinberger from Riemannian geometry, given in [FW1] (resp. [FW2]). However, the mechanism giving rigidity is different here. Further, the type of generality achieved in the theorems in [FW2] seems not to be possible in the simplicial setting, since counterexamples abound, as the previous remark indicates.

One consequence of Theorem 1.2 is the following. Suppose B_Γ has more than one top-dimensional simplex; this can always be achieved by passing to a finite index subgroup of Γ. Now build a new triangulation C of $|B_\Gamma|$ by subdividing the top-dimensional simplices of B_Γ, so that the resulting triangulations on some pair of such simplices are not simplicially isomorphic. Then Theorem 1.2 implies that $[\mathrm{Aut}(Y) : \Gamma] < \infty$.

Another way to think of this is that, if we paint the (open) top-dimensional simplices of B_Γ with colors, and if we use at least 2 distinct colors, the group of color-preserving automorphisms of the universal cover of B_Γ is discrete, and contains Γ as a subgroup of finite index. This result is actually an ingredient in the proof of Theorem 1.2, and so is proven first. Such a result does not hold when $\mathrm{rank}_K G = 1$. We give explicit examples of this failure in Section 5.

Homeo($\boldsymbol{X_G}$). The ideas we use to prove the results above can be used to compute the group $\mathrm{Homeo}(X_G)$ of homeomorphisms of the topological space X_G. To state the theorem, we will need to consider the topological group $\mathrm{Homeo}^+(D^n)$ of orientation-preserving homeomorphisms of the closed n-dimensional ball D^n.

Theorem 1.3. *Let K be a nonarchimedean local field, and let G be the K-points of an absolutely simple, connected algebraic K-group of adjoint form. Let X_G be the Bruhat-Tits building associated to X_G. Let $\mathcal{C}$ denote the set of maximal simplices of X_G. Then there is an isomorphism of topological groups*

$$\mathrm{Homeo}(X_G) \approx A \rtimes \mathrm{Aut}(X_G)$$

where A is torsion free, and is isomorphic to a subgroup of $\prod_{\sigma\in\mathcal{C}} \mathrm{Homeo}^+(D^{\dim(X_G)})$ (endowed with the box topology) that we identify at the end of Section 3.

Outline of paper. After giving some preliminary material on Euclidean buildings in §2, we prove the main results in §4.1. In §5 we give an explicit example of a B_Γ satisfying the hypotheses of Theorem 1.1 and Theorem 1.2 and a non-example in the rank one case. In particular A is torsion free.

Standing assumption. All simplicial structures considered in this paper are assumed to be locally finite.

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2 Geometry and automorphisms of Euclidean buildings

We now recall some facts from Bruhat-Tits theory which will be needed in this paper. We refer the reader to [AB], [We] and to [Ti2] for these facts and definitions of terms.

2.1 The building $\mathbf{X_G}$

Let K be a nonarchimedean local field. Let $\mathbf{G}$ be the adjoint form of an absolutely almost simple, connected, simply connected algebraic group defined over K with positive K-rank. Let $G = \mathbf{G}(K)$.

The Bruhat-Tits theory associates a contractible simplicial complex X_G to G on which G acts by simplicial automorphisms. This is easiest to describe if we work with the simply connected cover of $\mathbf{G}$. So let $\widetilde{\mathbf{G}}$ be the simply connected cover of $\mathbf{G}$ and let $\widetilde{G} = \widetilde{\mathbf{G}}(K)$. Let

$$r := \mathrm{rank}_K(\mathbf{G})$$

An *Iwahori subgroup* I of $\widetilde{G}$ is the normalizer of a Sylow pro-p-subgroup of $\widetilde{G}$. These subgroups are conjugate to each other since the Sylow subgroups are conjugate. The *Euclidean (or affine) building* X_G associated with G is a simplicial complex defined as follows. The vertices of X_G correspond bijectively with maximal compact subgroups of $\widetilde{G}$. A collection of maximal compact subgroups gives a simplex in X_G precisely when their intersection contains an Iwahori subgroup. X_G is a contractible simplicial complex whose dimension equals $\text{rank}_K\mathbf{G}$. In particular, if $\text{rank}_K\mathbf{G} = 1$ then X_G is a tree.

