

# On the asymptotic geometry of abelian-by-cyclic groups

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## 1 Introduction

Gromov’s Polynomial Growth Theorem [Gro81] states that the property of having polynomial growth characterizes virtually nilpotent groups among all finitely generated groups.

Gromov’s theorem inspired the more general problem (see, e.g. [GdlH91]) of understanding to what extent the asymptotic geometry of a finitely-generated solvable group determines its algebraic structure—in short, are solvable groups quasi-isometrically rigid? In general they aren’t: very recently A. Dioubina [Dio99] has found a solvable group which is quasi-isometric to a group which is not virtually solvable; these groups are finitely generated but not finitely presentable. In the opposite direction, first steps in identifying quasi-isometrically rigid solvable groups which are not virtually nilpotent were taken for a special class of examples, the solvable Baumslag-Solitar groups, in [FM98] and [FM99b].

The goal of the present paper is to show that a much broader class of solvable groups, the class of finitely-presented, nonpolycyclic, abelian-by-cyclic groups, is characterized among all finitely-generated groups by its quasi-isometry type. We also give a complete quasi-isometry classification of the groups in this class; such a classification for nilpotent groups remains a major open question. Motivated by these results, we offer a conjectural picture of quasi-isometric classification and rigidity for polycyclic abelian-by-cyclic groups in §10.1.

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The proofs of these results lead one naturally from a geometry of groups problem to the theory of dynamical systems via the asymptotic behavior of certain flows and their associated foliations.

## 1.1 Abelian-by-cyclic groups

A group  $\Gamma$  is *abelian-by-cyclic* if there is an exact sequence

$$1 \rightarrow A \rightarrow \Gamma \rightarrow Z \rightarrow 1$$

where  $A$  is an abelian group and  $Z$  is an infinite cyclic group. If  $\Gamma$  is finitely generated, then  $A$  is a finitely generated module over the group ring  $\mathbf{Z}[Z]$ , although  $A$  may not be finitely generated as a group.

By a result of Bieri and Strebel [BS78], the class of finitely presented, torsion-free, abelian-by-cyclic groups may be described in another way. Consider an  $n \times n$  matrix  $M$  with integral entries and  $\det M \neq 0$ . Let  $\Gamma_M$  be the ascending HNN extension of  $\mathbf{Z}^n$  given by the monomorphism  $\phi_M$  with matrix  $M$ . Then  $\Gamma_M$  has a finite presentation

$$\langle t, a_1, \dots, a_n \mid [a_i, a_j] = 1, ta_it^{-1} = \phi_M(a_i), i, j = 1, \dots, n \rangle$$

where  $\phi_M(a_i)$  is the word  $a_1^{m_1} \cdots a_n^{m_n}$  and the vector  $(m_1, \dots, m_n)$  is the  $i^{\text{th}}$  column of the matrix  $M$ . Such groups  $\Gamma_M$  are precisely the class of finitely presented, torsion-free, abelian-by-cyclic groups (see [BS78] for a proof involving a precursor of the Bieri-Neumann-Strebel invariant, or [FM99b] for a proof using trees). The group  $\Gamma_M$  is polycyclic if and only if  $|\det M| = 1$ ; this is easy to see directly, and also follows from [BS80].

## 1.2 Statement of results

The first main theorem in this paper gives a classification of all finitely-presented, nonpolycyclic, abelian-by-cyclic groups up to quasi-isometry. It is easy to see that any such group has a torsion-free subgroup of finite index, so is commensurable (hence quasi-isometric) to some  $\Gamma_M$ . The classification of these groups is actually quite delicate—the standard quasi-isometry invariants (ends, growth, isoperimetric inequalities, etc.) do not distinguish any of these groups from each other, except that the size of the matrix  $M$  can be detected by large scale cohomological invariants of  $\Gamma_M$ .

Given  $M \in \text{GL}(n, \mathbf{R})$ , the *absolute Jordan form* of  $M$  is the matrix obtained from the Jordan form for  $M$  over  $\mathbf{C}$  by replacing each diagonal entry with its absolute value, and rearranging the Jordan blocks in some canonical order.

**Theorem 1.1 (Classification theorem).** *Let  $M_1$  and  $M_2$  be integral matrices with  $|\det M_i| > 1$  for  $i = 1, 2$ . Then  $\Gamma_{M_1}$  is quasi-isometric to  $\Gamma_{M_2}$  if and only if there are positive integers  $r_1, r_2$  such that  $M_1^{r_1}$  and  $M_2^{r_2}$  have the same absolute Jordan form.*

**Remark.** Theorem 1.1 generalizes the main result of [FM98], which is the case when  $M_1, M_2$  are positive  $1 \times 1$  matrices; in that case the result of [FM98] says even more, namely that  $\Gamma_{M_1}$  and  $\Gamma_{M_2}$  are quasi-isometric if and only if they are commensurable. When  $n \geq 2$ , however, it's not hard to find  $n \times n$  matrices  $M_1, M_2$  such that  $\Gamma_{M_1}, \Gamma_{M_2}$  are quasi-isometric but not commensurable. Polycyclic examples are given in [BG96], and the same ideas are used to produce nonpolycyclic examples in [FM99c].

The following theorem shows that the algebraic property of being a finitely-presented, nonpolycyclic, abelian-by-cyclic group is in fact a large-scale geometric property.

**Theorem 1.2 (Quasi-isometric rigidity).** *Let  $\Gamma = \Gamma_M$  be a finitely presented abelian-by-cyclic group, determined by an  $n \times n$  integer matrix  $M$  with  $|\det M| > 1$ . Let  $G$  be any finitely generated group quasi-isometric to  $\Gamma$ . Then there is a finite normal subgroup  $N \subset G$  such that  $G/N$  is abstractly commensurable to  $\Gamma_N$ , for some  $n \times n$  integer matrix  $N$  with  $|\det N| > 1$ .*

**Remark.** Theorem 1.2 generalizes the main result of [FM99b], which covers the case when  $M$  is a positive  $1 \times 1$  matrix. The latter result was given a new proof in [MSW], and in §9 we follow the methods of [MSW] in proving Theorem 1.2.

**Remark.** The “finitely presented” hypothesis in Theorem 1.2 cannot be weakened to “finitely generated”. Dioubina shows [Dio99] that the wreath product  $\mathbf{Z} \wr \mathbf{Z}$ , an abelian-by-cyclic group of the form  $\mathbf{Z}[\mathbf{Z}]$ -by- $\mathbf{Z}$ , is quasi-isometric to the wreath product  $(\mathbf{Z} \oplus F) \wr \mathbf{Z}$  whenever  $F$  is a finite group. But  $(\mathbf{Z} \oplus F) \wr \mathbf{Z}$  has no nontrivial finite normal subgroups, and when  $F$  is nonabelian it is not abstractly commensurable to an abelian-by-cyclic group.

One of the key technical results used to prove Theorem 1.1 is the following theorem, which we believe is of independent interest. It describes a rigidity phenomenon for 1-parameter subgroups of  $\mathrm{GL}(n, \mathbf{R})$  which generalizes work of Benardete [Ben88] (see also [Wit90]).

A 1-parameter subgroup  $M^t$  of  $\mathrm{GL}(n, \mathbf{R})$  determines a 1-parameter family of quadratic forms  $Q_M(t) = (M^{-t})^T(M^{-t})$  on  $\mathbf{R}^n$ , where the superscript  $T$  denotes transpose. Each  $Q_M(t)$  determines a norm  $\|\cdot\|_{M,t}$  and a distance function  $d_{M,t}$  on  $\mathbf{R}^n$ .

**Theorem 5.11 (One-parameter subgroup rigidity)** *Let  $M^t, N^t$  be 1-parameter subgroups of  $\mathrm{GL}(n, \mathbf{R})$ , such that  $M = M^1$  and  $N = N^1$  have no eigenvalues on the unit circle. If there exists a bijection  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  and constants  $K \geq 1, C \geq 0$  such that for each  $t \in \mathbf{R}$  and  $p, q \in \mathbf{R}^n$*

$$-C + \frac{1}{K} \cdot d_{M,t}(p, q) \leq d_{N,t}(f(p), f(q)) \leq K \cdot d_{M,t}(p, q) + C$$

*then  $M$  and  $N$  have the same absolute Jordan form.*

The proof of Theorem 5.11 is given in §6, and shows that in fact  $f$  is a homeomorphism with a reasonably high degree of regularity; see Proposition 6.3.

**Remark.** The case of Theorem 5.11 when  $f$  is the identity map follows from a theorem of D. Benardete [Ben88]. See also D. Witte [Wit90]. Benardete’s theorem determines precisely when two one-parameter subgroups of  $\mathrm{GL}(n, \mathbf{R})$  diverge, and it applies as well to matrices with eigenvalues on the unit circle.

### 1.3 Homogeneous spaces

Using coarse topological and geometrical methods, we reduce the study of quasi-isometries of  $\Gamma_M$  to that of a certain Lie group  $G_M$ .

After squaring  $M$  if necessary, we can assume that  $\det M > 0$  and that  $M$  lies on a 1-parameter subgroup  $M^t$  of  $\mathrm{GL}(n, \mathbf{R})$ . The group  $\Gamma_M$  is a cocompact subgroup of the solvable Lie group  $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$ , where  $\mathbf{R}$  acts on  $\mathbf{R}^n$  by the 1-parameter subgroup  $M^t$ . The group  $\Gamma_M$  is discrete in  $G_M$  if and only if  $\det M = 1$ . See section 4 for details.

The groups  $G_M$ , with their left invariant metrics, give a rich and familiar collection of examples, including: all real hyperbolic spaces, when  $M$  is a constant times the identity; many negatively curved homogeneous spaces, when  $M$  has all eigenvalues  $> 1$  in absolute value; and 3-dimensional SOLV geometry, when  $M$  is a  $2 \times 2$  hyperbolic matrix of determinant 1. The negatively curved examples associated to a real diagonal matrix with all eigenvalues  $> 1$  were studied by Pansu [Pan89a] (and later Gromov [Gro93]), who analyzed their quasi-isometric geometry using the idea of “conformal dimension”.

We should mention also the result of Heintze [Hei74] that the class of connected, negatively curved homogeneous spaces consists precisely of those spaces of the form  $N \rtimes \mathbf{R}$  where  $N$  is a nilpotent Lie group, and the action of  $\mathbf{R}$  on the Lie algebra has all eigenvalues strictly outside the unit circle.

## 1.4 Outline of proofs

After preliminary sections, §3 on Linear Algebra, and §4 on The Solvable Lie Group  $G_M$ , the proof of Theorem 1.1 can be divided into 3 main parts: §5,6 on the Dynamics of  $G_M$ ; §7 on Quasi-Isometries of  $\Gamma_M$  via Coarse Topology; and §8 on Finding the Integers, where the pieces of the proof are put together. The proof of Theorem 1.2 is contained in §9 on Quasi-Isometric Rigidity. Finally we pose some conjectures and problems in §10.

### §5,6: Dynamics of $G_M$

In these two sections we classify the Lie groups  $G_M$  up to *horizontal-respecting* quasi-isometry, that is up to quasi-isometries  $\phi: G_M \rightarrow G_N$  which take each set of the form  $\mathbf{R}^m \times \{t\}$  to a set of the form  $\mathbf{R}^n \times \{h(t)\}$  for some function  $h$  called the *induced time change*.

#### **Theorem 5.2' (Horizontal respecting quasi-isometries: special case)**

*Let  $M, N$  lie on 1-parameter subgroups  $M^t, N^t$  of  $\mathrm{GL}(n, \mathbf{R})$ , and suppose that  $\det M, \det N > 1$ . If there exists a horizontal respecting quasi-isometry  $\phi: G_M \rightarrow G_N$ , then there exist real numbers  $r, s > 0$  so that  $M^r$  and  $N^s$  have the same absolute Jordan form.*

**Remark.** In the special case where  $M, N$  are diagonalizable with all eigenvalues  $> 1$ , this can be extracted from work of Pansu [Pan89a] *without* the assumption that  $\phi$  is horizontal respecting. This special case was later reconsidered by Gromov (see [Gro93] Section 7.C), as an application of his “inf $\delta$ im” invariant. Our statement and proof of Theorem 5.2 is inspired in part by the ideas of exponential growth rates built into the inf $\delta$ im invariant (see also comments after Proposition 5.8).

In §5 we give a slightly more general version of this statement, Theorem 5.2.

The proof of Theorem 5.2 uses a certain dynamical system on  $G_M$ , the “vertical flow” which flows upward at unit speed along flow lines of the form  $(\text{point}) \times \mathbf{R} \subset \mathbf{R}^m \rtimes_M \mathbf{R}$ . When  $M$  has no eigenvalues on the unit circle this is a hyperbolic or Anosov flow, and in general it is a partially hyperbolic flow. We prove Theorem 5.2 in several steps, using stronger and stronger dynamical properties of flows in  $G_M$ .

**Step 1 (Foliations Rigidity, Proposition 5.4)** Using the Shadowing Lemma from hyperbolic dynamics we show that  $\phi$  coarsely respects three dynamically defined foliations of  $G_M$  and  $G_N$ : the weak stable, weak unstable, and center foliations. This, together with a result of Bridson-Gersten that depends in turn on work of Pansu (see Corollary 5.6), allows reduction to the case where  $M, N$  have no eigenvalues on the unit circle.

**Step 2 (Time rigidity, Proposition 5.8)** We show that the induced time change map of  $\phi$  is actually an *affine* map between the time parameters of  $G_M$  and  $G_N$ . After taking a real power of  $N$  and composing with a vertical translation, we can assume that  $\phi$  preserves the time parameter, that is  $h(t) = t$ .

**Step 3 (One-parameter subgroup rigidity, Theorem 5.11)** From Step 2,  $\phi$  induces a quasi-isometry between corresponding level sets of the time parameter on  $G_M, G_N$ , which reduces the proof to Theorem 5.11, One-Parameter Subgroup Rigidity. The latter theorem is proved in §6, by studying rigidity properties of certain flags of foliations of  $\mathbf{R}^n$  associated to the absolute Jordan form of  $M \in \mathrm{GL}(n, \mathbf{R})$ .

## §7: Quasi-Isometries of $\Gamma_M$ via Coarse Topology

Given an integer matrix  $M \in \mathrm{GL}(n, \mathbf{R})$  with  $\det M > 1$ , we study the geometry of  $\Gamma_M$  by constructing a contractible metric cell complex  $X_M$  on which  $\Gamma_M$  acts freely, properly discontinuously and cocompactly by isometries, so that  $\Gamma_M$  is quasi-isometric to  $X_M$ . Topologically,  $X_M$  is a product of  $\mathbf{R}^m$  with the homogeneous directed tree  $T_M$  with one edge entering and  $d$  edges leaving each vertex. Here  $d = \det M$ . Metrically, for every coherently oriented line  $\ell$  in  $T_M$ , the metric on  $X_M$  is such that  $\mathbf{R}^m \times \ell$  is isometric to  $G_M$ .

The main result of this section, Proposition 7.1, says that a quasi-isometry  $f: X_M \rightarrow X_N$  induces a quasi-isometry  $\phi: G_M \rightarrow G_N$  which respects horizontal foliations. This is proved using coarse geometric and topological methods. This is precisely where the condition  $\det M, \det N > 1$  is essential for our proof, since it gives that the trees  $T_M, T_N$  have nontrivial branching, and this branching allows us to show that  $f$  “remembers” the branch points (see Step 2 of §7.2).

While this proof is in the spirit of [FM98], further complications arise in this more general case (see §7.2). Also, for other applications (e.g. [FM99a], [MSW]), we shall derive Proposition 7.1 from a still more general result,

Theorem 7.7, which applies to many graphs of groups whose vertex and edge groups are fundamental groups of aspherical manifolds of fixed dimension.

### §8: Finding the integers

Given integer matrices  $M, N \in \text{GL}(n, \mathbf{R})$  with  $|\det M|, |\det N| > 1$  such that  $\Gamma_M$  and  $\Gamma_N$  are quasi-isometric, a simple argument allows us to reduce to the case of positive determinant, and then the results of §5–7 combine to show that there are positive real numbers  $r, s$  so that  $M^r$  and  $N^s$  have the same absolute Jordan form. We need to show that *integral*  $r, s$  exist. This is done by showing that a quasi-isometry  $X_M \rightarrow X_N$  induces a bilipschitz homeomorphism between certain self-similar Cantor sets attached to  $X_M$  and  $X_N$ . Applying a theorem of Cooper on bilipschitz types of these Cantor sets allows us to conclude that  $(\det M)^p = (\det N)^q$  for some integers  $p, q \geq 1$ , from which the desired conclusion follows.

### §9: Quasi-Isometric Rigidity

To prove Theorem 1.2, we use the coarse topology results from §7 to show that a group quasi-isometric to some  $\Gamma_M$  admits a quasi-action on a tree of  $n$ -dimensional Euclidean spaces. We then use the results of [MSW] to convert this quasi-action into a true action on a tree, whose edge and vertex stabilizers are finitely generated groups quasi-isometric to  $\mathbf{Z}^n$ . The proof is completed by invoking well-known quasi-isometry invariants, combined with a brief study of injective endomorphisms of virtually abelian groups.

