Topology and arithmetic of resultants, I: spaces of rational maps

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Abstract

We consider the interplay of point counts, singular cohomology, étale cohomology, eigenvalues of the Frobenius and the Grothendieck ring of varieties for two families of varieties: spaces of rational maps and moduli spaces of marked, degree d rational curves in $\mathbb{P}^n$. We deduce as special cases algebro-geometric and arithmetic refinements of topological computations of Segal, Cohen–Cohen–Mann–Milgram, Vassiliev and others.

1 Introduction

The starting point of this paper is the idea that topological theorems about algebraic varieties should have algebro-geometric proofs, and that such proofs should yield arithmetic information. More precisely, suppose that X is a (not necessarily projective) algebraic variety defined over $\mathbb{Z}$. We can then either extend scalars or reduce modulo a prime p in order to view X through three lenses:

Topological: The $\mathbb{C}$-points $X(\mathbb{C})$ form a complex algebraic variety. Attached to $X(\mathbb{C})$ are its compactly supported singular cohomology groups $H^*_c(X(\mathbb{C});\mathbb{C})$ and Betti numbers $b_i(X(\mathbb{C}))$.

Geometric: Let $q = p^d$ be any positive power of p. The $\overline{\mathbb{F}}_q$-points $X(\overline{\mathbb{F}}_q)$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of the finite field $\mathbb{F}_q$. The set $X(\overline{\mathbb{F}}_q)$ comes equipped with an action of the Frobenius Frob_q, acting on the coordinates of affine charts via $x \mapsto x^q$. Attached to this setup we have:

- The associated compactly supported étale cohomology groups $H^*_{et,c}(X/\overline{\mathbb{F}}_q;\mathbb{Q}_\ell)$. These are representations of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.
- The eigenvalues of Frob_q acting on $H^*_{et,c}(X/\overline{\mathbb{F}}_q;\mathbb{Q}_\ell)$; these are called *weights*.

Arithmetic: The set $X(\mathbb{F}_q)$ of $\mathbb{F}_q$-points. As observed by Hasse, this set can be realized as the fixed set of $\mathrm{Frob}_q : X(\overline{\mathbb{F}}_q) \longrightarrow X(\overline{\mathbb{F}}_q)$.

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In this paper we consider the interplay of these three viewpoints applied in two concrete situations: spaces of rational maps and moduli spaces of marked, degree d rational curves in $\mathbb{P}^n$. We obtain as special cases algebro-geometric and arithmetic refinements of topological computations of Segal, Cohen–Cohen–Mann–Milgram, Vassiliev and others. Our main inspiration comes from the ideas in Segal's beautiful paper [9].

Spaces of rational maps. In this paper we consider the following families of spaces of (tuples of) polynomials.

Definition 1.1 (**$Poly_n^{d,m}$**)**.** Fix a field K with algebraic closure denoted $\overline{K}$. Fix $d, n \geq 0$ and $m \geq 1$. Define $Poly_n^{d,m}$ to be the set of all m-tuples $(f_1, \dots, f_m)$ of polynomials $f_i \in K[z]$ such that:

1. Each f_i is monic of degree d.
2. The set of polynomials $\{f_1, \dots, f_m\}$ has no common root in $\overline{K}$ of multiplicity n or greater.

The classical theory of discriminants and resultants tells us that $Poly_n^{d,m}$ is an algebraic variety defined over $\mathbb{Z}$; see §3 below. The varieties $Poly_n^{d,m}$ include well-known classes of varieties as special cases. These include the following examples.

1. $Poly_1^{d,m}$ can be identified with the space $Rat^*_{d,m-1}$ of all degree d rational maps $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$ that take a given basepoint in $\mathbb{P}^1$ to a given basepoint in $\mathbb{P}^{m-1}$. This is so because an m-tuple of degree d polynomials $(f_0(z), \dots, f_{m-1}(z))$ determines, and is determined by, a *rational map* $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$ such that $\phi(\infty) = [1 : \cdots : 1]$ and $\phi^*\mathcal{O}(1) \cong \mathcal{O}(d)$ via
$$\phi(z) := [f_0(z) : \cdots : f_{m-1}(z)].$$
2. $Poly_2^{d,1}$ is the space of monic, degree d, square-free polynomials, studied in detail by Arnol'd [1] and many others. More generally, $Poly_n^{d,1}$ is the space of degree d polynomials with no root of multiplicity n. These spaces were studied in detail by Vassiliev [10] and others.

In the theorem that follows, $K_0(\mathrm{Var}_K)$ denotes the Grothendieck ring of K-varieties and $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var}_K)$ denotes the Lefschetz motive; see §2.

Theorem 1.2 (**Arithmetic of $Poly_n^{d,m}$**)**.** *Let $m \geq 1$ and $d \geq n \geq 1$.*

1. **Motive/point count:**
$$[Poly_n^{d,m}] = \mathbb{L}^{dm} - \mathbb{L}^{(d-n)m+1}$$
in $K_0(\mathrm{Var}_K)$, and thus
$$|Poly_n^{d,m}(\mathbb{F}_q)| = q^{dm} - q^{(d-n)m+1}.$$
2. **Betti numbers :**
$$b_i(Poly_n^{d,m}(\mathbb{C})) = \begin{cases} 1 & i = 0,\ 2nm-3 \\ 0 & else \end{cases}$$

3. **Comparison Theorem:** *There are isomorphisms of graded vector spaces*

$$H^*_{et,c}(Poly^{d,m}_{n/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\mathbb{C}\cong H^*_c(Poly^{d,m}_n(\mathbb{C});\mathbb{C}),$$

4. **Weights:** *There are isomorphisms of Galois representations*

$$H^i_{et,c}(Poly^{d,m}_{n/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\cong\begin{cases}\mathbb{Q}_\ell((n-d)m-1) & i=2(d-n)m+3\\ \mathbb{Q}_\ell(-dm) & i=2dm\\ 0 & else\end{cases}$$

Our approach to (3) of Theorem 1.2 is not the standard one – by compactifying with normal crossings divisors. Instead, we use the combinatorics of a "root cover" and the ℓ-adic cohomology of linear subspace arrangements due to Björner-Ekedahl [3].

The numerics of Theorem 1.2 shows that two rather different-looking varieties have common arithmetic features.

Corollary 1.3 (**Comparing $Rat^*_{d,n}$ and $Poly^{d(n+1),1}_{n+1}$).**

1. *There are equalities in the Grothendieck ring of varieties*

$$[Rat^*_{d,n}]=[Poly^{d(n+1),1}_{n+1}].$$

In particular, $Rat^*_{d,n}$ and $Poly^{d(n+1),1}_{n+1}$ have equal point counts over all finite fields.

2. *There are isomorphisms of graded Galois representations*

$$H^*_{et,c}(Rat^*_{d,n/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\cong H^*_{et,c}(Poly^{d(n+1),1}_{n+1/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell).$$

These results have topological precursors. Cohen–Cohen–Mann–Milgram [4] showed that for any generalized homology theory $\mathbb{E}_*$,

$$\mathbb{E}_*(Rat^*_{d,1}(\mathbb{C}))\cong\mathbb{E}_*(Poly^{2d,1}_2(\mathbb{C})).$$

Building on this, Vassiliev [10] showed that

$$\mathbb{E}_*(Rat^*_{d,n}(\mathbb{C}))\cong\mathbb{E}_*(Poly^{d(n+1),1}_{n+1}(\mathbb{C})).$$

Guest–Kozlowski–Yamaguchi [6] showed that, for $n>1$, Vassiliev's isomorphism is actually induced by a homotopy equivalence

$$Rat^*_{d,n}(\mathbb{C})\simeq Poly^{d(n+1),1}_{n+1}(\mathbb{C}).$$

Corollary 1.3 lifts these equivalences to the ℓ-adic cohomology of the associated varieties over finite fields. Are there algebraic explanations for all of this?