We will need the following properties of X_G.

1. X_G is *thick*; that is, any i-simplex of X_G with $i < r := \dim(X_G)$ is contained in at least three $(i+1)$-simplices.

2. Given any apartment (maximal flat) A in X_G, any $(r-1)$-dimensional simplex lying in A is contained in precisely two r-simplices of A.

3. Any two simplices of X_G are contained in a common apartment.

2.2 The action of G

The groups $\widetilde{G}$ and G act simplicially on X_G by conjugation. The stabilizer in $\widetilde{G}$ of any vertex of X_G is a maximal compact subgroup of $\widetilde{G}$. There are $r+1$ orbits of vertices of X_G under the $\widetilde{G}$-action. In this way each vertex is given a *type*. The action of $\widetilde{G}$ on X_G is type-preserving, and is transitive on the set of chambers (simplicies of maximal dimension) in X_G.

Let G^+ be the normal subgroup of G generated by all the unipotent radicals of K-parabolic subgroups of G. The group G^+ is the image of $\widetilde{G}$ under the covering map $\widetilde{\mathbf{G}} \to \mathbf{G}$. For example, if $G = \text{PGL}_n(K)$ then $G^+ = \text{PSL}_n(K)$; see e.g. [Ma, Chapter I]. The covering map restricted to the unipotent subgroups is injective since the kernel of the covering map is the center of $\widetilde{\mathbf{G}}$. The subgroup G^+ is cocompact in G, and indeed is finite index when $\text{char}(K) = 0$. Further, G^+ acts by type-preserving automorphisms on X_G.

Denote by $\text{Aut}_{\text{alg}}(G)$ the group of algebraic automorphisms of G. This group is the semidirect product of G with the group of automorphisms of the Dynkin diagram for (the Lie algebra corresponding to) G. This is a group of order 2 (if Dynkin diagram is not D_4) and is S_3 (if Dynkin diagram is D_4); see Theorem 2.8 and the discussion on page 90 of [PR]. Let $\text{Aut}_G(K)$ denote the group of field automorphisms σ of K such that ${}^\sigma\mathbf{G}$ and $\mathbf{G}$ are K-isomorphic, where ${}^\sigma\mathbf{G}$ is the group obtained from $\mathbf{G}$ by applying σ to the defining equations. The group G is a locally compact topological group under the topology coming from that of K. We then have (see [BT]) that the group of automorphisms of G, which we denote by $\mathrm{Aut}(G)$, is an extension of $\mathrm{Aut}_{\mathrm{alg}}(G)$ by $\mathrm{Aut}_G(K)$; that is, the sequence

$$1 \to \mathrm{Aut}_{\mathrm{alg}}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}_G(K) \to 1$$

is exact. If $\mathbf{G}$ is a K-split algebraic group, then $\mathrm{Aut}(G) = \mathrm{Aut}_{\mathrm{alg}}(G) \rtimes \mathrm{Aut}(K)$, see [Ti1, 5.8, 5.9, 5.10] and references there.

From the description of X_G given above, one sees that the group $\mathrm{Aut}(G)$ acts on the X_G by simplicial automorphisms, giving a representation

$$\rho : \mathrm{Aut}(G) \to \mathrm{Aut}(X_G).$$

The central theorem about automorphisms of buildings is the following.

Theorem 2.1 (Tits [Ti1])**.** *Assume that* $\mathrm{rank}_K \mathbf{G} > 1$. *Then the representation*

$$\rho : \mathrm{Aut}(G) \to \mathrm{Aut}(X_G)$$

is an isomorphism.

Note that G, which is a subgroup of finite index in $\mathrm{Aut}_{\mathrm{alg}}(G)$, is a normal subgroup of $\mathrm{Aut}(X_G)$. The group $\mathrm{Aut}(X_G)$ is a locally compact group with respect to the compact-open topology. This topology coincides with the topology on $\mathrm{Aut}(X_G)$ determined by the property that the sequences of neighborhoods about the identity map correspond to sets of automorphisms that are the identity on larger and larger balls in X_G. On the other hand, the groups G and $\mathrm{Aut}_G(K)$ inherit a topology from the topology on K. The isomorphism given in Theorem 2.1 is an isomorphism of topological groups.