### §10: Questions

We make some conjectures concerning possible extensions of this work to the polycyclic case. Also, we state some problems on the quasi-isometry group of  $\Gamma_M$ .

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## 2 Preliminaries

This brief section reviews some basic material; see for example [GdlH91].

Given  $K \geq 1, C \geq 0$ , a  $(K, C)$  *quasi-isometry* between metric spaces is a map  $f: X \rightarrow Y$  such that:

1. For all  $x, x' \in X$  we have

$$\frac{1}{K} \cdot d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq K \cdot d_X(x, x') + C$$

2. For all  $y \in Y$  we have  $d_Y(y, f(X)) \leq C$ .

If  $f$  satisfies (1) but not necessarily (2) then it is called a  $(K, C)$  *quasi-isometric embedding*. If  $f$  satisfies only the right hand inequality of (1) then  $f$  is  $(K, C)$  *coarsely lipschitz*, and if in addition  $C = 0$  then  $f$  is  *$K$ -lipschitz*.

A *coarse inverse* of a quasi-isometry  $f: X \rightarrow Y$  is a quasi-isometry  $g: Y \rightarrow X$  such that, for some constant  $C' > 0$ , we have  $d(g \circ f(x), x) < C'$  and  $d(f \circ g(y), y) < C'$  for all  $x \in X$  and  $y \in Y$ . Every  $(K, C)$  quasi-isometry  $f: X \rightarrow Y$  has a  $K, C'$  coarse inverse  $g: Y \rightarrow X$ , where  $C'$  depends only on  $K, C$ : for each  $y \in Y$  define  $g(y)$  to be any point  $x \in X$  such that  $d(f(x), y) \leq C$ .

A fundamental fact observed by Efremovich, by Milnor [Mil68], and by Švarc, which we use repeatedly without mention, states that if a group  $G$  acts properly discontinuously and cocompactly by isometries on a proper geodesic metric space  $X$ , then  $G$  is finitely generated, and  $X$  is quasi-isometric to  $G$  equipped with the word metric.

Given a metric space  $X$  and  $A, B \subset X$ , we denote the *Hausdorff distance* by

$$d_{\mathcal{H}}(A, B) = \inf\{r \in [0, \infty] \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}$$

The following lemma says that an ambient quasi-isometry induces a quasi-isometry between subspaces of a certain type. A map  $\sigma: S \rightarrow X$  between geodesic metric spaces is *uniformly proper* if there is function  $\rho: [0, \infty) \rightarrow [0, \infty)$  with  $\lim_{t \rightarrow \infty} \rho(t) = +\infty$ , and constants  $K \geq 1, C \geq 0$ , such that for all  $x, y \in S$  we have:

$$\rho(d_S(x, y)) \leq d_X(\sigma(x), \sigma(y)) \leq K \cdot d_S(x, y) + C$$

The function  $\rho$  and the constants  $K, C$  are called *uniformity data* for  $\sigma$ .

**Lemma 2.1.** *Given geodesic metric spaces  $X, Y, S, T$ , a quasi-isometry  $f: X \rightarrow Y$ , and uniformly proper maps  $\sigma: S \rightarrow X$  and  $\tau: T \rightarrow Y$ , suppose that*



$d_{\mathcal{H}}(f\sigma(S), \tau(T)) < \infty$ . Then  $S, T$  are quasi-isometric. To be explicit, any function  $g: S \rightarrow T$  such that  $d_Y(f\sigma(x), \tau g(x))$  is uniformly bounded is a quasi-isometry; the quasi-isometry constants for  $g$  depend only on those for  $f$ , the uniformity data for  $\sigma$  and  $\tau$ , and the bound for  $d_Y(f\sigma(x), g\tau(x))$ .

*Proof.* Pick  $K \geq 1$ ,  $C \geq 0$  and  $\rho: [0, \infty) \rightarrow [0, \infty)$  such that  $f$  is a  $(K, C)$  quasi-isometry,  $d_Y(f\sigma(x), g\tau(x)) \leq C$ , and  $\rho, K, C$  are uniformity data for  $\sigma, \tau$ .

Consider  $x, y \in S$  such that  $d_S(x, y) \leq 1$ . We have  $d_Y(f\sigma(x), f\sigma(y)) \leq K^2 + KC + C$ , and so  $d_Y(\tau g(x), \tau g(y)) \leq K^2 + KC + 3C$ , from which it follows that  $\rho(d_T(g(x), g(y))) \leq K^2 + KC + 3C$ . Since  $\lim_{t \rightarrow \infty} \rho(t) = \infty$  we obtain a bound  $d_T(g(x), g(y)) \leq A$  depending only on  $K, C, \rho$ . The usual ‘‘rubber band’’ argument, using geodesics in  $S$  divided into subsegments of length 1 with a terminal subsegment of length  $\leq 1$ , suffices to prove that  $g$  is  $(K', C')$  coarsely lipschitz, with  $K', C'$  depending only on  $K, C, \rho$ .

For any  $\xi \in T$  there is a point  $\bar{g}(\xi) \in S$  such that  $d_Y(f\sigma\bar{g}(\xi), \tau(\xi)) \leq C$ . For any  $\xi, \eta \in T$  with  $d(\xi, \eta) \leq 1$  we have

$$\begin{aligned} d_Y(f\sigma\bar{g}(\xi), f\sigma\bar{g}(\eta)) &\leq d_Y(f\sigma\bar{g}(\xi), \tau(\xi)) + d_Y(\tau(\xi), \tau(\eta)) + d_Y(f\sigma\bar{g}(\eta), \tau(\eta)) \\ &\leq K + 3C \end{aligned}$$

and so  $\rho(d_S(\bar{g}(\xi), \bar{g}(\eta))) \leq d_X(\sigma\bar{g}(\xi), \sigma\bar{g}(\eta)) \leq K^2 + 4KC$ . As above we obtain an upper bound for  $d_S(\bar{g}(\xi), \bar{g}(\eta))$  and the rubber band argument shows that  $\bar{g}$  is coarsely lipschitz.

For any  $x \in S$ , setting  $\xi = g(x) \in T$ , we have

$$\begin{aligned} d_Y(f\sigma(x), f\sigma\bar{g}(\xi)) &\leq d_Y(f\sigma(x), \tau g(x)) + d_Y(\tau(\xi), f\sigma\bar{g}(\xi)) \\ &\leq 2C \end{aligned}$$

It follows that  $d_X(\sigma(x), \sigma\bar{g}(\xi)) \leq 3KC$  and so  $\rho(d_S(x, \bar{g}g(x))) = \rho(d_S(x, \bar{g}(\xi))) \leq 3KC$ , yielding an upper bound for  $d_S(x, \bar{g}g(x))$ . Similarly,  $d_Y(\xi, g\bar{g}(\xi))$  is bounded for all  $\xi \in T$ .

Knowing that  $g: S \rightarrow T$  and  $\bar{g}: T \rightarrow S$  are coarse lipschitz maps which are coarse inverses of each other, it easily follows that  $g$  is a quasi-isometry, with quasi-isometry constants depending only on the coarse lipschitz constants for  $g$  and  $\bar{g}$ , and on the coarse inverse constants for  $g, \bar{g}$ .  $\diamond$

### 3 Linear Algebra

In this section we collect some basic results about canonical forms of matrices, and growth of vectors under the action of a matrix.

Let  $\mathcal{M}(n, F)$  denote all  $n \times n$  matrices over a field  $F$ , and let  $\text{GL}(n, F)$  be the group of invertible matrices. Let  $\text{GL}_0(n, \mathbf{R})$  be the identity component of  $\text{GL}(n, \mathbf{R})$ , consisting of all matrices of positive determinant.

#### 3.1 Jordan Forms

A matrix  $J \in \mathcal{M}(k, \mathbf{C})$  is a *Jordan block* if it has the form  $J = J(k, \lambda) = \lambda \cdot \text{Id} + N$  where  $\lambda \in \mathbf{C}$  and  $N_{ij} = \delta(i+1, j)$ , so  $N$  is the  $k \times k$  matrix with 1's on the superdiagonal and 0's elsewhere.

A matrix  $M \in \mathcal{M}(n, \mathbf{C})$  is in *Jordan form* if it is in block diagonal form

$$M = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & J_I \end{pmatrix}$$

where each  $J_i$  is a Jordan block. Every matrix in  $\mathcal{M}(n, \mathbf{C})$  is conjugate, via an invertible complex matrix, to a matrix in Jordan form, unique up to permutation of the Jordan blocks. When all eigenvalues are real, say  $J_i$  has eigenvalue  $\ell_i$ , we resolve the nonuniqueness by requiring  $\ell_1 \geq \ell_2 \geq \dots \geq \ell_I$ , and for each  $i = 1, \dots, I-1$ , if  $\ell_i = \ell_{i+1}$  then  $\text{rk}(J_i) \geq \text{rk}(J_{i+1})$ .

A matrix  $J \in \mathcal{M}(k, \mathbf{R})$  is a *real Jordan block* if it has one of the following two forms. The first form is an ordinary Jordan block  $J(k, \ell)$  where  $\ell \in \mathbf{R}$ . The second form, which requires  $k$  to be even, has a  $2 \times 2$  block decomposition of the form

$$J = J(k, a, b) = \begin{pmatrix} Q(a, b) & \text{Id} & \dots & 0 & 0 \\ 0 & Q(a, b) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & Q(a, b) & \text{Id} \\ 0 & 0 & \dots & 0 & Q(a, b) \end{pmatrix}$$

where  $\text{Id}$  is the identity,  $0$  is the zero matrix,  $Q(a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ , and  $b \neq 0$ .

A matrix  $M \in \mathcal{M}(n, \mathbf{R})$  is in *real Jordan form* if it is in block diagonal form as above where each block  $J_i$  is a real Jordan block. Every matrix in  $\mathcal{M}(n, \mathbf{R})$  is conjugate, via an invertible real matrix, to a matrix in real Jordan form, unique up to permutation of blocks.

The *absolute Jordan form* of  $M \in \mathcal{M}(n, \mathbf{R})$  is the matrix obtained from the Jordan form of  $M$  by replacing each diagonal entry  $\lambda$  by  $\ell = |\lambda|$ , and permuting the blocks to resolve the nonuniqueness. If  $M$  is invertible then the absolute Jordan form of  $M$  can be written in block diagonal form

$$\begin{pmatrix} J_M^+ & 0 & 0 \\ 0 & J_M^0 & 0 \\ 0 & 0 & J_M^- \end{pmatrix}$$

where the diagonal entries of  $J_M^+$  are  $> 1$ , of  $J_M^0$  are  $= 1$ , and of  $J_M^-$  are  $< 1$ . We call  $J_M^+$  the *expanding part* of the absolute Jordan form,  $J_M^0$  the *unipotent part*, and  $J_M^-$  the *contracting part*, and the block matrix  $\begin{pmatrix} J_M^+ & 0 \\ 0 & J_M^- \end{pmatrix}$  is called the *nonunipotent part*. Of course, one or more of these parts might be empty.

Note that the Jordan form of the real matrix  $J(k, a, b)$  is

$$\begin{pmatrix} J(k/2, a + bi) & 0 \\ 0 & J(k/2, a - bi) \end{pmatrix}$$

and so the absolute Jordan form of  $J(k, a, b)$  is

$$\begin{pmatrix} J(k/2, \sqrt{a^2 + b^2}) & 0 \\ 0 & J(k/2, \sqrt{a^2 + b^2}) \end{pmatrix}$$

Given  $M \in \mathcal{M}(n, \mathbf{R})$ , this process may be applied block-by-block to the real Jordan form of  $M$ , and the blocks then permuted, to obtain the absolute Jordan form of  $M$ .

Let  $\mathrm{GL}_\times(n, \mathbf{R})$  denote the set of all matrices in  $\mathrm{GL}(n, \mathbf{R})$  lying on a 1-parameter subgroup of  $\mathrm{GL}(n, \mathbf{R})$ , so  $\mathrm{GL}_\times(n, \mathbf{R}) \subset \mathrm{GL}_0(n, \mathbf{R})$ . It is well-known and easy to see, given a matrix  $M \in \mathrm{GL}(n, \mathbf{R})$ , that  $M \in \mathrm{GL}_\times(n, \mathbf{R})$  if and only if the negative eigenvalue Jordan blocks of  $M$  may be paired up so that the two blocks occurring in each pair are identical to each other, and this occurs if and only if  $M$  has a square root in  $\mathrm{GL}(n, \mathbf{R})$ . Thus, if  $M$  does not already lie on a 1-parameter subgroup then  $M^2$  does. We are therefore free to replace a matrix by its square in order to land on a 1-parameter subgroup.

Given a 1-parameter subgroup  $\rho(t)$  of  $\mathrm{GL}(n, \mathbf{R})$ , if  $M = \rho(1)$  then we will often abuse notation and write  $\rho(t) = M^t$ , despite the fact that  $M$  may not lie on a unique 1-parameter subgroup.

Given  $A \in \mathcal{M}(n, \mathbf{R})$  in Jordan form—no  $J(k, a, b)$  blocks—we say that  $\rho(t) = e^{At}$  is a 1-parameter *Jordan subgroup*. Notice that the matrices  $e^{At}$

are *not* themselves in Jordan form. For example when  $A = J(n, \ell) = \ell \cdot \text{Id} + N$  is a single  $n \times n$  Jordan block then  $e^{At}$  is obtained by multiplying the scalar  $e^{\ell t}$  times the matrix

$$e^{N \cdot t} = \sum_{i=0}^n \frac{1}{i!} N^i \cdot t^i = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} & \frac{t^n}{n!} \\ & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-1}}{(n-1)!} \\ & & 1 & \cdots & \frac{t^{n-3}}{(n-3)!} & \frac{t^{n-2}}{(n-2)!} \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & t \\ & & & & & 1 \end{pmatrix} \quad (3.1)$$

Nevertheless, for any Jordan form matrix  $J = \ell \cdot \text{Id} + N$  with  $\ell \in \mathbf{R}$ , the Jordan form of  $e^J$  is  $e^\ell \cdot \text{Id} + N$ .

Given a general 1-parameter subgroup  $e^{\mu t}$  in  $\text{GL}(n, \mathbf{R})$ , choose  $A$  so that  $A^{-1} \mu A$  is in real Jordan form, and so  $A^{-1} \mu A = \delta + \nu + \eta$  where  $\delta$  is diagonal,  $\nu$  is superdiagonal, and  $\eta$  is skew-symmetric. We then have

$$e^{\mu t} = (Ae^{(\delta+\nu)t}A^{-1})(Ae^{\eta t}A^{-1})$$

Since  $\eta$  is skew symmetric it follows that  $e^{\eta t}$  is in the orthogonal group  $\text{O}(n, \mathbf{R})$ . We have therefore proved (see [Wit90] for this particular formulation):

**Proposition 3.1 (1-parameter Real Jordan Form).** *Let  $M^t$  be a 1-parameter subgroup of  $\text{GL}(n, \mathbf{R})$ . There exists a 1-parameter Jordan subgroup  $e^{Jt}$ , a matrix  $A \in \text{GL}(n, \mathbf{R})$ , and a bounded 1-parameter subgroup  $P^t$  conjugate into the orthogonal group  $\text{O}(n, \mathbf{R})$ , such that  $e^J$  is the absolute Jordan form of  $M$ , and letting  $\overline{M}^t = A^{-1}e^{Jt}A$  we have*

$$M^t = \overline{M}^t P^t = P^t \overline{M}^t$$

**Remark.** In [Wit90] the subgroup  $\overline{M}^t$  is called the *nonelliptic part* of  $M^t$ , and  $P^t$  is called the *elliptic part*. These two 1-parameter subgroups, which commute with each other, are uniquely determined by  $M^t$ .

### 3.2 Growth of vectors under a linear transformation

Consider a 1-parameter subgroup  $M^t$  of  $\text{GL}(n, \mathbf{R})$ , with real Jordan form  $M^t = (A^{-1}e^{Jt}A)P^t = \overline{M}^t P^t$ . Let

$$0 < \lambda_1 < \cdots < \lambda_L$$

be the eigenvalues of  $\overline{M}$ . Let  $V_l = \ker((\lambda_l \cdot \text{Id} - \overline{M})^m)$  be the *root space* of the eigenvalue  $\lambda_l$ , where  $m$  is the multiplicity of  $\lambda_l$ . Let  $n_l$  be the *index of nilpotency* of  $\overline{M} \mid V_l$ , the smallest integer such that  $V_l = \ker((\lambda_l \cdot \text{Id} - \overline{M})^{n_l})$ . For  $i = 0, \dots, n_l - 1$  let  $V_{l,i} = \ker((\lambda_l \cdot \text{Id} - \overline{M})^{i+1})$ , so  $V_{l,0}$  is the eigenspace of  $\lambda_l$  and  $V_{l,n_l-1} = V_l$ . We thus have the *Jordan decomposition* of  $\overline{M}$ , which consists of the direct sum of root spaces

$$\mathbf{R}^n = V_1 \oplus \dots \oplus V_L$$

together with the *Jordan filtrations*

$$V_{l,0} \subset \dots \subset V_{l,n_l-1} = V_l, \quad l = 1, \dots, L$$

This decomposition is uniquely determined by  $\overline{M}$ , and hence by  $M$ .