Question 1.4. Are the varieties $Rat^*_{d,n}$ and $Poly^{d(n+1),1}_{n+1}$ isomorphic for $n\geq 2$? If not, what invariant distinguishes them?

Remarks 1.5.

1. Note that the requirement that $n \geq 2$ in Question 1.4 is necessary: it is *not* the case that $Rat^*_{d,1}$ is homotopy equivalent to $Poly_2^{2d,1}$, since $\pi_1(Rat^*_{d,1}(\mathbb{C})) \cong \mathbb{Z}$ but $Poly_2^{2d,1}(\mathbb{C}) \cong B_{2d}$, the braid group on $2d$ strands.

2. After seeing an earlier draft of this paper, Curt McMullen gave the following argument to show that Question 1.4 has a positive answer in the first nontrivial case, $d = 1$ and $n = 2$, as follows. The space $Poly_3^{3,1}$ is the complement in $\mathbb{A}^3$ of a twisted cubic, since $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$. On the other hand, the space $Rat^*_{1,2}$ is isomorphic to $\mathbb{A}^3 - \mathbb{A}^1$. Over $\mathbb{Z}[\frac{1}{3}]$, it is an exercise to write down an explicit isomorphism (involving division by 3) between these varieties. In characteristic 3, the twisted cubic above becomes $\{(0, 0, a^3)\}$, and its complement is isomorphic to $\mathbb{A}^3 - \mathbb{A}^1$ via the Frobenius $x \mapsto x^3$.

The moduli space of m-pointed rational curves in $\mathbb{P}^n$. Fix $d, n \geq 1$ and $m \geq 0$. Let $\mathcal{M}_{0,m}(\mathbb{P}^n, d)$ be the *moduli space of m-pointed, degree d rational curves in* $\mathbb{P}^n$; see §4 below for a precise definition. The spaces $\mathcal{M}_{0,m}(\mathbb{P}^n, d)$ are schemes defined over $\mathbb{Z}$. We state our results for the associated subvariety $\mathcal{M}^*_{0,m}(\mathbb{P}^n, d)$ of curves passing through a fixed base point in $\mathbb{P}^n$.

Theorem 1.6 (Arithmetic of $\boldsymbol{\mathcal{M}^*_{0,m}(\mathbb{P}^n, d)}$). *Let $m \geq 3$ and let $d, n \geq 1$. Then*

1. **Motive/point count:**

$$[\mathcal{M}^*_{0,m}(\mathbb{P}^n, d)] = \left(\prod_{i=2}^{m-2} (\mathbb{L} - i)\right) \left(\mathbb{L}^{d(n+1)} - \mathbb{L}^{(d-n-1)(n+1)+1}\right).$$

   in $K_0(\mathrm{Var}_K)$.

2. **Comparison Theorem:** *We have isomorphisms of graded vector spaces*

$$H^*_{et,c}(\mathcal{M}^*_{0,m}(\mathbb{P}^n; d)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong H^*_c(\mathcal{M}^*_{0,m}(\mathbb{P}^n; d)(\mathbb{C}); \mathbb{C}).$$

3. **Weights:** *Let*

$$\nu(a) := \sum_\sigma \prod_{j=1}^{2(m-3)-a} (\sigma(j) + 2),$$

   where the sum is over order preserving injections

$$\sigma \colon \{1, \ldots, 2(m-3) - a\} \hookrightarrow \{0, \ldots, m-4\}.$$

   Then, as a Galois representation, $H^i_{et,c}(\mathcal{M}^*_{0,m}(\mathbb{P}^n, d)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ consists precisely of

   (a) a direct summand

$$\mathbb{Q}_\ell(m - 3 + d(n+1) - i)^{\oplus \nu(2(m-3+d(n+1))-i))}$$

   if $m - 3 + 2d(n+1) \leq i \leq 2(m - 3 + d(n+1))$, and *(b) a direct summand*

$$\mathbb{Q}_\ell(m-1+d(n+1)-i)^{\oplus\nu(2(m-2+d(n+1))-(i+n))}$$

if $m+(2d-1)(n+1) \le i \le 2(m-3+d(n+1))-n+2,$

and nothing else.

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2 Preliminaries

In this section we review some background material that we will use later in the paper.

The Grothendieck ring of varieties. The Grothendieck ring $K_0(\mathrm{Var}_K)$ of varieties over the field K gives a useful framework for computing invariants like point counts. Recall that, as a group, $K_0(\mathrm{Var}_K)$ is generated by isomorphism classes of K-varieties $[X]$, modulo the relations $[X] = [Z] + [X - Z]$ for $Z \subset X$ a closed subvariety. Multiplication is induced by the Cartesian product of varieties, i.e. $[X][Y] = [X \times_K Y]$. It is elementary to prove that, for $K = \mathbb{F}_q$, the assignment

$$[X] \mapsto |X(\mathbb{F}_q)|$$

extends to a homomorphism $K_0(\mathrm{Var}_K) \longrightarrow \mathbb{Z}$. Thus, computations in $K_0(\mathrm{Var}_K)$ carry implications for point counts. Following convention, we denote by $\mathbb{L}$ the class $[\mathbb{A}^1] \in K_0(\mathrm{Var}_k)$ and refer to it as the *Lefschetz motive.*

The Grothendieck–Lefschetz Trace Formula Let Z be a scheme defined over $\mathbb{Z}$. We can reduce modulo q for any prime power q to obtain an algebraic variety defined over $\mathbb{F}_q$. We can then ask for the number $|Z(\mathbb{F}_q)|$ of $\mathbb{F}_q$-points of Z.

How can we compute $|Z(\mathbb{F}_q)|$? One approach begins with Hasse's fundamental observation that the set $Z(\mathbb{F}_q)$ is the set of fixed points of the *geometric Frobenius* morphism $\mathrm{Frob}_q \colon Z \longrightarrow Z$. In topology, the classical Lefschetz Fixed Point Theorem computes (in many cases) the number of fixed points of a continuous self-map $f : Z \longrightarrow Z$ of a triangulable topological space in terms of the traces of the induced maps $f^* : H^i(Z;\mathbb{Q}) \longrightarrow H^i(Z;\mathbb{Q})$ on the singular cohomology of Z. While singular cohomology is unsuitable for studying varieties over $\overline{\mathbb{F}}_q$, the *étale cohomology*, developed by Grothendieck and his school, is designed precisely for this purpose.

To set this up, fix a prime power q, a prime ℓ not dividing q, and let $\mathbb{Q}_\ell$ denote the ℓ-adic rationals. Attached to any variety Z defined over $\mathbb{F}_q$ are its étale cohomology groups

$H^i_{et}(X_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell), i \geq 0$ (cf. e.g. [8] or [5]). The key result we will use from this theory is the *Grothendieck–Lefschetz Trace Formula* [8, Theorem 25.1]. For smooth projective varieties Z defined over $\mathbb{F}_q$, this formula gives:

$$\begin{aligned} |Z(\mathbb{F}_q)| &= \#\,\mathrm{Fix}(\mathrm{Frob}_q : Z(\overline{\mathbb{F}}_q) \longrightarrow Z(\overline{\mathbb{F}}_q)) \\ &= \textstyle\sum_{i\geq 0}(-1)^i \,\mathrm{Trace}\left(\mathrm{Frob}_q : H^i_{et}(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \longrightarrow H^i_{et}(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\right) \end{aligned} \tag{2.1}$$

However the varieties we consider in this paper are not projective. To remedy this, we first note that Formula (2.1) holds for any Z of finite type if we replace $H^i_{et}(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ by compactly supported étale cohomology $H^i_{et,c}(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ (cf. [5, 6.1.1.1]). When Z is smooth, we can then apply Poincaré duality for étale cohomology [8, Theorem 24.1] to obtain

$$H^i_{et,c}(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \cong H^{2\dim(Z)-i}_{et}(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell(-\dim(Z)))^*$$

where * denotes the dual space. Plugging this in to (2.1) gives, for any smooth but not necessarily projective variety:

$$|Z(\mathbb{F}_q)| = q^{\dim(Z)} \sum_{i\geq 0}(-1)^i \,\mathrm{Trace}\left(\mathrm{Frob}_q : H^i_{et}(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)^* \longrightarrow H^i_{et}(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)^*\right) \tag{2.2}$$

3 Proof of Theorem 1.2

We remark that when $n > d$, the condition of having no common root of multiplicity n is clearly empty. When $n = d$, the condition that the degree d polynomials f_i have a common root of multiplicity $n = d$ is simply that there exists $z_0 \in K$ so that $f_i(z) = (z - z_0)^d$ for each i; the space of such polynomials is thus isomorphic to $\mathbb{A}^1$. We thus have

$$Poly^{d,m}_n \cong \begin{cases} \mathbb{A}^{dm} & \text{if } n > d \\ \mathbb{A}^{dm} - \mathbb{A}^1 & \text{if } n = d \end{cases} \tag{3.1}$$

Thus the most interesting case is when $d > n$.