2.3 Apartments and root subgroups

The apartments (maximal flats) in X_G correspond to maximal diagonalizable subgroups in $\widetilde{G}$. Suppose S is a maximal diagonalizable subgroup of $\widetilde{G}$, and let A be the corresponding apartment in X_G. Then S acts on A by translation. The root subgroups corresponding to S acts on X_G as follows. Any root subgroup determines a family of parallel hyperplanes in A. If u lies in the root subgroup it will fix a *half-apartment* of A, i.e. one component of the complement of some hyperplane P in A. Moreover, P is an intersection of apartments, and the action of the root group is transitive on the link of P (see §1.4 and §2.1 of [Ti2] or, alternatively, Proposition 18.17 of [We]). In particular we have the following.

Fact 2.2. *Let G be as above. Then for any $(r-1)$-simplex σ of X_G, and for any three r-simplices $\alpha_1, \alpha_2, \alpha_3$ having σ as their common intersection, there exists an element $\phi \in G^+$ so that $\phi(\alpha_1) = \alpha_1$ and $\phi(\alpha_2) = \alpha_3$.*

As an example consider $G = \mathrm{PGL}_2(\mathbf{Q}_p)$. Then X_G is a $(p+1)$-regular tree. Let ℓ be the apartment in X_G corresponding to the diagonal group of G. In this case ℓ is a bi-infinite geodesic in X_G. Let $\ell(0)$ be the vertex corresponding to $\mathrm{PGL}_2(\mathbf{Z}_p)$, i.e. the vertex corresponding to the standard lattice $\mathbf{Z}_p^2$. The geodesic ray $\ell([0,\infty))$ is a half-apartment based at $\ell(0)$. The above fact gives that subgroup of the root group fixing the half apartment $[0,\infty)$ acts transitively on the set of vertices adjacent to 0 minus the unique neighbor 1 of 0 in $[0,\infty)$. The representatives of this subgroup may be taken to be the representatives of nontrivial cosets in $\mathbf{Z}_p/p\mathbf{Z}_p$ if we identify the root group with the additive group $\mathbf{Q}_p$.

In this paper we will assume that $\mathrm{rank}_K \mathbf{G} > 1$. In this case work of Margulis, and of Venkataramana [Ve] in the positive characteristic case, implies that any lattice Γ in such a G is arithmetic. See [Ma].

3 Topological (non)rigidity of X_G

The following is a kind of topological rigidity result for X_G: it gives that the topological structure of X_G remembers the simplicial structure. It is worth mentioning that in section 3 we only need Y to be a locally compact simplicial complex homeomorphic to X_G.

Proposition 3.1. *Let $f : X_G \to X_G$ be a homeomorphism. Then f maps k-dimensional simplices of X_G onto k-dimensional simplices for each $0 \le k \le \dim(X_G)$. Hence there is a natural homomorphism*

$$\psi : \mathrm{Homeo}(X_G) \to \mathrm{Aut}(X_G).$$

Proof. To prove the proposition we first need to make the following.

Definition 3.2 (k-manifold point)**.** *For $0 \le k \le \dim(X_G)$ we define a k-*manifold point *by backwards induction, as follows: For $k = \dim(X_G)$ we define a k-manifold point to be any point $x \in X_G$ that has a neighborhood homeomorphic to $\mathbf{R}^k$. Now suppose we have defined ℓ-manifold points for each $\ell > k$. Let $X_G(\ell)$ denote the set of all ℓ-manifold* *points in X_G. Then $x \in X_G$ is called a k-manifold point if x has a neighborhood in $X_G \setminus \bigcup_{k<\ell\leq \dim X_G} X_G(\ell)$ which is homeomorphic to $\mathbf{R}^k$.*

As mentioned above, the semisimplicity of G gives that X_G is a *thick building*: for each $k < \dim(X_G)$, every k-dimensional simplex of X_G is the face of at least three $(k+1)$-dimensional simplices of X_G. From this we clearly have the following:

$x \in X_G$ is a k-manifold point if and only if x lies in the interior of a k-simplex of X_G.