**Proposition 3.2 (Growth of vectors).** *With the above notation, there exist constants  $A, B > 0$  with the following properties. Given  $l = 1, \dots, L$  with  $\lambda_l \geq 1$ :*

**Exponential Lower Bound** *If  $v \in V_l$  and  $t \geq 0$  then*

$$\|M^t v\| \geq A \lambda_l^t \|v\|$$

*In fact the same lower bound holds if  $v \in V_l \oplus V_{l+1} \oplus \dots \oplus V_L$ .*

**Exponential Polynomial Upper Bound** *Given  $i = 0, \dots, n_l - 1$ , if  $v \in V_{l,i}$  and  $t \geq 1$  then*

$$\|M^t v\| \leq B \lambda_l^t t^i \|v\|$$

*In fact the same upper bound holds if  $v \in (V_1 \oplus \dots \oplus V_{l-1}) \oplus V_{l,i}$ .*

**Exponential Polynomial Lower Bound** *Given  $i = 0, \dots, n_l - 1$ , if  $v \in V_{l,i} \setminus V_{l,i-1}$  then there exists  $C_v > 0$  such that if  $t \geq 1$  then*

$$\|M^t v\| \geq C_v \lambda_l^t t^i$$

*Given  $l = 1, \dots, L$  with  $\lambda_l \leq 0$ , similar statements are true with negative values of  $t$ .*

*Proof.* We start with the case when  $M^t = e^{Jt}$  is a 1-parameter Jordan subgroup, and the proposition follows by examining each Jordan block (3.1).

The second case we consider is when  $M^t$  has all positive real eigenvalues. By Proposition 3.1 we have  $M^t = A^{-1} e^{Jt} A$ , and Proposition 3.2 follows

immediately from the first case applied to  $e^{Jt}$ , together with the fact that  $A$  takes the Jordan decomposition of  $M^t$  to the Jordan decomposition of  $e^{Jt}$ .

In the general case, applying Proposition 3.1 we have  $M^t = (A^{-1}e^{Jt}A)P^t = \overline{M}^t P^t$ . We can then apply the second case to  $\overline{M}^t = A^{-1}e^{Jt}A$ . Since  $P^t$  commutes with  $\overline{M}^t$  it follows that  $P^t$  preserves the Jordan decomposition of  $\overline{M}^t$ . Proposition 3.2 then follows from the boundedness of  $P^t$ .  $\diamond$

## 4 The Solvable Lie Group $G_M$

Recall that  $\mathrm{GL}_\times(n, \mathbf{R})$  denotes those matrices in  $\mathrm{GL}(n, \mathbf{R})$  which lie on a 1-parameter subgroup of  $\mathrm{GL}(n, \mathbf{R})$ . Also, each matrix in  $\mathrm{GL}_\times(n, \mathbf{R})$  has positive determinant.

Given a matrix  $M \in \mathrm{GL}_\times(n, \mathbf{R})$  lying on a 1-parameter subgroup  $M^t$  of  $\mathrm{GL}(n, \mathbf{R})$ , we associate a solvable Lie group denoted  $G_M$ . This is the semidirect product  $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$  with multiplication defined by

$$(x, t) \cdot (y, s) = (x + M^t y, t + s)$$

for all  $(x, t), (y, s) \in \mathbf{R}^n \times \mathbf{R}$ . We will often identify  $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$  with the underlying set  $\mathbf{R}^n \times \mathbf{R}$ .

**Remark.** Although the Lie group  $G_M$  depends on more than just the matrix  $M = M^1$  itself—it depends on the entire 1-parameter subgroup  $M^t$ —we suppress this dependence in our notation  $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$ . This is justified by the fact that the quasi-isometry type of  $G_M$  depends only on  $M$ , not on the 1-parameter subgroup containing  $M$  (see the remark after Proposition 4.1). Henceforth, when we say something like “Given  $M \in \mathrm{GL}_\times(n, \mathbf{R}) \dots$ ”, we will either implicitly or explicitly choose a 1-parameter subgroup  $M^t < \mathrm{GL}(n, \mathbf{R})$  with  $M^1 = M$ , which in turn determines  $G_M$ .

If  $M$  has integer entries then there is a homomorphism  $\Gamma_M \rightarrow G_M$  taking the commuting generators  $a_1, \dots, a_n$  to the standard basis of the integer lattice  $\mathbf{Z}^n \times 0 \subset \mathbf{R}^n \times 0 \subset \mathbf{R}^n \times \mathbf{R}$ , and taking the stable letter  $t$  to the generator  $(0, 1) \in \mathbf{R}^n \times \mathbf{R}$ . The relator  $ta_i t^{-1} = \phi_M(a_i)$  is checked by noting that

$$(0, 1) \cdot (x, 0) \cdot (0, -1) = (Mx, 0), \quad \text{for all } x \in \mathbf{R}^n$$

Cocompactness of the image of this homomorphism is obvious. To see that  $\Gamma_M$  embeds in  $G_M$  one checks that in the abelian-by-cyclic extension  $1 \rightarrow A \rightarrow \Gamma_M \rightarrow \mathbf{Z} \rightarrow 1$ , the group  $A$  is identified with the nested union  $\mathbf{Z}^n \cup$

$M^{-1}(\mathbf{Z}^n) \cup M^{-2}(\mathbf{Z}^n) \cup \dots$ , in  $\mathbf{R}^n$ . This also shows that discreteness of  $\Gamma_M$  in  $G_M$  is equivalent to  $\det M = 1$ , which is equivalent to  $\mathbf{Z}^n = M(\mathbf{Z}^n)$ .

For the next several sections we will investigate the geometry of the solvable Lie group  $G_M$ . In this section we begin by showing that  $G_M$  and  $G_N$  are quasi-isometric if  $M, N$  have powers with the same absolute Jordan form. Later in §7 we will see that when  $M$  has integer entries, much of the geometry of  $\Gamma_M$  is reflected in the geometry of  $G_M$ .

We endow  $G_M$  with the left invariant metric determined by taking the standard Euclidean metric at the identity of  $G_M \approx \mathbf{R}^n \times \mathbf{R} = \mathbf{R}^{n+1}$ . At a point  $(x, t) \in \mathbf{R}^n \times \mathbf{R} \approx G_M$ , the tangent space is identified with  $\mathbf{R}^n \times \mathbf{R}$ , and the Riemannian metric is given by the symmetric matrix

$$\begin{pmatrix} Q_M(t) & 0 \\ 0 & 1 \end{pmatrix}$$

where  $Q_M(t) = (M^{-t})^T M^{-t}$ . For each  $t \in \mathbf{R}$ , the identification  $\mathbf{R}^n \approx \mathbf{R}^n \times t \subset G_M$  induces in  $\mathbf{R}^n$  the metric determined by the quadratic form  $Q_M(t)$ . This metric has distance formula

$$d_{M,t}(x, y) = \|M^{-t}(x - y)\|$$

### Remarks

1. When  $M$  is a  $1 \times 1$  matrix with entry  $a > 1$ , the group  $G_M$  is isomorphic to  $\text{Aff}(\mathbf{R})$ , the group of affine transformations of  $\mathbf{R}$ , and as a Riemannian manifold  $G_M$  is isometric to a scaled copy of the hyperbolic plane with constant sectional curvature depending on  $a$ .
2. The eigenvalues of  $M$  are greater than 1 in absolute value if and only if all sectional curvatures of  $G_M$  are negative (see [Hei74]).

**Proposition 4.1 (How the metric on  $G_M$  depends on choices).** *Given 1-parameter subgroups  $M^t, N^t$  in  $\text{GL}(n, \mathbf{R})$ , suppose there exist real numbers  $r, s > 0$  such that  $M^r$  and  $N^s$  have the same absolute Jordan form. Then the metric spaces  $G_M$  and  $G_N$  are quasi-isometric. To be explicit there exists  $A \in \text{GL}(n, \mathbf{R})$  and  $K \geq 1$  such that for each  $t \in \mathbf{R}$ , the map  $v \mapsto A(v)$  is a  $K$ -bilipschitz homeomorphism from the metric  $d_{M,t}$  to the metric  $d_{N, \frac{s}{r}t}$ ; it follows that the map from  $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$  to  $G_N = \mathbf{R}^n \rtimes_N \mathbf{R}$  given by*

$$(x, t) \mapsto \left( Ax, \frac{s}{r} \cdot t \right)$$

*is a bilipschitz homeomorphism from  $G_M$  to  $G_N$ , with bilipschitz constant  $\sup\{K, \frac{s}{r}, \frac{r}{s}\}$ .*

**Remark.** The absolute Jordan form of  $M^r$  is uniquely determined by  $M$  and  $r$ : it is the  $r^{\text{th}}$  power of the absolute Jordan form of  $M$ . It follows in particular that the quasi-isometry type of  $G_M$  depends only on the matrix  $M = M^1$ , not on the choice of 1-parameter subgroup  $M^t$ .

*Proof of Proposition 4.1.* We proceed in cases.

**Case 1:** Assume that  $N^t = e^{Jt}$  is the unique 1-parameter Jordan subgroup such that  $N = e^J$  is conjugate to the absolute Jordan form of  $M$ . Applying Proposition 3.1 we have

$$M^t = (A^{-1}N^tA)P^t$$

where  $A \in \text{GL}(n, \mathbf{R})$  and the 1-parameter subgroup  $P^t$  is bounded.

Choose  $t \in \mathbf{R}$  and  $v \in \mathbf{R}^n$ . We must show that the two numbers  $\|M^{-t}v\| = \|P^{-t}(A^{-1}N^{-t}A)v\|$  and  $\|N^{-t}Av\|$  have ratio bounded away from 0 and  $\infty$ , with bound independent of  $t, v$ . Setting  $u = N^{-t}Av$ , it suffices to show that  $\|P^{-t}A^{-1}u\|$  and  $\|u\|$  have bounded ratio. But this is clearly true, with a bound of

$$\left(\sup_t \|P^t\|\right) \cdot \text{Max}\left\{\|A\|, \frac{1}{\|A\|}\right\}$$

since the 1-parameter subgroup  $P^t$  is bounded.

**Case 2:** Assume that there exists  $a > 0$  such that  $M^t = N^{at}$  for all  $t$ . Then the metrics  $d_{M,t}$  and  $d_{N,at}$  are identical.

**General case:** Applying Case 2 we may assume that  $\det M = \det N$ . Applying Case 1 twice we may go from  $G_M$  to  $G_{e^J}$  to  $G_N$ , where  $e^J$  is conjugate to the absolute Jordan form of  $M$  and of  $N$ .  $\diamond$

## 5 Dynamics of $G_M$ , Part I: Horizontal Respecting Quasi-isometries

In this section we begin studying the asymptotic geometry of the solvable Lie groups  $G_M$  associated to 1-parameter subgroups  $M^t$  of  $\text{GL}(n, \mathbf{R})$ . As we saw in Section 4, the quasi-isometry type of  $G_M$  depends only on  $M$ , not on the choice of 1-parameter subgroup  $M^t$  passing through  $M$ ; see the remark after Proposition 4.1. We therefore continue to suppress the choice of 1-parameter subgroup in our notation. Further, we do not restrict the determinant to be  $> 1$ : the results of this section hold even when  $\det M = 1$ .



## 5.1 Definitions and statement of main theorem

Let  $X, Y$  be metric spaces. Let  $\mathcal{F}$  be a decomposition of  $X$ , that is, a collection of disjoint subsets of  $X$  whose union is  $X$ . Let  $\mathcal{G}$  be a decomposition of  $Y$ . Motivated by a foliation of a manifold, the elements of these decompositions are called *leaves* and the decomposition itself is called the *leaf space*. A quasi-isometry  $\phi: X \rightarrow Y$  is said to *coarsely respect* the decompositions  $\mathcal{F}, \mathcal{G}$  if there exists a number  $A \geq 0$  and a map of leaf spaces  $h: \mathcal{F} \rightarrow \mathcal{G}$  such that for each leaf  $L \in \mathcal{F}$  we have

$$d_{\mathcal{H}}(\phi(L), h(L)) \leq A$$

For example, consider the space  $G_M$ . The coordinate function  $G_M \approx \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$  given by  $(x, t) \mapsto t$  is called the *time function* of  $G_M$ . The level sets  $P_t \approx \mathbf{R}^n \times t$  form the *horizontal foliation* of  $G_M$ , whose leaves are called *horizontal leaves* of  $G_M$ , and whose leaf space is  $\mathbf{R}$ . Notice that  $d_{\mathcal{H}}(P_s, P_t) = |s - t|$ , and so the time function induces an isometry between the horizontal leaf space equipped with the Hausdorff metric and  $\mathbf{R}$ .

Consider another matrix  $N \in \text{GL}_{\times}(n, \mathbf{R})$ , and denote the horizontal leaves of  $G_N$  by  $P'_t$ .

**Definition (Horizontal respecting).** A quasi-isometry  $\phi: G_M \rightarrow G_N$  is said to be *horizontal respecting* if it coarsely respects the horizontal foliations of  $G_M, G_N$ . That is, there exists a function  $h: \mathbf{R} \rightarrow \mathbf{R}$  and  $A \geq 0$  such that  $d_{\mathcal{H}}(\phi(P_t), P'_{h(t)}) \leq A$  for all  $t \in \mathbf{R}$ .

The function  $h: \mathbf{R} \rightarrow \mathbf{R}$  is called an *induced time change* for  $\phi$ , with *Hausdorff constant*  $A$ .

If  $h, h'$  are two induced time changes for  $\phi$  then  $\sup_t |h(t) - h'(t)| \leq A + A' < \infty$ , where  $A, A'$  are Hausdorff constants for  $h, h'$  respectively. Also, if  $h: \mathbf{R} \rightarrow \mathbf{R}$  is an induced time change for  $\phi$  with Hausdorff constant  $A$ , if  $A' \geq 0$ , and if  $h': \mathbf{R} \rightarrow \mathbf{R}$  is any function satisfying  $\sup_{t \in \mathbf{R}} |h(t) - h'(t)| \leq A'$ , then  $h'$  is also an induced time change for  $\phi$ , with Hausdorff constant  $A + A'$ .

**Lemma 5.1.** *For each  $K, C, A$  there exists  $C'$  such that if  $\phi: G_M \rightarrow G_N$  is a horizontal respecting  $(K, C)$  quasi-isometry, and  $h: \mathbf{R} \rightarrow \mathbf{R}$  is an induced time change for  $\phi$  with Hausdorff constant  $A$ , then  $h$  is a  $(K, C')$  quasi-isometry of  $\mathbf{R}$ .*

*Proof.* We have  $|h(t) - h(s)| \leq d_{\mathcal{H}}(P_{h(t)}, P_{h(s)}) + 2A \leq K|t - s| + C + 2A$ . The reverse inequality is similar, and so  $h$  is a quasi-isometric embedding. Since  $\phi$  is coarsely onto, an easy argument shows  $h$  is coarsely onto.  $\diamond$

A  $(K, C')$  quasi-isometry  $h: \mathbf{R} \rightarrow \mathbf{R}$  induces a bijection of the two-point set  $\text{Ends}(\mathbf{R}) = \{-\infty, +\infty\}$ : given  $\eta_1, \eta_2 \in \text{Ends}(\mathbf{R})$ , we have  $h(\eta_1) = \eta_2$  if and only if  $h$  takes every sequence that diverges to  $\eta_1$  to a sequence that diverges to  $\eta_2$ . The following two properties of  $h$  are equivalent:

1.  $h$  induces the identity on  $\text{Ends}(\mathbf{R})$ .
2.  $h$  is *coarsely increasing*, that is there exists  $L > 0$  such that if  $t > s + L$  then  $h(t) > h(s)$ .

That (2) implies (1) is obvious. The other direction is true with any  $L > 2C'K$ , for if there existed  $t \geq s + L$  with  $h(t) \leq h(s)$ , then since  $h$  induces the identity on  $\text{Ends}(\mathbf{R})$  there would exist  $t' > t$  such that  $|h(s) - h(t')| \leq C'$ , but also  $|h(s) - h(t')| \geq |s - t'|/K - C' \geq L/K - C' > C'$ , a contradiction.

If  $h: \mathbf{R} \rightarrow \mathbf{R}$  is an induced time change of a horizontal respecting quasi-isometry  $\phi: G_M \rightarrow G_N$ , and if  $h$  satisfies the equivalent properties (1) and (2), then we say that  $\phi$  *coarsely respects the transverse orientation* of the horizontal foliations.

**Terminology (time vs. height):** In some contexts the vertical parameter which we have been calling “time” will also be called *height*, as sometimes this terminology is more suggestive, for example in discussing horizontal foliations.

Here is the main result, whose proof will occupy the remainder of this section and the next section.