Proof of Theorem 1.2. We prove the theorem by filtering the space of m-tuples of monic, degree d polynomials by closed subvarieties, and analyzing this filtration via topology and algebraic geometry, in a series of steps.

Step 1 (Building a filtration): Recording coefficients gives an isomorphism from the space of m-tuples $(f_1, \ldots, f_m)$ of monic, degree d polynomials to the affine space $\mathbb{A}^{dm}$. Parameterizing this space by the "roots" of the polynomials in the ordered m-tuple, we see that the ordered space of "roots" is isomorphic to $\overbrace{\mathbb{A}^d \times \cdots \times \mathbb{A}^d}^{m}$, and that $S_d \times \cdots \times S_d$ (m times) acts on this variety by permuting the roots. The quotient $\mathbb{A}^d \times \cdots \times \mathbb{A}^d/(S_d \times \cdots \times S_d)$ is thus a variety, and in fact is isomorphic to $\mathbb{A}^{dm}$, by Newton's theorem on symmetric polynomials.[1]

[1]We remind the reader that the identification of the space of m-tuples of monic polynomials with the quotient $\mathbb{A}^{dm}/(S_d)^m$ is an isomorphism of schemes. Over an algebraically closed field $\bar{K}$, the quotient map $\mathbb{A}^{dm} \longrightarrow \mathbb{A}^{dm}/(S_d)^m$ will be surjective on $\bar{K}$-rational points (since the roots of any polynomial over $\bar{K}$ are also in $\bar{K}$), but this will not be the case over a general field.

For any $k \geq 0$, denote by $R^{d,m}_{n,k}$ the space of m-tuples $(f_1, \dots, f_m)$ of monic, degree d polynomials for which there exists a monic $h \in K[z]$ with $\deg(h) \geq k$ and monic polynomials $g_i \in K[z]$ so that

$$f_i(z) = g_i(z)h(z)^n$$

for each $1 \leq i \leq m$. So for example $R^{d,m}_{n,0} = \mathbb{A}^{dm}$, and the $R^{d,m}_{n,k}$ give a descending filtration

$$\mathbb{A}^{dm} = R^{d,m}_{n,0} \supset R^{d,m}_{n,1} \supset \cdots \supset \emptyset. \tag{3.2}$$

We claim that each $R^{d,m}_{n,k}$ is a closed subvariety of $\mathbb{A}^{dm}$. To see this, let $\mathcal{S}_{d,m,n,k}$ be the collection of all m-tuples $\sigma = (\sigma_1, \dots, \sigma_m)$ of injections

$$\sigma_i : \{1, \dots, n\} \times \{1, \dots, k\} \longrightarrow \{1, \dots d\}$$

with the property that each σ_i is order-preserving on each $\{1, \dots, n\} \times \{j\}$ and such that $\sigma_i(1, a) < \sigma_i(1, b)$ for each $a < b$. For each $\sigma \in \mathcal{S}_{d,m,n,k}$, let L_σ be the linear subspace of $\mathbb{A}^{dm}$ defined by the equations

$$\{x_{\sigma_i(a,b)} = x_{\sigma_i(a',b)} : 1 \leq a, a' \leq n,\ 1 \leq i \leq m\} \cup \{x_{\sigma_i(a,b)} = x_{\sigma_j(a,b)} : 1 \leq i, j \leq m\}.$$

Note that each L_σ is an affine subspace of $\mathbb{A}^{dm}$ of dimension $k + m(d - nk)$. The action of S^m_d on $\mathbb{A}^d \times \cdots \times \mathbb{A}^d$ preserves the union of linear subspaces $\bigcup_{\sigma \in \mathcal{S}_{d,m,n,k}} L_\sigma$, and $R^{d,m}_{n,k}$ is the quotient of the union of linear subspaces by this action. Since the quotient of an affine variety by a finite group action is an affine variety, and since such quotient maps take closed invariant subvarieties to closed subvarieties, it follows that each $R^{d,m}_{n,k}$ is an closed subvariety of $\mathbb{A}^{dm}$.

Step 2 (Extracting common factors): Our goal in this step is to prove that, for each $k \geq 0$, there is an isomorphism of varieties:

$$R^{d,m}_{n,k} - R^{d,m}_{n,k+1} \cong Poly^{d-kn,m}_n \times \mathbb{A}^k. \tag{3.3}$$

The isomorphism (3.3) provides an algebraic map which extracts a common n-fold factor from a tuple of polynomials. This isomorphism will allow us to analyze $Poly^{d,m}_n$ recursively. Note that the case $k = 0$ follows by definition:

$$Poly^{d,m}_n := R^{d,m}_{n,0} - R^{d,m}_{n,1}.$$

Now assume $k \geq 1$. By the definition of $R^{d,m}_{n,k}$:

$$R^{d,m}_{n,k} - R^{d,m}_{n,k+1} \cong [\bigcup_{\sigma \in \mathcal{S}_{d,m,n,k}} L_\sigma - \bigcup_{\tau \in \mathcal{S}_{d,m,n,k+1}} L_\tau]/(S_d)^m \tag{3.4}$$

For $\sigma \in \mathcal{S}_{d,m,n,k}$ and $\tau \in \mathcal{S}_{d,m,n,k+1}$, we declare $\tau < \sigma$ if for each $1 \leq i \leq m$, we have that σ_i equals the restriction of τ_i to some set $\{1, \dots, n\} \times \{1, \dots, \hat{j}, \dots, k+1\}$. Note that $L_\tau \subset L_\sigma$ if and only if $\tau < \sigma$. Thus:

$$\bigcup_{\sigma \in \mathcal{S}_{d,m,n,k}} L_\sigma - \bigcup_{\tau \in \mathcal{S}_{d,m,n,k+1}} L_\tau = \bigcup_{\sigma \in \mathcal{S}_{d,m,n,k}} [L_\sigma - \bigcup_{\tau < \sigma} L_\tau]$$

Let L_1 denote the linear subspace corresponding to the m-tuple $(\sigma_1, \dots, \sigma_m)$ with $\sigma_i(a,b) = (b-1)n + a$ for all i. We think of L_1 as the "standard subspace". Under the $(S_d)^m$ action on $\bigcup_{\sigma \in \mathcal{S}_{d,m,n,k}} L_\sigma$, any subspace L_σ can be taken to any other L_τ, and the stabilizer of any single L_σ is isomorphic to the wreath product $[(S_n \wr S_k) \times S_{d-nk}]^m$. In particular, we have a transitive action of $(S_d)^m$ on $\mathcal{S}_{d,m,n,k}$.