Since being a k-manifold point is clearly a topological property for any fixed k, it follows that any homeomorphism $f : X_G \to X_G$ maps k-manifold points to themselves, and therefore f maps open k-simplices into open k-simplices, for each $0 \leq k \leq \dim(X_G)$. Applying the same argument to f^{-1}, we see that f maps each open k-simplex of X_G homeomorphically onto an open k-simplex of X_G.

Since f is a homeomorphism it preserves adjacencies between simplices, and so f induces a simplicial automorphism of X_G. This association of f to the simplicial automorphism it induces is clearly a homomorphism. $\diamond$

In contrast to rigidity, it is easy to see that the kernel of ψ is huge. Indeed it clearly contains the infinite product, over all maximal simplices σ, of the group of homeomorphisms of the closed $\dim(X_G)$-disk which are the identity on $\partial\sigma$. On the other hands we have the following.

Proposition 3.3. *The kernel of ψ is torsion free.*

Proof. Suppose $\varphi \in \ker(\psi)$ and that φ has finite order. We will argue inductively on the dimension $k \geq 0$ that φ is the identity on the k-skeleton of X_G. Since $\psi(\varphi) = \mathrm{id}$, we get $\varphi(v) = v$ for any vertex $v \in X_G$. Now assume that φ is identity on each j-simplex of X_G for each $j < k$. Let D be any k-simplex of X_G. Since $\psi(\varphi) = \mathrm{id}$, we have from the definition of ψ that $\varphi(D) \subseteq D$. By induction we have that $\varphi(x) = x$ for each $x \in \partial D$.

Let $\tau := \varphi|_D$. Suppose that $\tau \neq \mathrm{id}$. Since φ is torsion, after raising τ to a power we can (and will) assume that τ has order p for some prime p.

Since we have a p-group $\langle\tau\rangle$ acting on a closed disk D, we can apply Smith Theory to this action. The pair $(D, \partial D)$ is of course a homology k-ball. By Smith's Theorem (see, e.g. [Br], Theorem III.5.2), the pair $(\mathrm{Fix}(\tau), \mathrm{Fix}(\tau|_{\partial D})$ is a mod-p homology r-ball for some

$0 \leq r \leq k$. Since $\tau|_{\partial D} = \text{id}$ by the induction hypothesis, we have that $\text{Fix}(\tau|_{\partial D}) = \partial D$, it follows that $r = k$.

Now suppose that $\text{Fix}(\tau) \neq D$. Pick $x \in D$ in the complement of $\text{Fix}(\tau)$. Then radial projection away from x to ∂D gives a homotopy equivalence of pairs

$$(\text{Fix}(\tau), \text{Fix}(\tau|_{\partial D})) \simeq (\partial D, \partial D).$$

But this contradicts the fact that $(\text{Fix}(\tau), \text{Fix}(\tau|_{\partial D}))$ is a mod-p homology k-disk with $k > 0$, since as such, we have

$$H_k(\text{Fix}(\tau), \text{Fix}(\tau|_{\partial D}); \mathbf{Z}/p\mathbf{Z}) = 0 \neq H_k(D, \partial D; \mathbf{Z}/p\mathbf{Z}).$$

Thus it must be that $\text{Fix}(\tau) = D$; that is, $\tau = \text{id}$. We have just proven that $\varphi|_D = \text{id}$ for each k-simplex D of X_G, so by the induction on k we have $\varphi = \text{id}$, as desired. $\diamond$

The homomorphism $\psi : \text{Homeo}(X_G) \to \text{Aut}(X_G)$ gives an exact sequence

$$1 \to A \to \text{Homeo}(X_G) \xrightarrow{\psi} \text{Aut}(X_G) \to 1$$

where A is the kernel of ψ. Proposition 3.3 says precisely that A is torsion free. The injective homomorphism

$$A \to \prod_{\sigma \in \mathcal{C}} \text{Homeo}^+(D^{\dim(X_G)})$$

given by $f \mapsto (f|_\sigma)_{\sigma \in \mathcal{C}}$ obviously has image those $(h_\sigma)_{\sigma \in \mathcal{C}}$ such that h_τ agrees with h_η on $\tau \cap \eta$. This proves Theorem 1.3.

4 Proving extremal symmetry

4.1 Proof of Theorem 1.1

We begin the proof of Theorem 1.1 with the following.