**Theorem 5.2 (Horizontal respecting quasi-isometries).** *Let  $\phi: G_M \rightarrow G_N$  be a quasi-isometry which coarsely respects the transversely oriented horizontal foliations of  $G_M$  and  $G_N$ . Then there exist real numbers  $r, s > 0$  so that  $M^r$  and  $N^s$  have the same absolute Jordan form.*

Our proof of Theorem 5.2 proceeds in steps, following the outline given in the introduction.

## 5.2 Step 1a: Hyperbolic dynamics and the Shadowing Lemma

The Lie group  $G_M$  has a natural flow which fits into the theory of partially hyperbolic dynamical systems. From the dynamics we find that the flow has several invariant foliations, the “weak stable, weak unstable, and center” foliations. In §§5.2,5.3, by using the Shadowing Lemma ([HPS77], Lemma 7.A.2, page 133), we prove that a horizontal respecting quasi-isometry  $G_M \rightarrow G_N$  also respects the dynamically defined foliations of  $G_M, G_N$ .

From this result we obtain the first piece of our rigidity theorem, by showing that expanding, contracting, and unipotent parts of the absolute Jordan forms of  $M$  and  $N$  have the same ranks respectively, and that the unipotent parts are identical.

### 5.2.1 Dynamically defined foliations

Consider a 1-parameter subgroup  $M^t \in \mathrm{GL}(n, \mathbf{R})$ , with real Jordan form  $M^t = \overline{M}^t P^t$ . Consider the Jordan decomposition of  $\overline{M}$ , and group the root spaces according to whether the corresponding eigenvalue is  $< 1$ ,  $= 1$ , or  $> 1$  (alternatively, a logarithm which is  $< 0$ ,  $= 0$ , or  $> 0$ ), to obtain a decomposition  $\mathbf{R}^n = V^- \oplus V^0 \oplus V^+$ .

**Remark.** It might happen that one or two of the factors  $V^-$ ,  $V^0$ ,  $V^+$  is trivial, that is, 0-dimensional, for instance when all eigenvalues of  $M$  lie outside the unit circle.

Now consider the Lie group  $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$  determined by a 1-parameter subgroup  $M^t$ . Define the *vertical flow*  $\Phi$  on  $G_M$  to be

$$\Phi_t(x, s) = (x, s + t)$$

The tangent bundle  $TG_M$  has a  $\Phi$ -invariant splitting

$$TG_M = E^s \oplus E^c \oplus E^u$$

defined as follows. The tangent space at each point  $x \in G_M$  is identified with  $\mathbf{R}^n \oplus \mathbf{R}$ , and we take

$$E_x^s = V^+ \oplus 0, E_x^c = V^0 \oplus \mathbf{R}, E_x^u = V^- \oplus 0$$

It is evident from the construction that each of the distributions  $E^s \oplus E^c$ ,  $E^u \oplus E^c$ ,  $E^c$  is integrable, tangent to foliations denoted  $W^s$ ,  $W^u$ ,  $W^c$ . We call these foliations the (weak) *stable*, *unstable*, and *center foliations*, respectively. The stable and unstable foliations are transverse, and the intersection of any stable leaf with any unstable leaf is a center leaf.

Applying the Exponential Lower Bound from Proposition 3.2, there exist constants  $A > 0$ ,  $\lambda > 1$  such that:

- If  $v \in E^u$ , then for  $t \geq 0$  we have  $\|D\Phi_t v\| \geq A\lambda^t \|v\|$ , and for  $t \leq 0$  we have  $\|D\Phi_t v\| \leq \frac{1}{A}\lambda^t \|v\|$ .
- If  $v \in E^s$ , then for  $t \leq 0$  we have  $\|D\Phi_t v\| \geq A\lambda^{-t} \|v\|$ , and for  $t \geq 0$  we have  $\|D\Phi_t v\| \leq \frac{1}{A}\lambda^{-t} \|v\|$ .

Also, applying the Exponential-Polynomial Upper Bound from Proposition 3.2, there exists  $B > 0$  and an integer  $n \geq 1$  such that:

- If  $v \in E^c$ , then for  $|t| \geq 1$  we have  $\|D\Phi_t v\| \leq B |t|^n \|v\|$ .

When we want to emphasize the dependence of the  $V$ 's and  $E$ 's on the 1-parameter subgroup  $M^t$ , we will append a subscript, e.g.  $V_M^+$ ,  $E_M^s$ , etc.

### 5.2.2 Shadowing Lemma

Consider a flow  $\Phi$  on a metric space  $X$ . We write  $x \cdot t$  as an abbreviation for  $\Phi_t(x)$ . Given  $\epsilon, T > 0$ , an  $\epsilon, T$  pseudo-orbit of  $\Phi$  consists of a sequence of flow segments  $(x_i \cdot [0, t_i])$ , where the index set  $i$  runs over an interval in  $\mathbf{Z}$ , such that  $d_X(x_i \cdot t_i, x_{i+1}) < \epsilon$  and  $t_i > T$  for all  $i$ .

**Lemma 5.3 (Shadowing Lemma).** *Consider a 1-parameter subgroup  $M^t$  of  $\mathrm{GL}(n, \mathbf{R})$ , and let  $\Phi$  be the vertical flow on  $G_M$ . For every  $\epsilon, T > 0$  there exists  $\delta, \epsilon', T' > 0$  such that every  $\epsilon, T$  pseudo-orbit of  $\Phi$  is  $\delta$ -shadowed by an  $\epsilon', T'$  pseudo-orbit of  $\Phi$  which is contained in some center leaf. That is, if  $(x_i \cdot [0, t_i])$  is an  $\epsilon, T$  pseudo-orbit, then there is an  $\epsilon', T'$  pseudo-orbit  $(y_i \cdot [0, t_i])$  contained in some center leaf so that  $d(x_i \cdot t, y_i \cdot t) < \delta$  for all  $i$  and all  $t \in [0, t_i]$ .*

*Proof.* By construction, the foliations  $W^s$  and  $W^u$  are coordinate foliations in  $\mathbf{R}^{n+1}$ ; this shows that the flow  $\Phi$  has a “global product structure” in the language of hyperbolic dynamical systems. The lemma now follows the proof of the Shadowing Lemma in [HPS77], Lemma 7.A.2, page 133. A direct proof is also easy to work out, and is left to the reader.  $\diamond$

### 5.3 Step 1b: Foliations rigidity

The Shadowing Lemma implies further rigidity properties of horizontal respecting quasi-isometries:

**Proposition 5.4 (Foliations rigidity).** *Suppose  $\phi: G_M \rightarrow G_N$  is a quasi-isometry which coarsely respects the horizontal foliations and their transverse orientations. Then  $\phi$  also coarsely respects the weak unstable foliations  $W_M^u, W_N^u$ , the weak stable foliations  $W_M^s, W_N^s$ , and the center foliations  $W_M^c, W_N^c$ . In particular*

- $\dim(V_M^-) = \dim(V_N^-)$
- $\dim(V_M^+) = \dim(V_N^+)$
- $\dim(V_M^0) = \dim(V_N^0)$

## Remarks

1. In the case where neither  $M$  nor  $N$  has any eigenvalue on the unit circle, the center foliations of both  $G_M$  and  $G_N$  are simply the foliations by vertical flow lines, and Proposition 5.4 says that  $\phi$  respects these foliations. But in the general case, it is not true that  $\phi$  always respects the foliations by vertical flow lines. For a simple counterexample, consider the  $1 \times 1$  matrix  $M = N = (1)$ , which gives  $\Gamma_M = \Gamma_N = \mathbf{Z}^2$ . There exist horizontal respecting quasi-isometries of  $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$  which do not respect the vertical foliation.

2. If all eigenvalues of  $M$  and  $N$  are outside the unit circle, then both  $G_M$  and  $G_N$  are negatively curved, and the proposition follows from a standard fact: a quasigeodesic in a negatively curved space  $X$  is Hausdorff close to a geodesic (this was the approach taken in [FM98] in the case of a  $1 \times 1$  matrix  $M$ , where  $G_M$  is isometric to a scaled copy of the hyperbolic plane). This “fact” is unavailable when  $X = G_M$  is not negatively curved, forcing us to study horizontal respecting quasi-isometries via the Shadowing Lemma.

Before proving Proposition 5.4, we use it to obtain some pieces of our classification theorem. Since  $\text{rk}(J_M^-) = \dim(V_M^-)$  etc., we immediately have:

**Corollary 5.5.** *If there is a quasi-isometry from  $G_M$  to  $G_N$  which coarsely respects the transversely oriented horizontal foliations, then  $\text{rk}(J_M^-) = \text{rk}(J_N^-)$ ,  $\text{rk}(J_M^0) = \text{rk}(J_N^0)$ , and  $\text{rk}(J_M^+) = \text{rk}(J_N^+)$ .*  $\diamond$

We also have:

**Corollary 5.6.** *The unipotent blocks of the absolute Jordan forms of  $M$  and  $N$  are identical.*

*Proof.* Let  $L$  be some center leaf of  $G_M$ , of dimension  $k$ . From Proposition 5.4 it follows that  $\phi(L)$  is Hausdorff close to some center leaf  $L'$  of  $G_N$ , also of dimension  $k$ . By composition with nearest point projection (which moves points a uniformly bounded amount) we get an induced map  $L \rightarrow L'$ . By Lemma 2.1 this map is a quasi-isometry. By Proposition 4.1,  $L$  and  $L'$  are quasi-isometric to the nilpotent Lie groups  $\mathbf{R}^{k-1} \rtimes_{J_M^0} \mathbf{R}$  and  $\mathbf{R}^{k-1} \rtimes_{J_N^0} \mathbf{R}$ , respectively. As Bridson and Gersten have shown [BG96], Pansu’s invariant [Pan89b] may be used to prove that  $J_M^0 = J_N^0$ .  $\diamond$

*Proof of Proposition 5.4.* We begin with:

**Claim 5.7.** *For each vertical flow line  $\gamma = \Phi_{\mathbf{R}}(x)$  in  $G_M$ , there exists a center leaf  $\tau_\gamma$  in  $G_N$  such that  $\phi(\gamma)$  is contained in the  $\alpha$ -neighborhood of  $\tau_\gamma$ , where the constant  $\alpha > 0$  does not depend on  $\gamma$ .*

Before proving the claim, we apply it to prove the proposition as follows.

Consider any two vertical flow lines  $\gamma_1, \gamma_2$  in  $G_M$ . By the claim we have that  $\phi(\gamma_1)$  and  $\phi(\gamma_2)$  lie, respectively, in bounded neighborhoods of center leaves  $\sigma_1$  and  $\sigma_2$  of  $G_N$ . Since  $h(t) \rightarrow \pm\infty$  as  $t \rightarrow \pm\infty$ , for each choice of sign  $+$  or  $-$  the following two statements are equivalent, and the second statement implies the third:

1. The distance between the points  $\gamma_1 \cap P_t$  and  $\gamma_2 \cap P_t$  in  $P_t$  stays bounded as  $t \rightarrow \pm\infty$ .
2. The distance between the points  $\phi(\gamma_1) \cap P_{h(t)}$  and  $\phi(\gamma_2) \cap P_{h(t)}$  in  $P_{h(t)}$  stays bounded as  $t \rightarrow \pm\infty$ .
3. The Hausdorff distance between the sets  $\sigma_1 \cap P_{h(t)}$  and  $\sigma_2 \cap P_{h(t)}$  in  $P_{h(t)}$  stays bounded as  $t \rightarrow \pm\infty$ .

Using  $+$  signs, the first statement is equivalent to saying that  $\gamma_1, \gamma_2$  are contained in the same unstable leaf of  $G_M$ , and the third statement is equivalent to saying that  $\sigma_1, \sigma_2$  are contained in the same unstable leaf of  $G_N$ . It follows that  $\phi$  takes every unstable leaf of  $G_M$  into a bounded neighborhood of an unstable leaf of  $G_N$ . Applying the same argument to a coarse inverse  $\bar{\phi}$  of  $\phi$  gives the opposite inclusion. Since  $d(\bar{\phi} \circ \phi, \text{Id}) < \infty$  it follows that the image under  $\phi$  of any unstable leaf of  $G_M$  lies a bounded Hausdorff distance from an unstable leaf of  $G_N$ , that is,  $\phi$  coarsely preserves the unstable foliations. A similar argument using  $-$  signs shows that  $\phi$  coarsely preserves stable foliations. By taking intersections of stable and unstable leaves it follows that  $\phi$  coarsely preserves center foliations.

The final statements about dimensions follow from the fact that dimension is a quasi-isometry invariant, for leaves of the foliations in question; see [Ger93] or [BW97].

It remains to prove the claim. Applying Lemma 5.1, we have an induced time change  $h: \mathbf{R} \rightarrow \mathbf{R}$  which is a  $(K, C')$ -quasi-isometry with Hausdorff constant  $A$ , where  $C'$  depends only on  $K, C, A$ . Furthermore by Lemma 5.1 and the comments following it, the map  $h$  is coarsely increasing: there exists  $L = L(K, C, A) > 0$  such that if  $t \geq s + L$  then  $h(t) > h(s)$ .

We can furthermore increase  $L$ , depending only on  $K, C', A$ , so that:

$$\begin{aligned} \text{if } t' \geq t + L, \quad x \in P_{t'}, \quad y \in P_t, \quad \phi(x) \in P_{s'} \text{ and } \phi(y) \in P_s, \\ \text{then } s' \geq s + 1. \end{aligned} \quad (5.1)$$

In fact taking  $L > (C' + 2A + 1)K$  will do, for then we have

$$\begin{aligned} h(t') &\geq h(t) + (t' - t)/K - C' \geq h(t') + L/K - C' \\ &\geq h(t) + 2A + 1 \end{aligned}$$

and, since  $P_{s'}$  is  $A$ -Hausdorff close to  $P_{h(t')}$  and  $P_s$  is  $A$ -Hausdorff close to  $P_{h(t)}$ , it follows that  $s' \geq s + 1$ .

To prove the claim, we first show that  $\phi(\gamma)$  is Hausdorff close to some pseudo-orbit in  $G_N$ , and then we apply the Shadowing Lemma to show that the pseudo-orbit lies in a bounded neighborhood of some center leaf.

To be more precise, fix a point  $x_0 \in \gamma$  and consider the sequence  $x_i = \Phi_{i \cdot L}(x_0)$  for  $i \in \mathbf{Z}$ . Let  $y_i = \phi(x_i)$ , and let  $s_i$  be such that  $y_i \in P_{s_i}$ . From (5.1) it follows that  $s_{i+1} \geq s_i + 1$ . Let  $t_i = s_{i+1} - s_i \geq 1$ .

We claim that there exists  $\epsilon > 0$ , depending ultimately only on  $K, C$ , so that  $(y_i \cdot [0, t_i])$  is an  $\epsilon, 1$  pseudo-orbit; in other words,  $d(y_i \cdot t_i, y_{i+1})$  is bounded. To see why, first note that

$$\begin{aligned} d(y_i \cdot t_i, y_i) &= t_i = s_{i+1} - s_i \\ &\leq 2A + h(L \cdot (i + 1)) - h(L \cdot i) \\ &\leq 2A + KL + C' \end{aligned}$$

and then

$$d(y_i, y_{i+1}) \leq K \cdot d(x_i, x_{i+1}) + C \leq KL + C$$

so we may take  $\epsilon = 2A + 2KL + C + C'$ .

Applying the Shadowing Lemma, there exists  $\beta, \epsilon', T'$  such that  $(y_i \cdot [0, t_i])$  is  $\beta$ -Hausdorff close to an  $\epsilon', T'$  pseudo-orbit  $(y'_i, [0, t_i])$  contained in some center leaf of  $G_N$ . On the other hand, since every point of  $\gamma$  is within distance  $L$  of some  $x_i$ , it follows that  $\phi(\gamma)$  is uniformly Hausdorff close to  $(y_i \cdot [0, t_i])$ , and so it is also uniformly close to the pseudo-orbit  $(y'_i \cdot [0, t_i])$ .  $\diamond$

## 5.4 Step 2: Time rigidity

The main result of this subsection says that a horizontal respecting quasi-isometry has an induced time change function which is be affine.

**Proposition 5.8 (Time rigidity).** *Consider the Lie groups  $G_M, G_N$  where  $M, N \in \text{GL}_\times(n, \mathbf{R})$  each have an eigenvalue of absolute value greater than 1. Then there exists  $m \in \mathbf{R}_+$  with the following properties. For all  $K \geq 1$ ,  $C, A \geq 0$  there exists  $A' \geq 0$  such that if  $\phi: G_M \rightarrow G_N$  is a  $(K, C)$  quasi-isometry which coarsely respects horizontal foliations and their transverse*

orientations, with an induced time change of Hausdorff constant  $A$ , then there exists  $b \in \mathbf{R}$  such that  $h(t) = mt + b$  is an induced time change with Hausdorff constant  $A'$ . In fact,  $m$  can be computed as follows: Let  $\alpha$  (resp.  $\beta$ ) be the least eigenvalue greater than 1 of the absolute Jordan form of  $M$  (resp.  $N$ ); the numbers  $\alpha, \beta$  exist by the assumption on eigenvalues. Then  $m = \log \alpha / \log \beta$ .