Our choice of L_1 identifies $\mathcal{S}_{d,m,n,k}$ with a subset of $(S_d)^m$. Namely, for $\sigma \in \mathcal{S}_{d,m,n,k}$, define $g_\sigma \in (S_d)^m$ to be the m-tuple of permutations $(g_{\sigma_1}, \dots, g_{\sigma_m})$ where for $1 \le j \le m$, and for $1 \le i \le k$, the element g_{σ_j} sends the ordered set $((i-1)n+1, \dots, in)$ to the ordered set $(\sigma_j(1,i), \dots, \sigma_j(n,i))$, and where g_{σ_j} maps the ordered set $(nk+1, \dots, d)$ to the ordered set $(1, \dots, d) - \sigma_j(\{1, \dots, n\} \times \{1, \dots, k\})$.[2]

Each g_σ gives an isomorphism $L_\sigma - \bigcup_{\tau < \sigma} L_\tau \longrightarrow L_1 - \bigcup_{\tau<1} L_\tau$. Taken together, all of the g_σ give a map

$$\coprod_{\sigma \in \mathcal{S}_{d,m,n,k}} [L_\sigma - \bigcup_{\tau<\sigma} L_\tau] \longrightarrow L_1 - \bigcup_{\tau<1} L_\tau$$

We claim that this map descends to a well-defined map

$$\bigcup_{\sigma \in \mathcal{S}_{d,m,n,k}} [L_\sigma - \bigcup_{\tau<\sigma} L_\tau] \longrightarrow (L_1 - \bigcup_{\tau<1} L_\tau)/(S_k \times (S_{d-nk})^m). \tag{3.5}$$

If we denote points in the right hand side of (3.5) by $[x]$, then, by the transitivity of the $(S_d)^m$-action on $\mathcal{S}_{d,m,n,k}$, this claim amounts to proving that $[g_\sigma(x)] = [x]$ for any σ and any $x \in [L_1 - \bigcup_{\tau<1} L_\tau] \cap [L_\sigma - \bigcup_{\tau<\sigma} L_\tau]$.

Recall that we have been considering $\mathbb{A}^{dm}$ as the product $\overbrace{\mathbb{A}^d \times \cdots \times \mathbb{A}^d}^{m}$, and note that we have been implicitly denoting the coordinates of a point $x \in \mathbb{A}^{dm}$ by x_{ij} (or $x_{\sigma_i(a,b)}$) where the i denotes a factor of $\mathbb{A}^d$ and the j (or $\sigma_i(a,b)$) denotes the coordinate in $\mathbb{A}^d$. Under this notation, observe that the map $x \mapsto [g_\sigma(x)]$ is a rule for choosing:

1. k unordered blocks

$$\{x_{\sigma_1(1,i)} = \cdots = x_{\sigma_1(n,i)} = \cdots = x_{\sigma_m(1,i)} = \cdots = x_{\sigma_m(n,i)}\}_{i=1}^k$$

   of n equal coordinates out of each of the d blocks of coordinates in the m-tuple

$$(x_{11}, \dots, x_{1d}, \dots, x_{m1}, \dots, x_{md}),$$

   and

2. an m-tuple of unordered blocks of $d-nk$ coordinates $(\{x_{\sigma_1(i)}\}_{i=1}^{d-nk}, \dots, \{x_{\sigma_m(i)}\}_{i=1}^{d-nk})$.

Because $x \in L_\sigma - \bigcup_{\tau<\sigma} L_\tau$, there are at most $nk+n-1$ coordinates in each of the d-tuples $(x_{i1}, \dots, x_{id})$ from which g_σ can draw the k blocks of n equal coordinates. Further, there are at most k distinct values that the blocks of n equal coordinates can take. Because $x \in L_1 - \bigcup_{\tau<1} L_1$, we also know that for all $1 \le j \le m$, these k values are given by the values

[2] Stated in terms of group actions, we have just observed that the ordering on $\{1, \dots, d\}$ and on $(\{1, \dots, n\} \times \{1, \dots, k\}) \coprod \{1, \dots, d-nk\}$ determines a canonical section of the quotient map $(S_d)^m \longrightarrow ((S_d)^m / \operatorname{Stab}(L_1)) \cong \mathcal{S}_{d,m,n,k}$.

$\{x_{j,(a-1)n+1} = \cdots = x_{j,an}\}_{a=1}^{k}$. Therefore, up to reordering, we conclude that the two sets of k-values are equal

$$\begin{aligned}&\{x_{\sigma_1(1,a)} = \cdots = x_{\sigma_1(n,a)} = \cdots = x_{\sigma_m(1,a)} = \cdots = x_{\sigma_m(n,a)}\}_{a=1}^{k}\\ &= \{x_{1,(a-1)n+1} = \cdots = x_{1,an} = \cdots = x_{m,(a-1)n+1} = \cdots = x_{m,an}\}_{a=1}^{k}.\end{aligned}$$

It remains to show that for $1 \le j \le m$, the excess values $\{x_{\sigma_j(i)}\}_{i=1}^{d-nk}$ are, up to reordering, given by the values $\{x_{ji}\}_{i=nk+1}^{d}$.

Because there are at most $nk+n-1$ coordinates in each of the d-tuples $(x_{i1}, \ldots, x_{id})$ from which g_σ can draw the k blocks of n equal coordinates, and because $x \in L_1$, we know that, for each $1 \le j \le m$, the permutation g_{σ_j} is the identity on all but $n-1$ of the $d-nk$ coordinates $(x_{j,nk+1}, \ldots, x_{j,d})$. For each $1 \le j \le m$, let $I_j \subset \{(j, nk+1), \ldots, (j,d)\}$ denote the subset of indices of the coordinates $(x_{j,nk+1}, \ldots, x_{j,d})$ on which g_{σ_j} is not the identity. It suffices to show that that, for each $1 \le j \le m$, the permutation g_{σ_j} preserves both the set of values V that appear in $\{x_{j,a}\}_{a\in I_j}$ and their multiplicity.

Because $x \in L_\sigma - \bigcup_{\tau<\sigma} L_\tau$, it follows that V consists precisely of those values v (of the k blocks of n equal coordinates) that are taken by $\ell_v > n$ of the coordinates $\{x_{j,i}\}_{i=1}^{d}$ for $n \nmid \ell_v$. Further, for each $v \in V$, the multiplicity of v in the set $\{x_{j,a}\}_{a\in I_j}$ is equal to the remainder of ℓ_v when divided by n. Both of these are invariant under permutations of the coordinates. We therefore conclude that, for each $1 \le j \le m$, the excess values $\{x_{\sigma_j(i)}\}_{i=1}^{d-nk}$ are, up to reordering, given by the values $\{x_{ji}\}_{i=nk+1}^{d}$. We have thus shown that $[x] = [g_\sigma(x)]$, and therefore that the union of the g_σ descends to a well-defined map as in (3.5). By construction, the map (3.5) factors through a map

$$\left(\bigcup_{\sigma\in\mathcal{S}_{d,m,n,k}} [L_\sigma - \bigcup_{\tau<\sigma} L_\tau]\right)/(S_d)^m \longrightarrow (L_1 - \bigcup_{\tau<1} L_\tau)/(S_k \times (S_{d-nk})^m).$$

By inspection, this map is inverse to the map induced by the inclusion

$$L_1 - \bigcup_{\tau<1} L_\tau \longrightarrow \bigcup_{\sigma\in\mathcal{S}_{d,m,n,k}} [L_\sigma - \bigcup_{\tau<\sigma} L_\tau].$$

Now, L_1 is isomorphic to $\mathbb{A}^k \times (\mathbb{A}^{(d-nk)})^m$ as an $S_k \times (S_{d-nk})^m$-variety. Under this isomorphism:

$$L_1 - \bigcup_{\tau<1} L_\tau \cong \mathbb{A}^k \times [(\mathbb{A}^{d-nk})^m - \bigcup_{\nu\in\mathcal{S}_{d-nk,m,n,1}} L_\nu]$$

It follows that the right-hand side of (3.4) (and therefore the left-hand side!) is isomorphic to

$$(\mathbb{A}^k/S_k) \times (R_{n,0}^{d-nk,m} - R_{n,1}^{d-nk,m}) = \mathbb{A}^k \times Poly_n^{d-nk,m}$$

as desired.