The following was suggested to us by G. Prasad. It suggests that Theorem 1.1 is actually a consequence of the topology of X_G rather than rigidity of the lattice Γ.

Proposition 4.1. *Let Y be a locally finite simplicial complex whose geometric realization is homeomorphic to the geometric realization of X_G. Then Y is a simplicial subdivision of X_G.*

Proof. For a simplicial complex Z we denote by Z^k the k-skeleton of Z. We need to show that $(X_G)^k \subset Y^k$. This statement is obvious for $k = \dim X_G$. We now assume that the statement is proven for each $i > k$, and we show it is then true for k. Note that

$$(X_G)^k = \bigcup_{\ell \leq k} (X_G)^\ell.$$

Since the property of having a neighborhood homeomorphic to $\mathbf{R}^\ell$ is clearly invariant by homeomorphism, we see that if $\ell \leq k$ and $x \in (X_G)^\ell$ then x is not an interior point of a j-simplex of Y for any $j > k$. Hence $X_G^k \subset Y^k$, as we wanted to show. $\diamond$

With the above in hand we can now prove Theorem 1.1.

Proof. [Proof of Theorem 1.1] Let Y be the lift of the simplicial structure of C to the universal cover Y of C. The given homeomorphism of C with the geometric realization $|B_\Gamma|$ lifts to a Γ-equivariant homeomorphism $Y \to X_G$. By Proposition 4.1 this homeomorphism is just the inclusion of a simplicial subdivision. Any $\phi \in \mathrm{Aut}(C)$ lifts to a Γ-equivariant element $\widetilde{\phi} \in \mathrm{Aut}(Y)$. Since Y is a simplicial subdivision of X_G, the simplicial automorphism $\widetilde{\phi}$ induces a unique element $\iota(\widetilde{\phi}) \in \mathrm{Aut}(X_G)$. Thus we have a homomorphism

$$\iota : \mathrm{Aut}(Y) \to \mathrm{Aut}(X_G)$$

which is clearly injective. Since ι is Γ-equivariant and injective, it follows that ι induces an injective homomorphism $\bar{\iota} : \mathrm{Aut}(C) \to \mathrm{Aut}(B_\Gamma)$, as desired. $\diamond$

It is worth mentioning that the injective homomorphism $\iota : \mathrm{Aut}(Y) \to \mathrm{Aut}(X_G)$ is continuous and proper with respect to the compact-open topology. We will identify $\mathrm{Aut}(Y)$ with $\iota(\mathrm{Aut}(Y))$ and consider $\mathrm{Aut}(Y)$ as a subgroup of $\mathrm{Aut}(X_G)$ in the sequel.

4.2 Characterizing X_G among all simplicial structures

For the rest of this section we assume that G, Γ and B_Γ are as in the statement of Theorem 1.2, in particular $\mathrm{rank}_K\, G \geq 2$.

Note that any simplicial automorphism of B_Γ induces an automorphism of $\pi_1(B_\Gamma) = \Gamma$, well-defined up to conjugacy. We thus have a homomorphism

$$\nu : \mathrm{Aut}(B_\Gamma) \to \mathrm{Out}(\Gamma)$$

where Out(Γ) is the group of *outer automorphisms* of Γ, i.e. the quotient of Aut(Γ) by inner automorphisms. The following is probably well-known to experts.

Theorem 4.2. *Let G, Γ and B_Γ be as above. Then ν, defined above, is an isomorphism.*

Proof. We first show that ν is injective. To see this, note that as B_Γ is a $K(\Gamma, 1)$ space, any $f_* \in \mathrm{Out}(\Gamma)$ is induced by some self-homotopy equivalence f of B_Γ, unique up to free homotopy. Suppose $f_* \in \ker(\nu)$, so that f is homotopically trivial. Suppose $f \in \ker(\nu)$. Since B_Γ is aspherical and f_* acts trivially (up to conjugation) on $\pi_1(B_\Gamma)$, it follows that f is freely homotopic to the identity map. Metrize B_Γ so that it has the path metric induced by giving each simplex the standard Euclidean metric; X_Γ then inherits a unique path metric making the covering $X_\Gamma \to B_\Gamma$ a local isometry.