### Remarks

1. In the case of self-quasi-isometries of  $\text{Aff}(R) = G_{(e^1)} = \mathbf{H}^2$  which coarsely respect the horizontal foliation, this result is part of Proposition 5.3 of [FM98], where the conclusion is that the induced time change is a translation of  $\mathbf{R}$ .
2. One of the delicate points in Gromov's development of the  $\text{infdim}$  invariant is the rescaling problem discussed at the beginning of Section 7.C<sub>1</sub> of [Gro93]: rates of exponential growth change when the parameter is rescaled. Time Rigidity allows us to avoid the rescaling problem altogether, by showing that the time parameter is "natural" with respect to quasi-isometries.

*Proof.* This proof will define a sequence of constants which will depend on  $K, C, A$  and on the matrices  $M$  and  $N$ . We will indicate the dependence on  $K, C, A$  by writing, for example,  $C_1 = C_1(K, C, A)$ , but we will suppress the dependence on  $M, N$ . Although each constant in the sequence will depend on previous constants in the sequence, by induction it will ultimately depend only on  $K, C, A, M, N$ .

**Claim 5.9.** *For each fixed time  $t_0$ , and for each  $t \leq t_0$ , we have*

$$h(t) \geq m(t - t_0) + h(t_0) - C_1$$

for some  $C_1 = C_1(K, C, A) \geq 0$ .

Accepting this claim for the moment, we prove the proposition. The idea is simply that the conclusion of the claim, applied to both  $h$  and its coarse inverse  $\bar{h}$ , with  $t_0 \rightarrow +\infty$ , implies the proposition.

Let  $s$  be a time parameter for  $G_N$ . Let  $\bar{\phi}: G_N \rightarrow G_M$  be a coarse inverse for  $\phi$ , also a quasi-isometry which coarsely respects the horizontal foliations and their transverse orientations, and with an induced time change  $\bar{h}(s)$ . The constants for  $\bar{\phi}$  and  $\bar{h}$  depend only on  $K, C, A$ . The claim therefore applies as well to  $\bar{h}$  and we obtain, for each fixed time  $s_0$  and each  $s \leq s_0$ ,

$$\bar{h}(s) \geq \frac{1}{m}(s - s_0) + \bar{h}(s_0) - C_2$$



for some  $C_2 = C_2(K, C, A) \geq 0$ .

It is clear  $\bar{h}$  is a coarse inverse for  $h$ , that is:

$$|\bar{h}(h(t)) - t| \leq C_3, \quad |h(\bar{h}(s)) - s| \leq C_3$$

for some  $C_3 = C_3(K, C, A) \geq 0$ .

Also, by Lemma 5.1 and the comments after it, the map  $h$  is coarsely increasing: there exists  $L = L(K, C, A) \geq 0$  such that if  $t' > t + L$  then  $h(t') > h(t)$ .

We reverse the inequality in the claim as follows. Fix  $t_0$ . Let  $s_0 = t_0$ . Consider for the moment some  $t \leq t_0 - L$ . Letting  $s = h(t)$  it follows that  $s \leq s_0$  and so we have

$$\bar{h}(h(t)) \geq \frac{1}{m}(h(t) - h(t_0)) + \bar{h}(h(t_0)) - C_2$$

But  $t + C_3 \geq \bar{h}(h(t))$  and  $\bar{h}(h(t_0)) \geq t_0 - C_3$  and so we obtain

$$\begin{aligned} t &\geq \frac{1}{m}(h(t) - h(t_0)) + t_0 - (2C_3 + C_2) \\ h(t) &\leq m(t - t_0) + h(t_0) + m(2C_3 + C_2) \end{aligned}$$

This has been derived only for  $t \leq t_0 - L$ , but for  $t_0 - L \leq t \leq t_0$  we obtain a similar inequality with another constant in place of  $m(2C_3 + C_2)$ . Therefore, for all  $t \leq t_0$  we obtain

$$m(t - t_0) + h(t_0) - C_4 \leq h(t) \leq m(t - t_0) + h(t_0) + C_4$$

for some  $C_4 = C_4(K, C, A)$ . Note that this is true *for all*  $t_0$ , with  $C_4$  independent of  $t_0$ .

In particular, taking  $t_0 = 0$ , for all  $t \leq 0$  we obtain

$$mt + h(0) - C_4 \leq h(t) \leq mt + h(0) + C_4$$

Now take any  $t_1 \geq 0$ , and since  $0 \leq t_1$  we obtain

$$m(0 - t_1) + h(t_1) - C_4 \leq h(0) \leq m(0 - t_1) + h(t_1) + C_4$$

and so

$$mt_1 + h(0) - C_4 \leq h(t_1) \leq mt_1 + h(0) + C_4$$

Taking  $b = h(0)$ , this proves that  $mt + b$  is an induced time change for  $\phi$ , with Hausdorff constant  $A' = C_4 + A$ .

Now we turn to the proof of Claim 5.9.

Let  $M^t = \overline{M}^t Q^t$ ,  $N^t = \overline{N}^t Q^t$  be the real Jordan forms. Let  $U$  (resp.  $U'$ ) be the root space with eigenvalue 1 for  $\overline{M}$  (resp.  $\overline{N}$ ). Let  $W$  (resp.  $W'$ ) be the direct sum of root spaces with eigenvalue  $\geq 1$  for  $\overline{M}$  (resp.  $\overline{N}$ ). Recall that  $\alpha$  is the smallest eigenvalue  $> 1$  for  $\overline{M}$ , and  $\beta$  is the smallest eigenvalue  $> 1$  for  $\overline{N}$ . Let  $V$  be the direct sum of  $U$  and the eigenspace with eigenvalue  $\alpha$  for  $\overline{M}$ . We have  $U \subset V \subset W$ ; let  $\mathcal{F}(U), \mathcal{F}(V), \mathcal{F}(W)$  be the corresponding foliations of  $G_M \approx \mathbf{R}^n \times \mathbf{R}$ , whose leaves are parallel to  $U \times \mathbf{R}, V \times \mathbf{R}, W \times \mathbf{R}$  respectively. We also have  $U' \subset W'$ ; let  $\mathcal{F}(U'), \mathcal{F}(W')$  be the corresponding foliations of  $G_N$ .

Here is the idea for proving Claim 5.9. Each leaf of  $\mathcal{F}(V)$  is foliated by leaves of  $\mathcal{F}(U)$ . Because  $V$  is the direct sum of  $U$  with the  $\alpha$  eigenspace of  $\overline{M}$ , it follows that as  $t \rightarrow -\infty$  distinct leaves of  $\mathcal{F}(U)$  in  $\mathcal{F}(V)$  diverge from each other *exactly* as  $\alpha^{-t}$ , measured in the time  $t$  horizontal plane of  $G_M$ . This is a consequence of the Exponential Lower Bound and the Exponential·Polynomial Upper Bound in Proposition 3.2; notice that it is critical here that  $V$  not be the direct sum of  $U$  with the  $\alpha$  *root space*, for then Exponential·Polynomial Upper Bound would be at best  $\alpha^{-t}$  times some polynomial, which would mess up the following calculations. Mapping over via the quasi-isometry  $\phi: G_M \rightarrow G_N$ , distinct leaves of  $\mathcal{F}(U)$  in a single leaf of  $\mathcal{F}(V)$  must (coarsely) map to distinct leaves of  $\mathcal{F}(U')$  in a single leaf of  $\mathcal{F}(W')$ , which as  $s \rightarrow -\infty$  diverge from each other *at least as fast as*  $\beta^{-s}$ , by the Exponential Lower Bound. The time change map  $t \mapsto h(t) = s$  therefore cannot grow slower than  $s = \frac{\log \alpha}{\log \beta} \cdot t$ , as  $t \rightarrow -\infty$ .

To make this precise, pick a leaf  $L_V$  of  $\mathcal{F}(V)$ , contained in some leaf  $L_W$  of  $\mathcal{F}(W)$ . We use the symbol  $\gamma$  to denote a general leaf of  $\mathcal{F}(U)$ , which we will typically take to be a subset of  $L_V$ . By Proposition 5.4, there exists a leaf  $L_{W'}$  of  $\mathcal{F}(W')$  such that

$$d_{\mathcal{H}}(f(L_W), L_{W'}) \leq C_5 = C_5(K, C, A)$$

and for each leaf  $\gamma$  of  $\mathcal{F}(U)$  there exists a leaf  $\gamma'$  of  $\mathcal{F}(U')$  such that

$$d_{\mathcal{H}}(f(\gamma), \gamma') \leq C_5$$

Moreover, if  $\gamma \subset L_V$  then  $\gamma' \subset L_{W'}$ , because  $L_V \subset L_W$  and so  $\gamma'$  stays in a bounded neighborhood of  $L_{W'}$ , but any leaf of  $\mathcal{F}(U')$  which is not a subset of  $L_{W'}$  has points which are arbitrarily far from  $L_{W'}$ .

Let  $P_t$  be the horizontal subset of  $G_M$  at height  $t \in \mathbf{R}$ , and let  $d_t$  denote Hausdorff distance in  $P_t$  between closed subsets of  $P_t$ . Let  $P'_s$  be

the horizontal subset of  $G_N$  at height  $s \in \mathbf{R}$ , and let  $d'_s$  denote Hausdorff distance in  $P'_s$ .

Since the Hausdorff distance in  $G_N$  between  $\phi(P_t)$  and  $P'_{h(t)}$  is at most  $A$ , the vertical projection from  $\phi(P_t)$  to  $P'_{h(t)}$  induces a quasi-isometry between  $P_t$  and  $P'_{h(t)}$ ; the multiplicative constant of this quasi-isometry is  $K$ , and its additive constant depends only on  $K, C, A$ . It follows that there exists a ‘‘coarseness constant’’  $C_6 = C_6(K, C, A)$  so that for any  $t$ , and for any  $x, y \in P_t$  with  $d_t(x, y) \geq C_6$ , if  $x', y' \in P'_{h(t)}$  are the vertical projections of  $\phi(x), \phi(y)$  then

$$\frac{1}{2K}d_t(x, y) \leq d'_{h(t)}(x', y') \leq 2Kd_t(x, y) \quad (5.2)$$

To prove Claim 5.9, fix a time  $t_0$  and let  $s_0 = h(t_0)$ . Let  $\gamma_1, \gamma_2$  be two leaves of  $\mathcal{F}(U)$  contained in  $L_V$ , and let  $\gamma'_i$  be the unique leaf of  $\mathcal{F}(U')$  within bounded Hausdorff distance of  $\phi(\gamma_i)$ ; this bound depends only on  $K, C, A$ , as shown in Proposition 5.4.

In  $G_M$ , apply the Exponential Lower Bound and the Exponential·Polynomial Upper Bound of Proposition 3.2, and so for all  $t \leq t_0$  we have

$$\begin{aligned} A \cdot \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) &\leq d_t(\gamma_1 \cap P_t, \gamma_2 \cap P_t) \\ &\leq B \cdot \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) \end{aligned}$$

where  $A, B$  depend only on  $G_M$  (note that  $t = t_0$  gives  $A \leq 1 \leq B$ ).

We want the distance between  $\gamma_1$  and  $\gamma_2$  in  $P_t$  to be greater than the coarseness constant  $C_6$ , for each  $t \leq t_0$ , in order that property (5.2) may be applied. We therefore impose a condition on  $\gamma_1$  and  $\gamma_2$ , namely that

$$d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) \geq \frac{C_6}{A}$$

which implies, for all  $t \leq t_0$ , that

$$d_t(\gamma_1 \cap P_t, \gamma_2 \cap P_t) \geq C_6$$

and so

$$\begin{aligned} \frac{1}{2K} \cdot d_t(\gamma_1 \cap P_t, \gamma_2 \cap P_t) &\leq d'_{h(t)}(\gamma'_1 \cap P'_{h(t)}, \gamma'_2 \cap P'_{h(t)}) \\ &\leq 2K \cdot d_t(\gamma_1 \cap P_t, \gamma_2 \cap P_t) \end{aligned}$$

which implies

$$\begin{aligned} \frac{A}{2K} \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) &\leq d'_{h(t)}(\gamma'_1 \cap P'_{h(t)}, \gamma'_2 \cap P'_{h(t)}) \\ &\leq 2BK \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) \end{aligned}$$

Next, applying the Exponential Lower Bound of Proposition 3.2 in  $G_N$ , for each  $s \leq s_0$  we have

$$d'_s(\gamma'_1 \cap P'_s, \gamma'_2 \cap P'_s) \geq A \cdot \beta^{-s+s_0} d'_{s_0}(\gamma'_1 \cap P'_{s_0}, \gamma'_2 \cap P'_{s_0})$$

Taking  $s = h(t)$ , and using the fact that  $s_0 = h(t_0)$ , this implies

$$\beta^{-h(t)+h(t_0)} d'_{h(t_0)}(\gamma'_1 \cap P'_{h(t_0)}, \gamma'_2 \cap P'_{h(t_0)}) \leq \frac{2BK}{A} \cdot \alpha^{-t+t_0} \cdot d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0})$$

Therefore,

$$\beta^{-h(t)+h(t_0)} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) \leq \frac{4BK^2}{A} \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0})$$

Now divide both sides by  $d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0})$ , and take logarithms, obtaining

$$(-h(t) + h(t_0)) \log(\beta) \leq \log\left(\frac{4BK^2}{A}\right) + (-t + t_0) \log(\alpha)$$

and so

$$h(t) \geq \frac{\log(\alpha)}{\log(\beta)}(t - t_0) + h(t_0) - \frac{\log\left(\frac{4BK^2}{A}\right)}{\log(\beta)}$$

proving Claim 5.9 and therefore completing the proof of Proposition 5.8.  $\diamond$

## 5.5 Interlude: The induced boundary map

The *upper boundary*  $\partial^u G_M$  is defined to be the leaf space of the weak stable foliation, identified with  $V^-$ . The *lower boundary*  $\partial_\ell G_M$  is the leaf space of the weak unstable foliation, identified with  $V^+$ . The *internal boundary*  $\partial_{\text{int}} G_M$  is defined as

$$\partial_{\text{int}} G_M = \partial_\ell G_M \times \partial^u G_M = V^+ \times V^- \approx \mathbf{R}^n / V^0$$

which is identified with the leaf space of the center foliation.

As a consequence of Proposition 5.4, a quasi-isometry  $\phi: G_M \rightarrow G_L$  which respects the transversely oriented horizontal foliations induces a bijection

$$\partial_{\text{int}} \phi: \partial_{\text{int}} G_M \rightarrow \partial_{\text{int}} G_L$$

which preserves the factors, that is,

$$\partial_{\text{int}} \phi = \partial_\ell \phi \times \partial^u \phi: \partial_\ell G_M \times \partial^u G_M \rightarrow \partial_\ell G_L \times \partial^u G_L$$

Recall the 1-parameter family of metrics  $d_{M,t}$  on  $\mathbf{R}^n$  given by the quadratic form  $Q_{M,t} = (M^{-t})^T M^{-t}$ . The internal boundary  $\partial_{\text{int}} G_M$  is identified with  $\mathbf{R}^n/V^0$  and with  $V^+ \times V^-$ , and we consider two 1-parameter families of metrics.

First, regarding points of  $\mathbf{R}^n/V^0$  as affine subspaces parallel to  $V^0$ , there is a 1-parameter family of Hausdorff metrics induced from  $d_{M,t}$  which we denote  $dh_{M,t}$ . Second, restrict the action of  $M^t$  to the subspace  $V^+ \times V^-$  to get a 1-parameter subgroup of  $\text{GL}(V^+ \times V^-)$ , and by choosing a basis for  $V^+ \times V^-$  we obtain a 1-parameter subgroup  $\hat{M}^t$  of  $\text{GL}(k, \mathbf{R})$ , where  $k$  is the dimension of  $V^+ \times V^-$ . We obtain a 1-parameter family of metrics  $d_{\hat{M},t}$ . There is a canonical identification  $V^+ \times V^- \approx \mathbf{R}^n/V^0$ , and with respect to this identification the metrics  $d_{\hat{M},t}$  and  $dh_{M,t}$  are bilipschitz equivalent, with a uniform bilipschitz constant independent of  $t$ .

Note that the absolute Jordan form of  $\hat{M}$  is identical with the nonunipotent part of the absolute Jordan form of  $M$ , and similarly for  $N$ .

**Lemma 5.10.** *Given two 1-parameter subgroups  $M^t, N^t$  of  $\text{GL}(n, \mathbf{R})$ , for all  $K \geq 1, C, A \geq 0$ , there exist  $K' \geq 1, C' \geq 0$  with the following properties. If  $\phi: G_M \rightarrow G_N$  is a  $K, C$  quasi-isometry which coarsely respects the transversely oriented horizontal foliations, with Hausdorff constant  $A$ , then for every  $t \in \mathbf{R}$  the induced map  $\partial_{\text{int}} \phi: \partial_{\text{int}} G_M \rightarrow \partial_{\text{int}} G_N$  is a  $K', C'$  quasi-isometry from the metric  $d_{\hat{M},t}$  to the metric  $d_{\hat{N},h(t)}$ .*

*Proof.* With what we know, the proof is mostly a matter of chasing through definitions.