Step 3 (Computing $[Poly_n^{d,m}] \in K_0(\mathrm{Var}_K)$): We have shown that there is a descending filtration of closed subvarieties:

$$\mathbb{A}^{dm} = R_{n,0}^{d,m} \supset R_{n,1}^{d,m} \supset \cdots \supset \emptyset.$$

As a result, $\mathbb{A}^{dm}$ admits a disjoint decomposition by locally closed subvarieties

$$\mathbb{A}^{dm} = \coprod_{k\geq 0} (R^{d,m}_{n,k} - R^{d,m}_{n,k+1}).$$

Taking classes in the Grothendieck ring $K_0(\mathrm{Var}_K)$ gives

$$\mathbb{L}^{dm} = \sum_{k\geq 0} ([R^{d,m}_{n,k}] - [R^{d,m}_{n,k+1}]), \tag{3.6}$$

where we write $\mathbb{L}$ for the class $[\mathbb{A}^1] \in K_0(\mathrm{Var}_K)$.

Plugging in the expression from Equation (3.3) into Equation (3.6) then gives the following recursive formula in the ring $K_0(\mathrm{Var}_K)$:

$$[Poly^{d,m}_n] = \mathbb{L}^{dm} - \sum_{k\geq 1} [Poly^{d-kn,m}_n] \cdot \mathbb{L}^k \tag{3.7}$$

It is left to prove the claimed result, namely that for $d > n \geq 1$, this recursion is solved by $[Poly^{d,m}_n] = \mathbb{L}^{dm} - \mathbb{L}^{(d-n)m+1}$. We proceed by induction on d. This gives :

$$\begin{aligned}
[Poly^{d,m}_n] &= \mathbb{L}^{dm} - \textstyle\sum_{k\geq 1} [Poly^{d-kn,m}_n] \cdot \mathbb{L}^k \\
&= \mathbb{L}^{dm} - \left(\textstyle\sum_{k\geq 1}^{\lfloor \frac{d}{n} \rfloor -1} \left(\mathbb{L}^{(d-nk)m} - \mathbb{L}^{(d-n(k+1))m+1}\right) \cdot \mathbb{L}^k + \mathbb{L}^{(d-n\lfloor \frac{d}{n} \rfloor)m} \cdot \mathbb{L}^{\lfloor \frac{d}{n} \rfloor}\right) \\
&= \mathbb{L}^{dm} - \left(\textstyle\sum_{k\geq 1}^{\lfloor \frac{d}{n} -1 \rfloor} \left(\mathbb{L}^{(d-nk)m+k} - \mathbb{L}^{(d-n(k+1))m+k+1}\right) + \mathbb{L}^{(d-n\lfloor \frac{d}{n} \rfloor)m+\lfloor \frac{d}{n} \rfloor}\right) \\
&= \mathbb{L}^{dm} - \left(\mathbb{L}^{(d-n)m+1} - \mathbb{L}^{(d-n\lfloor \frac{d}{n} \rfloor)m+\lfloor \frac{d}{n} \rfloor} + \mathbb{L}^{(d-n\lfloor \frac{d}{n} \rfloor)m+\lfloor \frac{d}{n} \rfloor}\right) \\
&= \mathbb{L}^{dm} - \mathbb{L}^{(d-n)m+1}.
\end{aligned}$$

Applying the homomorphism $K_0(\mathrm{Var}_K) \longrightarrow \mathbb{Z}$ given by $[X] \mapsto |X(\mathbb{F}_q)|$ gives $Poly^{d,m}_n(\mathbb{F}_q) = q^{dm} - q^{(d-n)m+1}$. This proves Statement (1) of the theorem.

Step 4 (The comparison theorem): We now establish Statement (3). Artin's comparison theorem (see [2]) shows that the ℓ-adic cohomology of the variety $Poly^{d,m}_{n/\mathbb{C}}$ agrees with the singular cohomology of $Poly^{d,m}_n(\mathbb{C})$. We now claim that this agreement persists when we replace $\mathbb{C}$ by $\overline{\mathbb{F}}_q$. To see this, we will use transfer plus a result of Björner–Ekedahl on complements of linear subspace arrangements.

The relevant subspace arrangement $V_{\mathcal{A}} \subset \mathbb{A}^{dm}$ consists of all subspaces of the form

$$L_{\bar{\sigma}} = \{z_{1,\sigma_1(i)} = \cdots = z_{m,\sigma_m(i)} \mid 1 \leq i \leq n\}$$

for some collection of injections

$$\bar{\sigma} = \{\{1,\ldots,n\} \xrightarrow{\sigma_j} \{1,\ldots,d\}\}_{j=1}^m.$$

The action of $(S_d)^{\times m}$ on $\mathbb{A}^{dm}$ by permuting the coordinates preserves the arrangement $V_{\mathcal{A}}$, and, over an algebraically closed field $\overline{K}$, the variety $Poly_n^{d,m}$ is the quotient of this action on the complement $\mathbb{A}^{dm} - V_{\mathcal{A}}$. By transfer, we see that

$$H^i_{et,c}(Poly^{d,m}_{n/\bar{K}}; \mathbb{Q}_\ell) \cong H^i_{et,c}(\mathbb{A}^{dm}_{\bar{K}} - V_{\mathcal{A}/\bar{K}}; \mathbb{Q}_\ell)^{(S_d)^{\times m}}.$$

Further, if we denote by $L_{\mathcal{A}}$ the intersection lattice of the subspaces in $V_{\mathcal{A}}$, we see that the natural identification

$$L_{\mathcal{A}_{/\mathbb{C}}} \longrightarrow L_{\mathcal{A}_{/\overline{\mathbb{F}}_q}}$$

defines an $(S_d)^{\times m}$-equivariant isomorphism of lattices which also respects the natural dimension functions on each. By [3, Theorem 4.9(i)], the ℓ-adic cohomology of the complement of a subspace arrangement is functorially determined by the intersection lattice together with its dimension function. In particular, the isomorphism above defines an $(S_d)^{\times m}$-equivariant isomorphism

$$H^i_{et,c}(\mathbb{A}^{dm}_{\mathbb{C}} - V_{\mathcal{A}/\mathbb{C}}; \mathbb{Q}_\ell) \cong H^i_{et,c}(\mathbb{A}^{dm}_{\overline{\mathbb{F}}_q} - V_{\mathcal{A}/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell).$$

The restriction of this isomorphism to the subspaces of invariants gives the claimed comparison isomorphism, proving Statement (3).

Step 5 (Betti numbers): To prove Statement (2) we proceed by induction on d. For the base case, the statement of the theorem follows immediately from the isomorphism

$$Poly_n^{n,m} \cong \mathbb{A}^{nm} - \mathbb{A}^1.$$

Now suppose that we have shown the result for $j < d$. We will prove the induction step by first computing the Betti numbers at step d. We then use the comparison isomorphism to obtain the ranks of the ℓ-adic cohomology groups, and we deduce the weights from the point count and Grothendieck–Lefschetz.