Since since B_Γ is compact and f is homotopic to the identity, each track in this homotopy moves points of B_Γ some uniformly bounded distance D. Thus f has some lift $\widetilde{f} \in \mathrm{Aut}(X_G)$ such that $\widetilde{f}$ moves each point of X_G at most a distance D. We claim that the only element of $\mathrm{Aut}(X_G)$ that moves all points of X_G at most a uniformly bounded distance is the identity automorphism. Given this claim, it follows that $\widetilde{f}$, and hence f, is the identity, so that ν is injective.

The claim is well known, but for completeness we indicate a proof. The building X_G admits a nonpositively curved (in the CAT(0) sense) metric with the property that $\mathrm{Aut}(X_G) = \mathrm{Isom}(X_G)$. Now, the boundary ∂X_G of X_G as a nonpositively curved space, namely the set of Hausdorff equivalence classes of infinite geodesic rays, can be identified with the spherical Tits building associated to G (see [We, Theorem 8.24 and Chapter 28]) . By the nonpositive curvature condition, infinite geodesic rays in X_G either stay a uniformly bounded distance from each other, hence represent the same equivalence class in ∂X_G, or diverge with distance between point being unbounded. If an element $\phi \in \mathrm{Aut}(X_G)$ moves all points of X_G a uniformly bounded distance, it follows that ϕ induces the identity map on ∂X_G. But the natural homomorphism $\mathrm{Aut}(X_G) \to \mathrm{Aut}(\partial X_G)$ is injective (see [Ti1] or [We, Theorem 12.30 and Section 28.29]), from which it follows that ϕ is the identity, proving the claim.

We now show that ν is surjective, and thus is an isomorphism. To see this, note that by the assumptions on G, we can apply the Margulis Superrigidity Theorem (see [Ma]), proved in positive characteristic by Venkataramana [Ve], to the lattice Γ in G. This gives in particular that Γ satisfies *strong (Mostow-Prasad) rigidity*, which means that any automorphism of Γ can be extended to a continuous homomorphism of G. Note that the group of continuous automomorphisms of G is precisely $\mathrm{Aut}(X_G)$. Thus, given any $h \in \mathrm{Out}(\Gamma)$, there is some $h' \in \mathrm{Aut}(X_G)$ extending (a representative of) h, and so preserving Γ in G. Thus h' descends to the desired automorphism of B_Γ, proving that ν is surjective. We have thus shown that ν is an isomorphism. $\diamond$

The following result, crucial to our proof of Theorem 1.2, gives the consequence discussed at the end of the introduction.

Proposition 4.3 (Coloring rigidity)**.** *Let the notation and assumptions be as above. Then precisely one of the following holds:*

(i) $\mathrm{Aut}(Y)$ *is discrete.*

(ii) $G^+ \subseteq \iota(\mathrm{Aut}(Y))$, *where* ι *is the monomorphism in Proposition 3.3.*

Proof. Recall that $\iota(\mathrm{Aut}(Y))$ is a closed subgroup of $\mathrm{Aut}(X_G)$ with respect to the compact-open topology. The continuity of ι, together with the fact that ι is injective, implies that if $\iota(\mathrm{Aut}(Y))$ is discrete then $\mathrm{Aut}(Y)$ is discrete, in which case (i) would hold.

Assuming that (i) does not hold, we show that (ii) holds. Since (i) does not hold, the above paragraph shows that $\mathrm{Aut}(Y)$ is not discrete. Thus there is a sequence of elements $\varphi_n \in \mathrm{Aut}(Y)$ such that $\{g_n = \iota(\varphi_n)\}$ converges to the identity in $\mathrm{Aut}(X_G)$.

Note that $\Gamma \subseteq \mathrm{Aut}(Y)$ and, with this abuse of notation, $\iota(\Gamma) = \Gamma$. Note that $H := G \cap \iota(\mathrm{Aut}(Y))$ is a closed normal subgroup of $\iota(\mathrm{Aut}(Y))$ containing Γ. We claim that H is indiscrete. Assume the contrary and let g_n be as above. Since $\{g_n\}$ converges to the identity it follows that $g_n\gamma g_n^{-1} \to \gamma$ for any $\gamma \in \Gamma$. Since H is normal and discrete, and since $\Gamma \subset H$, it follows that $g_n\gamma g_n^{-1} = \gamma$ for n large enough.