The quasi-isometry  $\phi$  is a bounded distance from a quasi-isometry  $\psi: G_M \rightarrow G_N$  which takes the horizontal leaf  $P_t$  to the horizontal leaf  $P'_{h(t)}$ , and which simultaneously takes center leaves of  $G_M$  to center leaves of  $G_N$ . Now restrict the center foliations of  $G_M, G_N$  to  $P_t, P'_{h(t)}$ , and denote the respective leaf spaces as  $Q_t, Q'_{h(t)}$ .

In order to apply Lemma 2.1, consider each horizontal leaf  $P_t$  of  $G_M$  as a geodesic metric space, with respect to the Riemannian metric induced by restriction from  $G_M$ . The inclusion map  $P_t \hookrightarrow G_M$  is evidently  $(1, 0)$  coarsely lipschitz, and it is uniformly proper, with a uniformity function  $s(r) = a^r$  where  $a > 1$  is larger than the maximum of the absolute values of all eigenvalues of  $M$  and their multiplicative inverses. Note in particular that the coarse Lipschitz constants and the uniformity functions of the maps  $P_t \hookrightarrow G_M$  depend only on  $K, C, A$  and on the matrix  $M$ , but not on  $t$ . Similar remarks apply to the inclusion map  $P'_{h(t)} \hookrightarrow G_N$ . Applying Lemma 2.1, restricting  $\psi$  to  $P_t$  results in a map  $\psi_t: P_t \mapsto P'_{h(t)}$  which is

a quasi-isometry. There is in turn an induced map  $\theta_t: Q_t \mapsto Q'_{h(t)}$  which is a quasi-isometry with respect to the associated Hausdorff metric. The quasi-isometry constants of the maps  $\psi_t$  and  $\theta_t$  depend only on  $K, C, A$ .

Now consider the coordinate identifications  $G_M \approx \mathbf{R}^n \times \mathbf{R}$ ,  $G_N \approx \mathbf{R}^n \times \mathbf{R}$ . By construction of the left invariant metrics, for each  $t$  the space  $P_t$  is identified with  $\mathbf{R}^n \times t \approx \mathbf{R}^n$  with the metric  $d_{M,t}$ , and the space  $P'_{h(t)}$  is identified with  $\mathbf{R}^n$  with metric  $d_{N,h(t)}$ , and so the maps  $\psi_t: \mathbf{R}^n \rightarrow \mathbf{R}^n$  are uniform quasi-isometries from  $d_{M,t}$  to  $d_{N,h(t)}$  for all  $t$ . Also,  $Q_t$  is identified with  $\mathbf{R}^n/V_M^0$  with the associated Hausdorff metric  $dh_{M,t}$ , and  $Q'_{h(t)}$  is identified with  $\mathbf{R}^n/V_N^0$  with the associated Hausdorff metric  $dh_{N,h(t)}$ , and so the maps  $\theta_t: \mathbf{R}^n/V_M^0 \rightarrow \mathbf{R}^n/V_N^0$  are uniform quasi-isometries from  $dh_{M,t}$  to  $dh_{N,h(t)}$  for all  $t$ . This implies that  $\theta_t: V_M^+ \times V_M^- \rightarrow V_N^+ \times V_N^-$  is a quasi-isometry from  $d_{\hat{M},t}$  to  $d_{\hat{N},t}$  for all  $t$ . But for all  $t$  the map  $\theta_t$  is identical to  $\partial_{\text{int}}\phi: \partial_{\text{int}}G_M \rightarrow \partial_{\text{int}}G_N$ , proving the lemma.  $\diamond$

### 5.6 Step 3: Reduction of Theorem 5.2 to Theorem 5.11

Assume the hypotheses of Theorem 5.2, namely that we have 1-parameter subgroups  $M^t, N^t$ , and a quasi-isometry  $\phi: G_M \rightarrow G_N$  which coarsely respects the transversely oriented horizontal foliations. Applying Proposition 5.8, there is an induced time change of the form  $h(t) = mt + b$  with  $m > 0$ . Applying Proposition 4.1, there is a horizontal respecting quasi-isometry  $G_N \rightarrow G_{N^m}$  with an induced time change of the form  $s \mapsto s/m$ . By composition we obtain a horizontal respecting quasi-isometry  $G_M \rightarrow G_{N^m}$  with an induced time change of the form  $t \mapsto t + b'$ . Changing the coordinates in  $G_M$  by a translation of the time coordinate  $t$ , we have a horizontal respecting quasi-isometry  $G_M \rightarrow G_{N^m}$  for which the identity map  $t \mapsto t$  is an induced time change. Applying Lemma 5.10, we obtain a map  $\partial_{\text{int}}\phi: \mathbf{R}^n \rightarrow \mathbf{R}^n$  which, for each  $t$ , is a  $(K', C')$ -quasi-isometry from  $d_{\hat{M},t}$  to  $d_{\hat{N}^m,t}$ .

Now apply the following theorem (with  $N$  in place of  $N^m$ ), which will be proved in the next section:

**Theorem 5.11 (One-parameter subgroup rigidity).**

*Let  $M^t, N^t$  be 1-parameter subgroups of  $\text{GL}(n, \mathbf{R})$ , such that  $M = M^1$  and  $N = N^1$  have no eigenvalues on the unit circle. If there exists a bijection  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  and constants  $K \geq 1, C \geq 0$  such that for each  $t \in \mathbf{R}$  and  $p, q \in \mathbf{R}^n$  we have*

$$-C + \frac{1}{K}d_{M,t}(p, q) \leq d_{N,t}(f(p), f(q)) \leq Kd_{M,t}(p, q) + C$$

*then  $M$  and  $N$  have the same absolute Jordan form.*

Returning to the previous discussion, this theorem allows us to conclude that  $\hat{M}$  and  $\hat{N}^m$  have the same absolute Jordan form, and so the nonunipotent parts of the absolute Jordan forms of  $M, N^m$  are identical. We have already proved in Corollary 5.6 that the unipotent parts are identical, and so  $M$  and  $N^m$  have the same absolute Jordan forms, finishing the proof of Theorem 5.2.  $\diamond$

## 6 Dynamics of $G_M$ , Part II: 1-parameter subgroup rigidity

In this section we give a proof of Theorem 5.11.

Let  $M^t, N^t$  be 1-parameter subgroups of  $\mathrm{GL}(n, \mathbf{R})$  with no eigenvalues on the unit circle. Let  $M^t = \overline{M}^t P^t, N^t = \overline{N}^t Q^t$  be the real Jordan forms, so  $\overline{M}$  and  $\overline{N}$  have all positive eigenvalues, none equal to 1. Let  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a bijection which satisfies

$$-C + \frac{1}{K} d_{M,t}(p, q) \leq d_{N,t}(f(p), f(q)) \leq K d_{M,t}(p, q) + C \quad (6.1)$$

for all  $t \in \mathbf{R}, p, q \in \mathbf{R}^n$ .

The bijection  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  must in fact be a homeomorphism. To see why, for each  $p \in \mathbf{R}^n, R > 0, T > 0$  let

$$F_{p,R}(T) = \{q \in \mathbf{R}^n \mid d_{M,t}(p, q) < R \text{ for all } t \in (-T, T)\}$$

In other words,  $F_{p,R}(T)$  is the intersection of open balls of radius  $R$  about  $p$  in each of the metrics  $d_{M,t}$ , for  $t \in (-T, T)$ . Since the eigenvalues of  $\overline{M}$  are all positive real numbers, none equal to 1, it follows from Proposition 3.2 that for each  $p \in \mathbf{R}^n$  and each  $R > 0$  the collection of sets  $F_{p,R}(T)$  as  $T$  ranges in  $(0, \infty)$  is a neighborhood basis for  $p$ , in the standard topology on  $\mathbf{R}^n$ . We define a similar neighborhood basis using matrix  $N$ , denoted  $G_{p,R}(T)$ . Since  $f(F_{p,R}(T)) \subseteq G_{f(p), KR+C}(T)$  for each  $p \in \mathbf{R}^n, R > 0, T > 0$ , it follows that  $f$  is continuous. The same argument applies to  $f^{-1}$ , and so  $f$  is a homeomorphism.

The idea of the proof of Theorem 5.11 is to show that  $f$  respects certain “flags of foliations” which are closely related to the Jordan decompositions of  $\mathbf{R}^n$  with respect to  $M^t$  and  $N^t$ . We begin by setting up the notation needed to define and study these foliations.

**Definition (Flags of foliations).** If  $V$  is a vector subspace of  $\mathbf{R}^n$ , define a foliation  $\mathcal{F}(V)$  of  $\mathbf{R}^n$  whose leaves are the affine subspaces of  $\mathbf{R}^n$  parallel

to  $V$ . Given a flag of subspaces  $V_1 \subset \cdots \subset V_r$ , it follows that if  $1 \leq i < j \leq r$  then each leaf of  $\mathcal{F}(V_i)$  is contained in some leaf of  $\mathcal{F}(V_j)$ ; we denote this relation by saying that  $\mathcal{F}(V_1) \prec \cdots \prec \mathcal{F}(V_r)$  is a *flag of foliations* of  $\mathbf{R}^n$ .

Recall the root space decompositions of  $\mathbf{R}^n$  with respect to  $\overline{M}$  and  $\overline{N}$ . We denote the eigenvalues of  $\overline{M}$  and  $\overline{N}$  by

$$0 < \mu_m^- < \cdots < \mu_1^- < 1 < \mu_1^+ < \cdots < \mu_r^+$$

and

$$0 < \nu_n^- < \cdots < \nu_1^- < 1 < \nu_1^+ < \cdots < \nu_s^+$$

respectively. The corresponding root space decompositions are denoted

$$V_m^- \oplus \cdots \oplus V_1^- \oplus V_1^+ \oplus \cdots \oplus V_r^+$$

and

$$W_n^- \oplus \cdots \oplus W_1^- \oplus W_1^+ \oplus \cdots \oplus W_s^+.$$

As in section 4 we set

$$\begin{aligned} V^- &= V_m^- \oplus \cdots \oplus V_1^-, & V^+ &= V_1^+ \oplus \cdots \oplus V_r^+ \\ W^- &= W_n^- \oplus \cdots \oplus W_1^-, & W^+ &= W_1^+ \oplus \cdots \oplus W_s^+ \end{aligned}$$

Define the *root space flags*

$$\begin{aligned} U_i^- &= V_i^- \oplus \cdots \oplus V_1^-, & i &= 1, \dots, m \\ U_j^+ &= V_1^+ \oplus \cdots \oplus V_j^+, & j &= 1, \dots, r \\ Y_i^- &= W_i^- \oplus \cdots \oplus W_1^-, & i &= 1, \dots, n \\ Y_j^+ &= W_1^+ \oplus \cdots \oplus W_j^+, & j &= 1, \dots, s \end{aligned}$$

and by convention we take  $U_0^-, U_0^+, Y_0^-, Y_0^+$  each to be the trivial subspace.

Associated to the root space flags we have *root space foliation flags*

$$\begin{aligned} \mathcal{F}(U_1^-) &\prec \cdots \prec \mathcal{F}(U_m^-) = \mathcal{F}(V^-) \\ \mathcal{F}(U_1^+) &\prec \cdots \prec \mathcal{F}(U_r^+) = \mathcal{F}(V^+) \\ \mathcal{F}(Y_1^-) &\prec \cdots \prec \mathcal{F}(Y_n^-) = \mathcal{F}(W^-) \\ \mathcal{F}(Y_1^+) &\prec \cdots \prec \mathcal{F}(Y_s^+) = \mathcal{F}(W^+) \end{aligned}$$



**Step 1:  $f$  respects contracting and expanding foliations.**

First we show that  $f(\mathcal{F}(V^-)) = \mathcal{F}(W^-)$  and  $f(\mathcal{F}(V^+)) = \mathcal{F}(W^+)$ .

Given  $p, q \in \mathbf{R}^n$  we have the following chain of equivalences:

1.  $p, q$  are in the same leaf of  $\mathcal{F}(V^+)$ .
2.  $d_{M,t}(p, q) = \|M^{-t}(p - q)\| \rightarrow 0$  as  $t \rightarrow +\infty$ .
3.  $d_{M,t}(p, q)$  is bounded for  $t \in [0, +\infty)$ .
4.  $d_{N,t}(f(p), f(q))$  is bounded for  $t \in [0, +\infty)$ .
5.  $d_{N,t}(f(p), f(q)) = \|N^{-t}(f(p) - f(q))\| \rightarrow 0$  as  $t \rightarrow +\infty$ .
6.  $f(p), f(q)$  are in the same leaf of  $\mathcal{F}(W^+)$ .

The equivalence of (1-3) follows from Proposition 3.2, and similarly for (4-6). The equivalence of (3) and (4) follows from (6.1). This shows  $f(\mathcal{F}(V^+)) = \mathcal{F}(W^+)$ . A similar argument with  $t \in (-\infty, 0]$  shows  $f(\mathcal{F}(V^-)) = \mathcal{F}(W^-)$ .

**Step 2:  $f$  respects root space foliation flags.**

Next we show:

**Claim 6.1.**  *$f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  respects the root space foliation flags, and corresponding root spaces have the same eigenvalues. More precisely we have:*

1.  $r = s$ .
2.  $\mu_j^+ = \nu_j^+$  for  $j = 1, \dots, r$ .
3.  $f(\mathcal{F}(U_j^+)) = \mathcal{F}(U_j^+)$  for  $j = 1, \dots, r$ .
4.  $m = n$ .
5.  $\mu_i^- = \nu_i^-$  for  $i = 1, \dots, m$ .
6.  $f(\mathcal{F}(U_i^-)) = \mathcal{F}(U_i^-)$  for  $i = 1, \dots, m$ .

*It follows that  $M, N$  have the same eigenvalues with the same multiplicities.*

We give the proof of 1,2,3; the proof of 4,5,6 is similar.

We know by Step 1 that  $f(\mathcal{F}(V^+)) = \mathcal{F}(W^+)$ . Consider points  $p, q$  in the same leaf of  $\mathcal{F}(V^+)$ , so  $f(p), f(q)$  are in the same leaf of  $\mathcal{F}(W^+)$ . From Proposition 3.2 it follows that as  $t \rightarrow -\infty$  both of the quantities  $d_{M,t}(p, q)$  and  $d_{N,t}(f(p), f(q))$  approach  $+\infty$ . It follows that for sufficiently large  $t$ , in the inequality (6.1) we can absorb the additive constant  $C$ , yielding

$$\frac{1}{K+1}d_{M,t}(p, q) \leq d_{N,t}(f(p), f(q)) \leq (K+1)d_{M,t}(p, q) \quad (6.2)$$

Define displacement vectors  $v = p - q$ ,  $w = f(p) - f(q)$ . Taking natural logarithms, dividing by  $t$ , and taking limsup, we have

$$\begin{aligned} \limsup_{t \rightarrow -\infty} \frac{\log(d_{M,t}(p, q))}{t} &= \limsup_{t \rightarrow -\infty} \frac{\log(d_{N,t}(f(p), f(q)))}{t} \\ \limsup_{t \rightarrow +\infty} \frac{\log \|M^t v\|}{t} &= \limsup_{t \rightarrow +\infty} \frac{\log \|N^t w\|}{t} \end{aligned} \quad (6.3)$$

To evaluate these limits, let  $I(p, q) = I(v)$  be the unique integer such that

$$v \in U_{I(v)}^+ - U_{I(v)-1}^+$$

or, equivalently, the unique integer such that  $p, q$  are in the same leaf of  $\mathcal{F}(U_{I(p,q)}^+)$  but not in the same leaf of  $\mathcal{F}(U_{I(p,q)-1}^+)$  (recall the convention that  $U_0^+ = 0$ , and so  $I(p, q) = 0$  if and only if  $p = q$ ). Define  $J(f(p), f(q)) = J(w)$  similarly by

$$w \in Y_{J(w)}^+ - Y_{J(w)-1}^+$$

Applying Proposition 3.2 we have

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \frac{\log \|M^t v\|}{t} &= \mu_{I(v)}^+ \\ \limsup_{t \rightarrow +\infty} \frac{\log \|N^t w\|}{t} &= \nu_{J(w)}^+ \end{aligned}$$

and so by (6.3) we have

$$\mu_{I(p,q)}^+ = \mu_{I(v)}^+ = \nu_{J(w)}^+ = \nu_{J(f(p), f(q))}^+$$

Since  $f$  is a bijection from each leaf of  $\mathcal{F}(V^+)$  to some leaf of  $\mathcal{F}(W^+)$ , items (1) and (2) of Claim 6.1 now follow, and it also follows that

$$I(p, q) = J(f(p), f(q))$$

for all  $p, q$  contained in the same leaf of  $\mathcal{F}(V^+)$ .