Step 5a (Inducting on the degree): Our argument is an extension of the arguments in Segal [9]. As in *loc. cit.* we construct, for all $d > n$, a continuous open embedding

$$Poly_n^{d-1,m}(\mathbb{C}) \times \mathbb{C}^m \longrightarrow Poly_n^{d,m}(\mathbb{C}) \tag{3.8}$$

by "bringing zeroes in from infinity". We will show by induction that this induces an isomorphism on compactly supported rational cohomology. For the base case, we have a map of cofiber sequences

$$\begin{tikzcd}
R^{n+1,m}_{n,1}(\mathbb{C})^+ \arrow[r] \arrow[d] & R^{n+1,m}_{n,0}(\mathbb{C})^+ \arrow[r] \arrow[d] & (Poly_n^{n+1,m}(\mathbb{C}))^+ \arrow[d] \\
(R^{n,m}_{n,1}(\mathbb{C}) \times \mathbb{C}^m)^+ \arrow[r] & (R^{n,m}_{n,0}(\mathbb{C}) \times \mathbb{C}^m)^+ \arrow[r] & (Poly_n^{n,m}(\mathbb{C}) \times \mathbb{C}^m)^+
\end{tikzcd}$$

where X^+ denotes the 1-point compactification of X. This is isomorphic to

$$\begin{tikzcd}
(\mathbb{C}^{m+1})^+ \arrow[r] \arrow[d] & (\mathbb{C}^{(n+1)m})^+ \arrow[r] \arrow[d] & (Poly_n^{n+1,m}(\mathbb{C}))^+ \arrow[d] \\
(\mathbb{C}^1 \times \mathbb{C}^m)^+ \arrow[r] & (\mathbb{C}^{nm} \times \mathbb{C}^m)^+ \arrow[r] & (Poly_n^{n,m}(\mathbb{C}) \times \mathbb{C}^m)^+
\end{tikzcd}$$

Because the first two vertical maps induce isomorphisms in compactly supported cohomology, the Five Lemma (applied to the map of long exact sequences in cohomology) shows that the right vertical map induces a cohomology isomorphism as well.

Step 5b (Computing $H_c^i(R_{n,k}^{d,m}(\mathbb{C});\mathbb{C})$): Suppose that we have shown that (3.8) induces an isomorphism in compactly supported singular rational cohomology for $j < d$. We will deduce the singular cohomology of $Poly_n^{d,m}(\mathbb{C})$ from the following claim.

Claim 1. Let $d \geq n$ and $k \leq \lfloor \frac{d}{n} \rfloor$. Then the compactly supported singular cohomology $R_{n,k}^{d,m}(\mathbb{C})$ is given by

$$H_c^i(R_{n,k}^{d,m}(\mathbb{C});\mathbb{C}) \cong \begin{cases} \mathbb{C} & i = 2(d-kn)m + 2k \\ 0 & \text{else} \end{cases}$$

We prove this claim by downward induction on k. For the base case, observe that

$$R_{n,1}^{n,m} \cong \mathbb{A}^1$$
$$R_{n,0}^{n,m} \cong \mathbb{A}^{nm}$$

so the statement follows. Similarly to (3.8), we also construct a continuous open embedding

$$R_{n,k}^{d-1,m}(\mathbb{C}) \times \mathbb{C}^m \longrightarrow R_{n,k}^{d,m}(\mathbb{C}) \tag{3.9}$$

by "bringing in zeroes from infinity". For $d \leq n+1$ and all k, we see that this induces an isomorphism on compactly supported cohomology. Similarly, supposing that $n \nmid d$, we see that the map induces an isomorphism on compactly supported cohomology for $k = \lfloor \frac{d}{n} \rfloor$.

Continuing to assume to $n \nmid d$, we now induct down on k. Suppose we have shown the claim for all $j < d$, and also assume that we have shown that the maps (3.9) induce isomorphisms in compactly supported rational cohomology for $j < d$ and all k. For the base case of the induction on k, let $a = \lfloor \frac{d}{n} \rfloor$. Then we have

$$\begin{aligned} R_{n,a}^{d,m} &\cong Poly_n^{d-an,m} \times \mathbb{A}^a \\ &\cong \mathbb{A}^{(d-an)m+a}, \end{aligned}$$

so the statement follows. Similarly, we observed above that the map (3.9) induces an isomorphism on compactly supported cohomology for $j = d$ and $k = a$. Now suppose we have shown that (3.9) induces such an isomorphism for $j = d$ and $k+1 > 1$. Observe that the "bringing in zeroes" maps fit together to give a continuous map of cofiber sequences

$$\begin{tikzcd} R_{n,k+1}^{d,m}(\mathbb{C})^+ \arrow[r] \arrow[d] & R_{n,k}^{d,m}(\mathbb{C})^+ \arrow[r] \arrow[d] & (Poly_n^{d-kn,m}(\mathbb{C}) \times \mathbb{C}^k)^+ \arrow[d] \\ (R_{n,k+1}^{d-1,m}(\mathbb{C}) \times \mathbb{C}^m)^+ \arrow[r] & (R_{n,k}^{d-1,m}(\mathbb{C}) \times \mathbb{C}^m)^+ \arrow[r] & (Poly_n^{d-1-kn,m}(\mathbb{C}) \times \mathbb{C}^k \times \mathbb{C}^m)^+ \end{tikzcd}$$

This gives rise to a map of long exact sequences

$$\begin{tikzcd} \cdots H_c^{i-2(k+m)}(Poly_n^{d-1-kn,m}(\mathbb{C});\mathbb{C}) \arrow[r] \arrow[d] & H_c^{i-2m}(R_{n,k}^{d-1,m}(\mathbb{C});\mathbb{C}) \arrow[r] \arrow[d] & H_c^{i-2m}(R_{n,k+1}^{d-1,m}(\mathbb{C});\mathbb{C}) \cdots \arrow[d] \\ \cdots H_c^{i-2k}(Poly_n^{d-kn,m}(\mathbb{C});\mathbb{C}) \arrow[r] & H_c^i(R_{n,k}^{d,m}(\mathbb{C});\mathbb{C}) \arrow[r] & H_c^i(R_{n,k+1}^{d,m}(\mathbb{C});\mathbb{C}) \cdots \end{tikzcd}$$

Our inductive hypotheses and the Five Lemma show that the claim holds for $R^{d,m}_{n,k}$ and that the map (3.9) is an equivalence for k. This concludes the induction step, and thus the claim, when $n \nmid d$.

When $d = an$ for $a > 1$, the induction proceeds as above, once we establish the cases $k = a$ and $k = a-1$. The claim about the cohomology of $R^{an,m}_{n,a}$ follows from the isomorphism

$$R^{an,m}_{n,a} \cong \mathbb{A}^a.$$

For $k = a - 1$, the identification

$$R^{an,m}_{n,a-1} - R^{an,m}_{n,a} \cong Poly^{n,m}_n \times \mathbb{A}^{a-1} \cong (\mathbb{A}^{nm} - \mathbb{A}^1) \times \mathbb{A}^{a-1}$$

gives rise to the long exact sequence in compactly supported cohomology

$$\cdots \longrightarrow H^{i-2(a-1)}_c(\mathbb{C}^{nm} - \mathbb{C}^1; \mathbb{C}) \longrightarrow H^i_c(R^{an,m}_{n,a-1}(\mathbb{C}); \mathbb{C}) \longrightarrow H^i_c(\mathbb{C}^a; \mathbb{C}) \xrightarrow{\partial} \cdots$$

This implies that

$$H^i_c(R^{an,m}_{n,a-1}(\mathbb{C}); \mathbb{C}) \cong \begin{cases} 0 & i < 2a \\ 0 & 2a+1 < i < 2nm + 2(a-1) \\ \mathbb{C} & i = 2nm + 2(a-1) \\ 0 & \text{else} \end{cases}$$

For the remaining cases, we have a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^{2a}_c(R^{an,m}_{n,a-1}(\mathbb{C}); \mathbb{C}) \longrightarrow H^{2a}_c(\mathbb{C}^a; \mathbb{C}) \xrightarrow{\partial} H^{2a+1}_c((\mathbb{C}^{nm} - \mathbb{C}^1) \times \mathbb{C}^{a-1}; \mathbb{C}) \\ \longrightarrow H^{2a+1}_c(R^{an,m}_{n,a-1}(\mathbb{C}); \mathbb{C}) \longrightarrow 0. \end{aligned}$$