By the assumption $\mathrm{rank}_K G \geq 2$, the group G has Kazhdan's property (T), and so the lattice Γ in G is finitely generated. Hence there exists some n_0 such that if $n > n_0$ then $g_n\gamma g_n^{-1} = \gamma$ for all $\gamma \in \Gamma$.

We now show g_n is the identity if $n > n_0$. Recall from Theorem 4.2 that $\mathrm{Aut}(B_\Gamma)$ and $\mathrm{Out}(\Gamma)$ are isomorphic. This together with the fact that g_n centralizes Γ if $n > n_0$, which we showed in the previous paragraph, imply that such g_n induces the trivial isometry of B_Γ. Note now that g_n centralizes Γ so the action of g_n on X_G is trivial. Thus g_n is identity if $n \geq n_0$, which is a contradiction. Hence H is indiscrete.

Recall that G/Γ has a finite G-invariant measure and $\Gamma \subset H$, hence G/H has a finite G-invariant measure, namely the direct image of the measure on G/Γ under the natural map $G/\Gamma \to G/H$. We showed above that H is an indiscrete subgroup of G. Now [Ma, Chapter II, Theorem 5.1] states that such a subgroup must contain G^+, as we wanted to show. $\diamond$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Recall from Proposition 4.1 that Y is a simplicial subdivision of X_G. There is nothing to prove if $\mathrm{Aut}(Y)$ is discrete, so suppose that this is not the case. By Proposition 4.3, there exists a subgroup $H \subseteq \mathrm{Aut}(Y)$ such that $\iota : H \to G^+$ is an isomorphism.

Let C be a chamber which is a fundamental domain for the standard action of G^+ on X_G. Then C is simplicially subdivided by Y. Since G^+ acts transitively on chambers, we have that if C' is any chamber of X_G then there is some $\varphi \in H$ such that $\iota(\varphi)(C) = C'$. Hence $\varphi(C) = C'$. The proof of the theorem is now complete. $\diamond$

5 Explicit examples

In this section we give explicit examples of the arithmetic complexes to which Theorem 1.1 and Theorem 1.2 apply. We then give examples non-example in the rank one case.

An explicit example where Theorem 1.1 and Theorem 1.2 apply. The explicit construction of these examples is given [LSV], using lattices constructed in [CS]. These examples were constructed as explicit examples of "*Ramanujan complexes*". Similar (explicit) constructions of complexes for which the above theorems holds are possible in characteristic zero using lattices constructed in [CMSZ1], [CMSZ2] and [MS].

Let $G = \mathrm{PGL}_3(\mathbf{F}_2((y)))$. We want to describe a quotient of X_G by a lattice Γ which is a congruence subgroup of a lattice Γ', where Γ' acts simply transitively on the vertices of X_G. Note that the building X_G is in fact a *clique complex* : a set of $(k+1)$ vertices are the vertices of a simplex if and only if every 2 of these vertices are the vertices of an edge. This property holds for quotient complexes as well. Thus, in order to describe the simplicial complex B_Γ it suffices to describe the Cayley graph of Γ'/Γ with an explicit set of generators.

Let t be a generator for the field of 16 elements whose minimal polynomial is t^4+t+1. In other words, $\mathbf{F}_{16} = \mathbf{F}_2[t]/(t^4+t+1)$. The following set S of seven matrices generates $\mathrm{PGL}_3(\mathbf{F}_{16})$. The clique complex corresponding to the Cayley graph of $\mathrm{PGL}_3(\mathbf{F}_{16})$ with respect to this set of generators is the complex obtained by taking the quotient of X_G by a lattice Γ, as above. This lattice is a congruence subgroup of a lattice Γ' which is constructed using a division algebra which splits at all places except at $1/y$ and $1/(y+1)$, at which it remains a division algebra.