We now prove item (3) of Claim 6.1 by induction on  $j$ . If  $p, q$  are in the same leaf of  $\mathcal{F}(U_1^+)$  then  $I(p, q) = 1$  and so  $J(p, q) = 1$  which implies that  $f(p), f(q)$  are in the same leaf of  $\mathcal{F}(Y_1^+)$ . A similar argument with  $f^{-1}$  proves that  $f(\mathcal{F}(U_1^+)) = \mathcal{F}(Y_1^+)$ , proving the base step of the induction. Now assume that  $f(\mathcal{F}(U_j^+)) = \mathcal{F}(Y_j^+)$ , and suppose  $p, q$  are in the same leaf of  $\mathcal{F}(U_{j+1}^+)$ . There are two cases to consider. If  $p, q$  lie in the same leaf of  $\mathcal{F}(U_j^+)$  then by the induction hypothesis  $f(p), f(q)$  lie in the same leaf of  $\mathcal{F}(Y_j^+)$ , in particular they lie in the same leaf of  $\mathcal{F}(Y_{j+1}^+)$ . If  $p, q$  do not lie in the same leaf of  $\mathcal{F}(U_j^+)$  then  $I(p, q) = j + 1$  and so  $J(f(p), f(q)) = j + 1$  and so  $f(p), f(q)$  lie on the same leaf of  $\mathcal{F}(Y_{j+1}^+)$ . A similar argument with  $f^{-1}$  shows that  $f(\mathcal{F}(U_{j+1}^+)) = Y_{j+1}^+$ , completing the induction.

As mentioned earlier, (4–6) are proved similarly, completing the proof of Claim 6.1.

### Step 3: $f$ respects Jordan foliation flags.

From Step 2, for each fixed  $j = 1, \dots, r$  the matrices  $M, N$  have  $\mu_j^+$  root spaces  $V_j^+, W_j^+$  respectively. As part of their root space flags we have

$$\begin{aligned} U_j^+ &= U_{j-1}^+ \oplus V_j^+ \\ Y_j^+ &= Y_{j-1}^+ \oplus W_j^+ \end{aligned}$$

Let  $c_j$  be the index of nilpotency of  $\mu_j \cdot I - M$ , and let  $d_j$  be the index of nilpotency of  $\mu_j \cdot I - N$ . Then we have Jordan filtrations

$$\begin{aligned} V_{j,0}^+ &\subset \dots \subset V_{j,c_j}^+ = V_j^+ \\ W_{j,0}^+ &\subset \dots \subset W_{j,d_j}^+ = W_j^+ \end{aligned}$$

and we set  $U_{j,k}^+ = U_{j-1}^+ \oplus V_{j,k}^+$  and  $Y_{j,k}^+ = Y_{j-1}^+ \oplus W_{j,k}^+$ , yielding subspace flags

$$\begin{aligned} U_{j-1}^+ &\subset U_{j,0}^+ \subset \dots \subset U_{j,c_j-1}^+ = U_j^+ \\ Y_{j-1}^+ &\subset Y_{j,0}^+ \subset \dots \subset Y_{j,d_j-1}^+ = Y_j^+ \end{aligned}$$

Corresponding to these subspace flags are foliation flags,

$$\begin{aligned} \mathcal{F}(U_{j-1}^+) &\prec \mathcal{F}(U_{j,0}^+) \prec \dots \prec \mathcal{F}(U_{j,c_j-1}^+) = \mathcal{F}(U_j^+) \\ \mathcal{F}(Y_{j-1}^+) &\prec \mathcal{F}(Y_{j,0}^+) \prec \dots \prec \mathcal{F}(Y_{j,d_j-1}^+) = \mathcal{F}(Y_j^+) \end{aligned}$$

called the *expanding Jordan foliation flags* associated to the corresponding root space foliations  $\mathcal{F}(U_j^+)$ ,  $\mathcal{F}(Y_j^+)$  respectively. The *contracting Jordan foliation flags* associated to each root space foliation  $\mathcal{F}(U_i^-)$ ,  $\mathcal{F}(Y_i^-)$  are similarly defined.

**Claim 6.2.**  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  respects the Jordan foliation flags associated to corresponding root space foliations. More precisely, for each  $j = 1, \dots, r$  we have:

1.  $c_j = d_j$ .
2.  $f(\mathcal{F}(U_{j,k}^+)) = \mathcal{F}(Y_{j,k}^+)$  for  $k = 0, \dots, c_j - 1$ .

and similarly for the contracting Jordan foliation flags.

From this claim, for each  $j = 1, \dots, r$  it immediately follows that  $\overline{M}, \overline{N}$  have the same Jordan blocks with eigenvalue  $\mu_j^+$ , and so the expanding parts of the Jordan forms for  $\overline{M}, \overline{N}$  are identical; similarly for the contracting parts. Since  $M, N$  have no eigenvalues on the unit circle, it now follows that  $M, N$  have the same absolute Jordan forms, completing the proof of Theorem 5.11.

*Proof of Claim 6.2.* Consider  $p, q \in \mathbf{R}^n$  in the same leaf of  $\mathcal{F}(U_j^+)$  but not in the same leaf of  $\mathcal{F}(U_{j-1}^+)$ , and so  $f(p), f(q)$  are in the same leaf of  $\mathcal{F}(Y_j^+)$  but not in the same leaf of  $\mathcal{F}(Y_{j-1}^+)$ . Define displacement vectors  $v = p - q$ ,  $w = f(p) - f(q)$ , and so  $v \in U_j^+ - U_{j-1}^+$  and  $w \in Y_j^+ - Y_{j-1}^+$ . We know that

$$\limsup_{t \rightarrow +\infty} \frac{\log \|M^t v\|}{t} = \frac{\log \|N^t w\|}{t} = \mu_j^+$$

We also know that (6.2) is true for  $t$  sufficiently close to  $-\infty$ , and so for  $t$  sufficiently close to  $+\infty$  we have

$$\frac{1}{K+1} \|M^t v\| \leq \|N^t w\| \leq (K+1) \|M^t v\| \quad (6.4)$$

By induction on  $k = 0, 1, \dots$ , we shall prove that  $v \in U_{j,k}^+$  if and only if  $w \in Y_{j,k}^+$ , or equivalently that  $f(\mathcal{F}(U_{j,k}^+)) = \mathcal{F}(Y_{j,k}^+)$ .

For the basis step  $k = 0$ , divide the inequality (6.4) by  $\mu^t$  to obtain, for all  $t$  sufficiently close to  $+\infty$ :

$$\frac{1}{K+1} \frac{\|M^t v\|}{\mu^t} \leq \frac{\|N^t w\|}{\mu^t} \leq (K+1) \frac{\|M^t v\|}{\mu^t} \quad (6.5)$$

By the Exponential Lower Bound and the Exponential·Polynomial Upper Bound of Proposition 3.2, the quantity  $\frac{\|M^t v\|}{\mu^t}$  is bounded for  $t \geq 0$  if and only if  $v \in U_{j,0}^+$ ; and the quantity  $\frac{\|N^t w\|}{\mu^t}$  is bounded on  $t \geq 0$  if and only if  $w \in Y_{j,0}^+$ . However by (6.5) the boundedness of these two quantities on  $t \geq 0$  are equivalent.

For the induction step, assume that  $f(\mathcal{F}(U_{j,k-1}^+)) = \mathcal{F}(Y_{j,k-1}^+)$ , that is,  $v \in U_{j,k-1}^+$  if and only if  $w \in Y_{j,k-1}^+$ . We must prove that  $v \in U_{j,k}^+ - U_{j,k-1}^+$  if and only if  $w \in Y_{j,k}^+ - Y_{j,k-1}^+$ . From (6.4), for  $t$  sufficiently close to  $+\infty$  we have

$$\frac{1}{K+1} \frac{\|M^t v\|}{\mu^t t^k} \leq \frac{\|N^t w\|}{\mu^t t^k} \leq (K+1) \frac{\|M^t v\|}{\mu^t t^k} \quad (6.6)$$

and

$$\frac{1}{K+1} \frac{\|M^t v\|}{\mu^t t^{k-1}} \leq \frac{\|N^t w\|}{\mu^t t^{k-1}} \leq (K+1) \frac{\|M^t v\|}{\mu^t t^{k-1}} \quad (6.7)$$

By the Exponential·Polynomial Upper and Lower Bounds of Proposition 3.2, the following two statements are equivalent:

- (1)  $v \in U_{j,k}^+ - U_{j,k-1}^+$
- (2) For  $t \geq 0$ , the quantity  $\frac{\|M^t v\|}{\mu^t t^k}$  is bounded, but the quantity  $\frac{\|M^t v\|}{\mu^t t^{k-1}}$  is not bounded.

Similarly, the following two statements are equivalent:

- (3)  $w \in Y_{j,k}^+ - Y_{j,k-1}^+$
- (4) For  $t \geq 0$ , the quantity  $\frac{\|N^t w\|}{\mu^t t^k}$  is bounded, but the quantity  $\frac{\|N^t w\|}{\mu^t t^{k-1}}$  is not bounded.

But by inequalities (6.6) and (6.7), statements (2) and (4) are equivalent, and so statements (1) and (3) are equivalent, completing the inductive proof of item 2 of Claim 6.2 for all  $k \geq 0$ .

The foliation flag  $\mathcal{F}(U_{j,0}^+) \prec \dots \prec \mathcal{F}(U_{j,k}^+) \prec \dots$  must terminate at  $\mathcal{F}(U_j^+)$  for the same value of  $k$  for which the flag  $\mathcal{F}(Y_{j,0}^+) \prec \dots \prec \mathcal{F}(Y_{j,k}^+) \prec \dots$  terminates at  $\mathcal{F}(Y_j^+)$ , proving that  $c_j = d_j$ , and completing the proof of Claim 6.2.  $\diamond$

Our proof of Theorem 5.11 actually provides for some regularity of  $f$ . We record the statement here, although it is not used at all in this paper.

**Proposition 6.3 (Regularity).** *With the assumptions as in Theorem 5.11,  $f$  is a homeomorphism which respects the contracting and expanding root space foliation flags of  $\overline{M}, \overline{N}$ , and for each corresponding pair of root space foliations  $f$  also respects the associated Jordan foliation flags.*  $\diamond$

**Remark.** Even stronger regularity properties should hold. For instance,  $f$  should satisfy lipschitz conditions in directions parallel to a root space, by arguments similar to the results of [FM98]. Understanding what happens transverse to root spaces will require new ideas.

## 7 Quasi-isometries of $\Gamma_M$ via Coarse Topology

Recall the notation for abelian-by-cyclic Lie groups: given  $M \in \mathrm{GL}_\times(m, \mathbf{R})$ , a 1-parameter subgroup  $M^t \subset \mathrm{GL}(m, \mathbf{R})$  with  $M^1 = M$  determines a Lie group denoted  $G_M = \mathbf{R}^m \rtimes_M \mathbf{R}$ .

This entire section will be devoted to a proof of the following.

**Proposition 7.1 (Induced quasi-isometries).** *Consider integral matrices  $M \in \mathrm{GL}_\times(m, \mathbf{R})$ ,  $N \in \mathrm{GL}_\times(n, \mathbf{R})$  and suppose that  $\det M, \det N > 1$ . If there exists a quasi-isometry  $f: \Gamma_M \rightarrow \Gamma_N$  then  $m = n$  and there exists a quasi-isometry  $\phi: G_M \rightarrow G_N$  which coarsely respects horizontal foliations and their transverse orientations. Furthermore, all associated constants for  $\phi$  depend only on those for  $f$ .*

### 7.1 A geometric model for $\Gamma_M$

Let  $M \in \mathrm{GL}_\times(m, \mathbf{R})$  be an integral matrix lying on a 1-parameter subgroup  $M^t$  of  $\mathrm{GL}(m, \mathbf{R})$  with  $M^1 = M$  and with associated Lie group  $G_M$ . We assume that  $\det M > 1$  and we denote  $d = \det M$ .

We start by constructing a contractible,  $(m+1)$ -dimensional metric complex  $X_M$  on which  $\Gamma_M$  acts properly discontinuously and cocompactly by isometries, and so the group  $\Gamma_M$  will be quasi-isometric to the geodesic metric space  $X_M$ .

The description of  $\Gamma_M$  as an ascending HNN extension shows that  $\Gamma_M$  is the fundamental group of the mapping torus of an injective endomorphism of the  $m$ -dimensional torus. Let  $X_M$  be the universal cover of this mapping

torus. Topologically, there is a fibration

$$\begin{array}{ccc} \mathbf{R}^{n-1} & \longrightarrow & X_M \\ & & \downarrow \\ & & T_M \end{array}$$

where  $T_M$  is the homogeneous directed tree with one edge coming into each vertex and  $d = \det M$  edges going out of each vertex. Hence  $X_M$  is a topological product  $X_M \approx \mathbf{R}^{n-1} \times T_M$ .

The action of  $\Gamma_M$  on  $X_M$  by deck transformations induces an action of  $\Gamma_M$  on  $T_M$ . This action is equivalent to the usual action of the HNN extension  $\Gamma_M$  on its Bass-Serre tree  $T_M$ .

Before constructing a metric on  $X_M$ , let us describe the essential properties of such a metric. These are best described by giving the isometry types of natural subcomplexes of  $X_M$ .

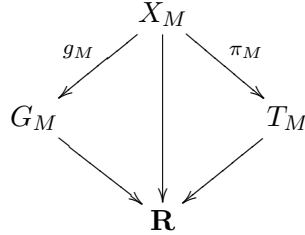
**Definition (Doubled horoballs).** We define a *doubled  $G_M$  horoball*, denoted  $H_M$ , to be the metric space obtained by identifying two copies of  $\{(x, t) \in G_M \mid t \geq 0\}$  along  $\{(x, 0) \in G_M\}$ , endowed with the path metric.

**Definition (Hyperplanes in  $X_M$ ).** Let  $P_\ell = \pi_M^{-1}(\ell)$ , where  $\ell$  is a bi-infinite line in the directed tree  $T_M$ . We call  $P_\ell$  a *hyperplane* in  $X_M$ . There are two cases to consider:

- $\ell$  is coherently oriented in  $T_M$ . In this case  $P_\ell$  is isometric to  $G_M$ , and we call  $P_\ell$  a *coherent hyperplane* in  $X_M$ .
- $\ell$  is not coherently oriented in  $T_M$ , and thus switches orientation precisely once. In this case  $P_\ell$  is isometric to  $H_M$ , and we call  $P_\ell$  an *incoherent hyperplane* in  $X_M$ .

This definition nearly determines a metric on  $X_M$ . To specify a metric on  $X_M$ , one proceeds as follows. Fix a path metric on  $T_M$  so that each edge has length 1. Fix a base vertex on  $T_M$ . These choices determine a unique height function  $T_M \rightarrow \mathbf{R}$  taking the base vertex to the origin, and taking each edge to a segment of length 1 via an orientation preserving isometry. We have also defined a height function  $G_M \rightarrow \mathbf{R}$ . Note that the height function on  $G_M$  was previously called the “time function”; we will use both terms.

The complex  $X_M$  is the fiber product of the two height functions  $T_M \rightarrow \mathbf{R}$ ,  $G_M \rightarrow \mathbf{R}$ , as shown in the following diagram:



There are induced projections  $g_M: X_M \rightarrow G_M$  and  $\pi_M: X_M \rightarrow T_M$ , and an induced height function  $X_M \rightarrow \mathbf{R}$ . There is a unique path metric on  $X_M$  so that each continuous cross section  $G_M \rightarrow X_M$  of  $g_M$  is a path-isometric embedding; and hence each coherent hyperplane in  $X_M$  is an isometrically embedded copy of  $G_M$ .

**Definition (horizontal leaf).** A *horizontal leaf*  $L$  in  $X_M$  is a subset of the form  $L = \pi_M^{-1}(v)$  where  $v \in T_M$ .

Note that the collection of horizontal leaves on  $X_M$ , equipped with the Hausdorff metric, forms a metric space which is isometric to  $T_M$  via the projection map  $\pi_M: X_M \rightarrow T_M$ .

Note that each hyperplane in  $X_M$  comes equipped with a foliation by horizontal leaves. For coherent hyperplanes  $P$  in  $X_M$ , which are isometric to  $G_M$ , the notion of horizontal leaf in  $P$  coincides with that of a horizontal leaf in  $G_M$ , given in §5.1.

## 7.2 Proof of Proposition 7.1

Let  $M, N$  be as in the statement of the proposition.

We begin by showing that  $M$  and  $N$  have the same size. Suppose that  $M \in \text{GL}(m, \mathbf{R})$  and  $N \in \text{GL}(n, \mathbf{R})$ . In §7.1 we constructed finite classifying spaces for  $\Gamma_M$  and  $\Gamma_N$  of dimensions  $m+1, n+1$  respectively, and by Lemma 5.2 of [FM99b] these numbers are the virtual cohomological dimensions of  $\Gamma_M, \Gamma_N$ . By a result of Block-Weinberger [BW97] and Gersten [Ger93], virtual cohomological dimension is a quasi-isometry invariant for groups with finite classifying spaces. It follows that  $m = n$ .