It suffices to show that the boundary map is an isomorphism. To see this, consider the closed embedding

$$\begin{aligned} \mathbb{A}^1 \times \mathbb{A}^{a-1} &\longrightarrow \mathbb{A}^{nm} \times \mathbb{A}^{a-1} \\ (z - \lambda, h) &\mapsto ((z-\lambda)^n, \cdots, (z-\lambda)^n, h) \end{aligned}$$

where we view $\mathbb{A}^1 \times \mathbb{A}^{a-1}$ as the variety of pairs of monic polynomials (f, h) with $\deg(f) = 1$ and $\deg(h) = a - 1$, and where we view $\mathbb{A}^{nm} \times \mathbb{A}^{a-1}$ as the variety of $m+1$-tuples of monic polynomials

$$(f_1, \cdots, f_m, h)$$

with $\deg(f_i) = n$ and $\deg(h) = a - 1$. By inspection,

$$\mathbb{A}^{nm} \times \mathbb{A}^{a-1} - \mathbb{A}^1 \times \mathbb{A}^{a-1} \cong Poly^{n,m}_n \times \mathbb{A}^{a-1}$$

and the assignments

$$\begin{aligned} (z - \lambda, h) &\mapsto (z - \lambda)h \\ (f_1, \cdots, f_m, h) &\mapsto (f_1 h^n, \cdots, f_m h^n) \end{aligned}$$

determine a map of cofiber sequences

$$\begin{tikzcd}
(\mathbb{C}^1 \times \mathbb{C}^{a-1})^+ \arrow[r] \arrow[d] & (\mathbb{C}^{nm} \times \mathbb{C}^{a-1})^+ \arrow[r] \arrow[d] & (Poly_n^{n,m}(\mathbb{C}) \times \mathbb{C}^{a-1})^+ \arrow[d, "\cong"] \\
R_{n,a}^{an,m}(\mathbb{C})^+ \arrow[r] & R_{n,a-1}^{an,m}(\mathbb{C})^+ \arrow[r] & (Poly_n^{n,m}(\mathbb{C}) \times \mathbb{C}^{a-1})^+
\end{tikzcd}$$

The left vertical map is an $(a-1)$–fold branched cover, so on the top degree of compactly supported cohomology, the map it induces is multiplication by $a-1$. In particular, this gives an isomorphism in rational cohomology, and by the Five Lemma applied to the map of long exact sequences, we see that the cohomology of $R_{n,a-1}^{an,m}(\mathbb{C})$ is as claimed.

Finally, to see that (3.9) is an isomorphism for $d = an$ and $k = a-1$, we apply the Five Lemma to the map of long exact sequences induced by the continuous map of cofiber sequences

$$\begin{tikzcd}
R_{n,a}^{an,m}(\mathbb{C})^+ \arrow[r] \arrow[d] & R_{n,a-1}^{an,m}(\mathbb{C})^+ \arrow[r] \arrow[d] & (Poly_n^{n,m}(\mathbb{C}) \times \mathbb{C}^{a-1})^+ \arrow[d] \\
* \arrow[r] & (R_{n,a-1}^{an-1,m}(\mathbb{C}) \times \mathbb{C}^m)^+ \arrow[r] & (Poly_n^{n-1,m}(\mathbb{C}) \times \mathbb{C}^{a-1} \times \mathbb{C}^m)^+
\end{tikzcd}$$

The downward induction on k now proceeds exactly as above, and this completes the proof of the claim.

To conclude the inductive step of the theorem, we consider the map of cofiber sequences

$$\begin{tikzcd}
R_{n,1}^{d,m}(\mathbb{C})^+ \arrow[r] \arrow[d] & (\mathbb{C}^{dm})^+ \arrow[r] \arrow[d] & (Poly_n^{d,m}(\mathbb{C}))^+ \arrow[d] \\
(R_{n,1}^{d-1,m}(\mathbb{C}) \times \mathbb{C}^m)^+ \arrow[r] & (\mathbb{C}^{(d-1)m} \times \mathbb{C}^m)^+ \arrow[r] & (Poly_n^{d-1,m}(\mathbb{C}) \times \mathbb{C}^m)^+
\end{tikzcd}$$

Applying the Five Lemma to the long exact sequence in cohomology, we see that the claim implies that the map (3.8) induces an isomorphism in compactly supported cohomology, and that

$$H_c^i(Poly_n^{d,m}(\mathbb{C}); \mathbb{C}) \cong \begin{cases} \mathbb{C} & i = 2(d-n)m+3 \\ \mathbb{C} & i = 2dm \\ 0 & \text{else} \end{cases} \tag{3.10}$$

This establishes Statement (2) of the theorem.

Step 6 (Computing the weights): Statement (3) applied to (3.10) gives that $H_{et,c}^i(Poly_{n/_{\overline{\mathbb{F}}_q}}^{d,m}; \mathbb{Q}_\ell)$ is one-dimensional for $i = 2dm$ and $i = 2(d-n)m+3$, and vanishes for all other i. Thus the trace of Frob_q on each of these cohomology groups is just the corresponding eigenvalue λ_i of Frob_q. When $i = 2dm$, Poincaré Duality implies that $\lambda_{2dm} = q^{dm}$. Plugging this information in to the Grothendieck-Lefschetz trace formula gives:

$$\begin{aligned} q^{dm} - q^{(d-n)m+1} = |Poly_n^{d,m}(\mathbb{F}_q)| &= \lambda_{2dm} - \lambda_{2(d-n)m+3} \\ &= q^{dm} - \lambda_{2(d-n)m+3} \end{aligned}$$

which implies that $\lambda_{2(d-n)m+3} = q^{(d-n)m+1}$, as claimed. This completes the proof of Statement (4) of the theorem. $\square$

4 The moduli space of m-pointed rational curves in $\mathbb{P}^n$

Fix $d, n \geq 1$ and $m \geq 0$. For a variety X, let $\mathrm{PConf}_m(X)$ denote the set of ordered m-tuples of distinct points in X. Let

$$\mathcal{M}_{0,m}(\mathbb{P}^n, d) := (\mathrm{PConf}_m(\mathbb{P}^1) \times Rat_d(\mathbb{P}^1, \mathbb{P}^n))/\,\mathrm{PGL}_2$$

where PGL_2 acts diagonally. $\mathcal{M}_{0,m}(\mathbb{P}^n, d)$ is the *moduli space of m-pointed, degree d rational curves in $\mathbb{P}^n$*. Note that the m marked points on $\mathbb{P}^1$ give an *ordered* m-tuple. Because PGL_2 acts triply transitively on $\mathbb{P}^1$, if $m \geq 1$ then we can always take (a representative of) any element of $\mathcal{M}_{0,m}(\mathbb{P}^n, d)$ to have ∞ as the first marked point of the rational curve. There is then a map

$$\mathrm{ev}_\infty : \mathcal{M}_{0,m}(\mathbb{P}^n, d) \longrightarrow \mathbb{P}^n$$

taking an equivalence class of marking and rational map $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ to $\phi(\infty)$. It is straightforward to show that the map ev_∞ is a Zariski-locally trivial fibration with fiber

$$\mathcal{M}^*_{0,m}(\mathbb{P}^n, d) := (\mathrm{PConf}_{m-1}(\mathbb{A}^1) \times Rat^*_d(\mathbb{P}^1, \mathbb{P}^n))/\mathbf{Aff}_1$$

where $\mathbf{Aff}_1$, the subgroup of PGL_2 fixing $\infty \in \mathbb{P}^1$, acts diagonally. We thus have, for $m \geq 1$, and for any field K:

$$[\mathcal{M}_{0,m}(\mathbb{P}^n, d)] = [\mathcal{M}^*_{0,m}(\mathbb{P}^n, d)] \cdot [\mathbb{P}^n] \in K_0(\mathrm{Var}_K) \tag{4.1}$$

Our goal now is to compute the motive/point count, Betti numbers, and weight filtration of $\mathcal{M}^*_{0,m}(\mathbb{P}^n, d)$ for $m \geq 3$; we intend to return to the cases $m = 1, 2$ in a sequel.