The set S consists of the following seven matrices:

$$\begin{pmatrix} t+t^3 & t^2 & t+t^2 \\ t & t^3 & 1+t+t^2 \\ t+t^2 & 1+t^2 & 1+t^3 \end{pmatrix} \quad \begin{pmatrix} 1+t+t^2+t^3 & t+t^2 & 1+t^2 \\ 1+t & t^2+t^3 & 1 \\ 1+t^2 & t & t^3 \end{pmatrix}$$

$$\begin{pmatrix} 1+t^2+t^3 & 1+t^2 & t \\ 1+t+t^2 & t+t^3 & t^2 \\ t & 1+t & t^2+t^3 \end{pmatrix} \quad \begin{pmatrix} t+t^2+t^3 & t & 1+t \\ 1 & 1+t+t^2+t^3 & t+t^2 \\ 1+t & 1+t+t^2 & t+t^3 \end{pmatrix}$$

$$\begin{pmatrix} 1+t^3 & 1+t & 1+t+t^2 \\ t^2 & 1+t^2+t^3 & 1+t^2 \\ 1+t+t^2 & 1 & 1+t+t^2+t^3 \end{pmatrix} \quad \begin{pmatrix} t^3 & 1+t+t^2 & 1 \\ t+t^2 & t+t^2+t^3 & t \\ 1 & t^2 & 1+t^2+t^3 \end{pmatrix}$$

$$\begin{pmatrix} t^2+t^3 & 1 & t^2 \\ 1+t^2 & 1+t^3 & 1+t \\ t^2 & x+x^2 & t+t^2+t^3 \end{pmatrix}$$

An example in the rank one case. We begin with an example of an (arithmetic) lattice Λ in $G = \mathrm{PGL}_2(\mathbf{Q}_5)$, given by a symmetric generating set of Λ with 6 elements, which acts simply transitively on X_G. In other words X_G, which is a 6-regular tree, is the Cayley graph of Λ. This lattice Λ is also used in [LPS] to construct explicit examples of "Ramanujan graphs". Let

$$\mathbf{H}(\mathbf{Z}) = \{\alpha = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : \ a_i \in \mathbf{Z}\}$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. For any $\alpha \in \mathbf{H}(\mathbf{Z})$ we let $\overline{\alpha} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ and let $N(\alpha) = \alpha\overline{\alpha}$. Let

$$\Lambda' = \{\alpha \in \mathbf{H}(\mathbf{Z}) : N(\alpha) = 5^k,\ k \in \mathbf{Z} \text{ and } \alpha \equiv_2 1\}.$$

Now let

$$\Lambda = \Lambda' / \sim$$

where

$$\alpha \sim \beta \quad \text{if} \quad 5^{k_1}\alpha = \pm 5^{k_2}\beta \quad \text{for some } k_1, k_2 \in \mathbf{Z}.$$

Note that Λ is an (arithmetic) subgroup of $\mathrm{PGL}_2(\mathbf{Q}_5)$ and $[\alpha][\overline{\alpha}] = 1$. It is easy to see, and is shown in [LPS, Section 3], that Λ is actually a free group on $\{\alpha_1, \alpha_2, \alpha_3\}$, where $N(\alpha_i) = 5$ and $a_0 > 0$ for each $i = 1, 2, 3$. We identify X_G with the Caley graph of Λ with respect to the generating set $S = \{\alpha_1, \overline{\alpha_1}, \alpha_2, \overline{\alpha_2}, \alpha_3, \overline{\alpha_3}\}$.

Now let Γ be the kernel of the map $\Lambda \to \mathbf{Z}/4\mathbf{Z}$ given by $\alpha_i \mapsto i$ for $i = 1, 2, 3$. Then $B_\Gamma = X_G/\Gamma$ is the Cayley graph of $\mathbf{Z}/4\mathbf{Z}$ with respect to this generating set; that is, it is the complete graph with 4 vertices. We now color the edges of B_Γ with 3 different colors so that the edges emanating from a vertex have 3 different colors, and we lift this to a coloring of X_G using the Γ action.

Fix an arbitrarily large ball in X_G. Consider the automorphism ϕ of the tree X_G which fixes this ball pointwise and flips two rays corresponding to α_1 and $\overline{\alpha_3}$ emanating from a vertex on the sphere and is the identity everywhere else. Then ϕ lies in the group of color-preserving automorphisms of this tree. As the large ball was chosen arbitrarily, this argument proves that the group of color-preserving automorphisms of X_G is not discrete. Of course we can replace different "colors" by different simplicial isomorphism types of triangulations of the corresponding simplices. We thus have a contrast with the conclusion of Theorem 1.2.

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