Proposition 4.1. *For $m \geq 3$ there is an isomorphism*

$$\mathcal{M}^*_{0,m}(\mathbb{P}^n, d) \cong \mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0, 1\}) \times Rat^*_d(\mathbb{P}^1, \mathbb{P}^n).$$

Proof. For $m \geq 3$, we define a map

$$\Psi : \mathrm{PConf}_{m-1}(\mathbb{A}^1) \times Rat^*_d(\mathbb{P}^1, \mathbb{P}^n) \longrightarrow \mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0, 1\}) \times Rat^*_d(\mathbb{P}^1, \mathbb{P}^n)$$

via

$$\Psi((z_1, \ldots, z_{m-1}), \phi) := (\beta(z_3), \ldots, \beta(z_{m-1}), \phi \circ \beta^{-1})$$

where β is the unique element of $\mathbf{Aff}_1$ so that $\beta(z_1) = 0$ and $\beta(z_2) = 1$. Note that we are making use of the simply transitive action of $\mathbf{Aff}_1$ on the space of pairs of distinct points in $\mathbb{A}^1$. Note also that for any $\alpha \in \mathbf{Aff}_1$:

$$\Psi(\alpha \cdot (z_1, \ldots, z_{m-1}, \phi \circ \alpha^{-1}) = \Psi((z_1, \ldots, z_{m-1}), \phi)$$

and so Ψ induces a map

$$\overline{\Psi} : \mathcal{M}^*_{0,m}(\mathbb{P}^n, d) \longrightarrow \mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0, 1\}) \times Rat^*_d(\mathbb{P}^1, \mathbb{P}^n).$$

The map $((z_3, \ldots, z_{m-1}), \phi) \mapsto ((0, 1, z_3, \ldots, z_{m-1}), \phi)$ is an inverse to $\overline{\Psi}$, and so $\overline{\Psi}$ is an isomorphism. $\square$

4.1 Proof of Theorem 1.6

Proof of Theorem 1.6. We begin with the motive/point count. Proposition 4.1 implies for $m \geq 3$ that:

$$[\mathcal{M}^*_{0,m}(\mathbb{P}^n, d)] = [\mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0,1\})][Rat^*_{d,n}].$$

The following proposition was explained to us by N. Gadish.

Proposition 4.2 (The class of $\mathbf{PConf}_{\boldsymbol{n}}\boldsymbol{(X)}$). *Let X be a variety defined over $\mathbb{Z}$. Then for any field K:*

$$[\mathrm{PConf}_r(X)] = \prod_{i=0}^{r-1}([X] - i) \in K_0(\mathrm{Var}_K).$$

Proof. We prove this by induction on r. For $r = 1$, there is nothing to show. Now assume we have shown it for $n < r$. Note that $\mathrm{PConf}_r(X)$ is the complement in X^r of a union of diagonal subspaces isomorphic to X^{r-1}. All of the iterated intersections of these subspaces are again isomorphic to X^i for $i < r$. By the inclusion–exclusion argument, this implies that $[\mathrm{PConf}_r(X)]$ is a polynomial in $[X]$ of degree r.

The key observation is that this polynomial is independent of X: indeed, it depends only on the combinatorics of the natural stratification of the fat diagonal in X^r, i.e. only on the combinatorics of partitions of $\{1, \cdots, r\}$. But, this same argument gives that $|\mathrm{PConf}_r(X)(\mathbb{F}_q)|$ is the *same* polynomial in $|X(\mathbb{F}_q)|$. Finally, by counting, we know that

$$|\mathrm{PConf}_r(X)(\mathbb{F}_q)| = \prod_{i=0}^{r-1}(|X(\mathbb{F}_q)| - i).$$

This proves the proposition. $\square$

Applying this formula with $X = \mathbb{A}^1 - \{0,1\}$, so that $[X] = \mathbb{L}^1 - 2$, gives

$$[\mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0,1\})] = \prod_{i=2}^{m-2}(\mathbb{L} - i).$$

Combining this with the calculation for $[Rat^*_{d,n}]$, we obtain the result.

For the Comparison Theorem, by Proposition 4.1, the Künneth isomorphism, and Theorem 1.2, we are reduced to producing an isomorphism

$$H^*_{et,c}(\mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0,1\})_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong H^*_c(\mathrm{PConf}_{m-3}(\mathbb{C} - \{0,1\}); \mathbb{C}).$$

Indeed, this follows from the same argument as for Theorem 1.2: the variety $\mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0,1\})$ is a complement of a hyperplane arrangement; the combinatorics of this arrangement are independent of the characteristic; and therefore, Björner–Ekedahl's results give the isomorphism.

We now compute the weights. By Proposition 4.1, Poincàre Duality and the Künneth isomorphism,

$$H^i_{et,c}(\mathcal{M}^*_{0,m}(\mathbb{P}^n, d)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong$$
$$\bigoplus_{a=0}^{2(m-3+d(n+1))-i} H^{2(m-3)-a}_{et,c}(\mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0,1\})_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \otimes H^{i+a-2(m-3)}_{et,c}(Rat^*_{d,n/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell).$$

The $Rat^*_{d,n}$ factor follows from Theorem 1.2. It remains to compute the first factor. Recall that because $\mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0,1\})$ is the complement of a hyperplane arrangement, its étale cohomology ring is generated in degree 1, and Frobenius acts on $H^i_{et}(\mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0,1\}); \mathbb{Q}_\ell)$ by q^i [7, Propositions 1, 2]. Applying Poincàre duality, we see that Frobenius acts on $H^i_{et}(\mathrm{PConf}_{m-3}(\mathbb{A}^1 - \{0,1\}); \mathbb{Q}_\ell)$ by q^{m-3-i}. It remains to compute the rank of the degree i component, and by the Comparison Theorem, it suffices to do this using singular cohomology over $\mathbb{C}$.

Ignoring the ring structure, the singular cohomology decomposes as a product

$$H^*(\mathrm{PConf}_{m-3}(\mathbb{C} - \{0,1\}); \mathbb{C}) \cong \bigotimes_{j=0}^{m-4} H^*(\mathbb{C} - \coprod_{j+2} *; \mathbb{C}).$$

Indeed, this follows from induction on m, and the Serre spectral sequence of the fibration

$$\mathrm{PConf}_{m-3}(\mathbb{C} - \{0,1\}) \longrightarrow \mathrm{PConf}_{m-4}(\mathbb{C} - \{0,1\}).$$

The key observation is that because the fundamental group of the base acts trivially on the cohomology of the fiber, we have

$$E_2^{p,q} = H^p(\mathrm{PConf}_{m-4}(\mathbb{C} - \{0,1\}); \mathbb{C}) \otimes H^q(\mathbb{C} - \coprod_{m-4} *; \mathbb{C}),$$

and, because the fibration admits a section, the spectral sequence degenerates on this page.

Using this decomposition, we see that the rank follows from the isomorphism of graded commutative rings

$$H^*(\mathbb{C} - \coprod_i *; \mathbb{C}) \cong \mathbb{C}[x_1, \cdots, x_i] / < x_a x_b = 0 >$$

with $|x_a| = 1$. In detail, we have

$$\begin{aligned} H^i_c(\mathrm{PConf}_{m-3}(\mathbb{C} - \{0,1\}); \mathbb{C}) &\cong H^{2(m-3)-i}(\mathrm{PConf}_{m-3}(\mathbb{C} - \{0,1\}); \mathbb{C}) \\ &\cong \bigoplus_{i_0 + \cdots + i_{m-4} = 2(m-3)-i} \bigotimes_{j=0}^{m-4} H^{i_j}(\mathbb{C} - \coprod_{j+2} *; \mathbb{C}) \\ &\cong \bigoplus_\sigma \bigotimes_{j=1}^{2(m-3)-i} H^1(\mathbb{C} - \coprod_{\sigma(j)+2} *; \mathbb{C}) \end{aligned}$$

(where the sum is as in the definition of ν above)

$$\begin{aligned} &\cong \bigoplus_\sigma \mathbb{C}^{\prod_{j=1}^{2(m-3)-i} (\sigma(j)+2)} \\ &\cong \mathbb{C}^{\sum_\sigma \prod_{j=1}^{2(m-3)-i} (\sigma(j)+2)} \end{aligned}$$

This completes the proof. □

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