Now  $\Gamma_M$  acts properly discontinuously, freely, and cocompactly on  $X_M$ . This action is by isometries, because  $\Gamma_M$  acts on  $G_M$ , on  $T_M$ , and on  $\mathbf{R}$  by isometries, and the fiber product diagram is equivariant with respect to these actions. It follows that  $\Gamma_M$  in any word-metric is quasi-isometric to



$X_M$ . Henceforth we will freely interchange  $\Gamma_M$  and  $X_M$  when discussing quasi-isometry type. The same discussion applies to  $\Gamma_N$  and  $X_N$ , and so the quasi-isometry  $f: \Gamma_M \rightarrow \Gamma_N$  gives a quasi-isometry (perhaps with bigger constants)  $f: X_M \rightarrow X_N$ .

Proposition 7.1 generalizes the case when  $M$  and  $N$  are  $1 \times 1$  matrices, done in §4 and §5 of [FM98]. The proof here is more difficult, and the steps must be proved in different order. In Steps 1 and 2 we prove (in a more general context; see Theorem 7.7) that a quasi-isometry  $X_M \rightarrow X_N$  coarsely respects hyperplanes and horizontal sets. However, we must still distinguish between coherent and incoherent hyperplanes. This is easy in the  $1 \times 1$  case handled in [FM98], where  $G_M$  and  $G_N$  are (scaled versions of)  $\mathbf{H}^2$ , and a doubled  $\mathbf{H}^2$  horoball is evidently not quasi-isometric to  $\mathbf{H}^2$ . In general we are unable to distinguish the quasi-isometry types of coherent and incoherent hyperplanes. To get around this, in Step 3, Proposition 7.11, we prove that there is no horizontal respecting quasi-isometry between a coherent and an incoherent hyperplane.

**Step 1. Quasi-isometrically embedded hyperplanes are close to hyperplanes:** Given integral matrices  $M, N \in \text{GL}_\times(n, \mathbf{R})$ , if  $P = G_M$  or  $H_M$ , then for all  $K \geq 1, C \geq 0$  there exists  $A \geq 0$  such that if  $\phi: P \rightarrow X_N$  is a  $K, C$ -quasi-isometric embedding then there is a unique hyperplane  $Q \subset X_N$  with  $d_{\mathcal{H}}(\phi(P), Q) \leq A$ .

This was proved for  $1 \times 1$  matrices in [FM98]. Our proof of Step 1, while following the same outline as the  $1 \times 1$  case, will actually apply in a much broader setting. The generalized versions of Steps 1 and 2, given in Theorems 7.3 and Theorem 7.7, are used for example in [FM99a] to study surface-by-free groups, and also in [MSW] to prove quasi-isometric rigidity theorems for various “homogeneous” graphs of groups (see the remark after Theorem 7.7).

The generalization of Step 1 given in Theorem 7.3 will require moving from the category of quasi-isometric embeddings into the category of uniformly proper embeddings. After a fair amount of work to establish the new setting, we then quote some theorems of coarse algebraic topology and follow the proof of [FM98].

Consider a finite graph  $\Gamma$  of finitely generated groups; each edge  $e$  is oriented, with initial and final vertices  $i(e), f(e)$ . We say that  $\Gamma$  is *geometrically homogeneous* if each edge-to-vertex injection is a quasi-isometry with respect to the word metric, or equivalently, has finite index image. Ideally we would like to have a version of Step 1 for any geometrically homogeneous graph

of groups in which each vertex and edge group is the fundamental group of a closed, aspherical  $n$ -manifold, or even more generally, an  $n$ -dimensional Poincaré duality group. This should come from a more careful reading of results in coarse algebraic topology such as [KK], but meanwhile we will use Theorems 7.5 and 7.6, which require us to impose additional assumptions on  $\Gamma$ .

Suppose that we have a category  $\mathcal{C}$  of aspherical, closed, smooth manifolds such that  $\mathcal{C}$  is closed under finite coverings and satisfies *smooth rigidity*, meaning that any homotopy equivalence between manifolds in  $\mathcal{C}$  is homotopic to a diffeomorphism. Such categories include: the  $n$ -torus,  $n \geq 1$ ; hyperbolic surfaces; all other irreducible, nonpositively curved, locally symmetric spaces, by Mostow's Rigidity Theorem [Mos73]; solvmanifolds, by earlier work of Mostow [Mos54]; nilmanifolds, by still earlier work of Malcev [Mal49]; and various generalizations due to Farrell and Jones [FJ89], [FJ97].

We shall assume that  $\Gamma$  is a geometrically homogeneous graph of groups where each vertex group  $\Gamma_v$  is the fundamental group of a manifold  $M_v$  in the category  $\mathcal{C}$ . Construct a *graph of aspherical manifolds*  $M_\Gamma$ , with fundamental group  $\pi_1\Gamma$ , as follows. For each edge  $e$ , the two injections  $\Gamma_e \rightarrow \Gamma_{i(e)}$ ,  $\Gamma_e \rightarrow \Gamma_{t(e)}$  determine two finite covering spaces of  $M_v$  each of whose fundamental group is identified with  $\Gamma_e$ , and so we obtain a diffeomorphism between the two covering spaces; identify these covering spaces and let  $M_e$  be the resulting smooth manifold. We have smooth, finite covering maps  $M_e \rightarrow M_{i(e)}$ ,  $M_e \rightarrow M_{t(e)}$  inducing the corresponding edge-to-vertex group injections. Form  $M_\Gamma$  from the disjoint union

$$\left( \bigcup_v M_v \right) \cup \left( \bigcup_e M_e \times e \right)$$

by gluing  $M_e \times i(e)$  to  $M_{i(e)}$  and  $M_e \times f(e)$  to  $M_{f(e)}$  via the finite covering maps  $M_e \rightarrow M_{i(e)}$  and  $M_e \rightarrow M_{f(e)}$ . From the construction of  $M_\Gamma$  we obtain a map  $M_\Gamma \rightarrow \Gamma$ , such that each fiber  $M_x$ ,  $x \in \Gamma$ , is a manifold in the category  $\mathcal{C}$ .

Let  $X_\Gamma$  be the universal cover of  $M_\Gamma$ . There is a  $\Gamma$ -equivariant fiber bundle  $X_\Gamma \rightarrow T_\Gamma$  over the Bass-Serre tree  $T_\Gamma$  of  $\Gamma$ , whose fiber is a contractible  $n$ -manifold. Any geodesic metric on  $M_\Gamma$  lifts to a  $\pi_1\Gamma$ -equivariant geodesic metric on  $X_\Gamma$ . Smoothness allows us to impose additional geometric structure on  $X_\Gamma$  which we now describe.

A geodesic metric space is *proper* if closed balls are compact. A *bounded geometry metric simplicial complex* is a simplicial complex  $\Sigma$  equipped with a proper, geodesic metric such that for some constants  $0 < C_1 < C_2$  each

positive dimensional simplex has diameter between  $C_1$  and  $C_2$ , and for some constant  $C > 0$  the link of each simplex has  $\leq C$  simplices. A subset  $S$  of  $\Sigma$  is *rectifiable* if for any  $p, q \in S$  there exists a path in  $S$  between  $p$  and  $q$  which is rectifiable in  $\Sigma$ , and which has the shortest  $\Sigma$ -length among all paths in  $S$  between  $p$  and  $q$ . The length of such a path defines a geodesic metric on  $S$ . A  $D$ -*homotopy* in  $\Sigma$  is a homotopy whose tracks all have diameter  $\leq D$ . The space  $\Sigma$  is *uniformly contractible* if there exists a function  $\delta: [0, \infty) \rightarrow [0, \infty)$ , such that for every bounded subset  $S \subset \Sigma$ , the inclusion map  $S \hookrightarrow \Sigma$  is  $\delta(\text{diam}(S))$ -homotopic to a constant map. More precisely we say that  $\Sigma$  is  $\delta$ -*uniformly contractible*.

Let  $T$  be a bounded geometry, metric simplicial tree, let  $X$  be a proper, geodesic metric, and let  $\pi: X \rightarrow T$  be a surjective map. Denote  $X_A = \pi^{-1}(A)$  for each  $A \subset T$ . The map  $\pi$  is called a *metric fibration* if:

1.  $X$  is a uniformly contractible, bounded geometry, metric simplicial complex.
2. For each subtree  $T' \in T$ , the subset  $X_{T'}$  is a subcomplex of  $X$  and is rectifiable in  $X$ .
3. For each  $t \in T$  the subspace  $X_t$  is uniformly contractible and is a bounded geometry, metric simplicial complex, with bounded geometry constants and uniform contractibility data independent of  $t$ .
4. The map  $\pi: X \rightarrow T$  is distance nonincreasing.
5. There is a homeomorphism  $\Theta: X \rightarrow F \times T$  such that:
  - (a) For all  $t \in T$ ,  $\Theta(X_t) = F \times t$ .
  - (b) For all  $x \in F$ , the map  $T \rightarrow x \times T \xrightarrow{\Theta^{-1}} X$  is a locally isometric embedding.
  - (c) There exists  $K \geq 1$  such that for all edges  $e$  of  $T$  and  $t \in e$ , the retraction  $r: e \rightarrow t$  induces a projection  $X_e \xrightarrow{\Theta} F \times e \xrightarrow{\text{Id} \times r} F \times t \xrightarrow{\Theta^{-1}} X_t$  which is  $K$ -lipschitz.

Each fiber  $X_t$ ,  $t \in T$ , is called a *horizontal leaf* in  $X$ . If  $L$  is a bi-infinite line in  $T$  then  $X_L$  is called a *hyperplane* in  $X$ . Items 4 and 5b combine to show that the map of item 5b is an isometric embedding; the image  $\Theta^{-1}(x \times T)$  is called a *vertical leaf* in  $X$ . For each subtree  $T' \subset T$ , the closest point retraction  $r: T \rightarrow T'$  induces a map

$$X \xrightarrow{\Theta} F \times T \xrightarrow{\text{Id} \times r} F \times T' \xrightarrow{\Theta^{-1}} X_{T'}$$

called *vertical projection* of  $X$  to  $X_{T'}$ .

**Remark.** Suppose  $\Gamma$  is a graph of groups taken from a category  $\mathcal{C}$  as above. Let  $M_\Gamma$  and  $X_\Gamma \rightarrow T_\Gamma$  be as constructed above starting from  $\Gamma$ . Then elementary constructions produce a metric and a simplicial structure on  $M_\Gamma$  which lifts to a  $\Gamma$ -equivariant metric and simplicial structure on  $X$  such that  $X \rightarrow T_\Gamma$  is a metric fibration. Item (1) follows by compactness of  $M_\Gamma$ .

**Remark.** The definition has some redundancy: item (1) is a formal consequence of item (3), as can be seen by elementary but mildly tedious arguments. But by the previous remark we may dispense with these arguments for the examples at hand.

The following lemma, applied to a bi-infinite lines in  $T$ , gives good geometric properties for hyperplanes:

**Lemma 7.2.** *If  $\pi: X \rightarrow T$  is a metric fibration then there exist functions  $\delta': [0, \infty) \rightarrow [0, \infty)$  and  $\rho: [0, \infty) \rightarrow [0, \infty)$  with  $\lim_{t \rightarrow \infty} \rho(t) = \infty$ , such that for any subtree  $T' \subset T$  we have:*

- (1) *The embedding  $X_{T'} \rightarrow X$  is  $\rho$ -uniformly proper.*
- (2) *The geodesic metric space  $X_{T'}$  is  $\delta'$ -uniformly contractible.*

*Proof.* To prove (1), consider  $x, y \in X_{T'}$ , let  $D = d_X(x, y)$ , and let  $\gamma: [0, D] \rightarrow X$  be a geodesic connecting  $x$  and  $y$ . Let  $N_D(T')$  be the  $D$ -neighborhood of  $T'$  in  $T$ , so  $\gamma \subset X_{N_D(T')}$ . Applying item 5 iteratively, projecting inward starting from the edges of  $N_D(T')$  furthest from  $T'$ , it follows that vertical projection  $X_{N_D(T')} \rightarrow X_{T'}$  distorts any distance  $r$  by at worst  $K^D r$ , and so  $d_{X_{T'}}(x, y) \leq K^D D$ .

To prove (2), suppose that  $A \subset X_{T'}$  and  $\text{diam}_{X_{T'}}(A) \leq R$  and so  $A$  is  $R'$ -homotopic to a constant in  $X$  where  $R'$  depends on  $R$  but not on  $A$ . This homotopy may then be mapped back to  $X_{T'}$  by vertical projection, distorting diameters of homotopy tracks by an amount bounded in terms of  $R'$  as we saw above. The result is an  $R''$ -homotopy of  $A$  to a constant in  $T'$ , with  $R''$  depending only on  $R$  and not on  $A$ .  $\diamond$

Here is our generalization of Step 1. It applies to any metric fibration of the form  $X_\Gamma \rightarrow T_\Gamma$ , where  $\Gamma$  is a finite, geometrically homogeneous graph of fundamental groups of manifolds in any of the categories  $\mathcal{C}$  described earlier.

**Theorem 7.3.** *Let  $\pi: X \rightarrow T$  be a metric fibration whose fibers are contractible  $n$ -manifolds for some  $n$ . Let  $P$  be a contractible  $(n + 1)$ -manifold which is a uniformly contractible, bounded geometry, metric simplicial complex. Then for any uniformly proper embedding  $\phi: P \rightarrow X$ , there exists a unique hyperplane  $Q \subset X$  such that  $\phi(P)$  and  $Q$  have finite Hausdorff distance in  $X$ . The bound on Hausdorff distance depends only on the metric fibration data for  $\pi$ , the uniform contractibility data and bounded geometry data for  $P$ , and the uniform properness data for  $\phi$ .*

*Proof.* Uniqueness of  $Q$  follows obviously from the fact that distinct hyperplanes in  $X$  have infinite Hausdorff distance.

For existence of  $Q$  we follow closely the proof of Proposition 4.1 of [FM98], concentrating on details needed to explicate the difference between the “quasi-isometric” setting of [FM98] and the present “uniformly proper” setting.

Using the bounded geometry of  $P$ , uniform contractibility of  $X$ , and uniform properness of  $\phi$ , we may replace  $\phi$  by a continuous, uniformly proper map, moving values of  $\phi$  a bounded distance. Henceforth we shall assume  $\phi$  is continuous.

Pick a topologically proper embedding of  $T$  in an open disc  $D$ . For each component  $U$  of  $D - T$ , the frontier of  $U$  in  $D$  is a bi-infinite line  $L(U)$  in  $T$ . There is a homeomorphism of pairs  $(\bar{U}, L(U)) \approx (L(U) \times [0, \infty), L_U \times 0)$ .

Consider the topologically proper embedding  $X \xrightarrow{\Theta} F \times T \hookrightarrow F \times D$ . Note that  $F \times D$  is a contractible  $(n + 2)$ -manifold. For each component  $U$  of  $D - T$  we have a homeomorphism

$$\begin{aligned} F \times \bar{U} &\xrightarrow{\approx} F \times (L(U) \times [0, \infty)) \xrightarrow{\approx} (F \times L(U)) \times [0, \infty) \\ &\xrightarrow[\Theta \times \text{Id}]{\approx} X_{L(U)} \times [0, \infty) \end{aligned}$$

The frontier of this set in  $F \times D$  is  $F \times L(U) \approx X_{L(U)}$ . Put a product metric and a product simplicial structure on  $X_{L(U)} \times [0, \infty)$  and glue to  $F \times L(U)$ . Doing this for each  $U$ , we impose a proper geodesic metric on  $F \times D$  for which the inclusion  $X \hookrightarrow F \times D$  is an isometric embedding.

The simplicial structure on  $F \times D$  evidently has bounded geometry. Also, the metric space  $F \times D$  is uniformly contractible. To see this, let  $A \subset F \times D$  have diameter  $\leq r$ . If  $A \cap X \neq \emptyset$  then homotoping along products lines of  $X_{L(U)} \times [0, \infty)$  for each  $U$  we obtain an  $r$ -homotopy of  $A$  into  $F \times T \approx X$ , and then we use uniform contractibility of  $X$ . Whereas if  $A \cap X = \emptyset$ , then  $A \subset F \times U \approx X_{L(U)} \times (0, \infty)$  for some component  $U$  of  $D - T$ ; there is an

$r$ -homotopy of  $A$  into some  $X_{L(U)} \times x$ , and the latter is uniform contractible by Lemma 7.2.

We now plug this setup into the coarse separation and packing methods of Farb–Schwartz [FS96] and Schwartz [Sch96]. We’ll use a generalization of the Coarse Separation Theorem with more easily applied hypotheses, due to Kapovich–Kleiner [KK]. We denote the  $r$ -ball about a subset  $A$  of a metric space  $M$  by  $B_r(A; M)$ . In a metric space  $Z$ , a subset  $U \subset Z$  is *deep in  $Z$*  if for each  $r > 0$  there exists  $x \in U$  such that  $B_r(x; Z) \subset U$ . A subset  $A \subset Z$  *coarsely separates  $Z$*  if for some  $D > 0$  there are at least two components of  $Z - N_D(A; Z)$  which are deep in  $Z$ ; the constant  $D$  is called a *coarse separation constant* for  $A$ . Note that if subsets  $A$  and  $B$  of  $Z$  have bounded Hausdorff distance from each other, then  $A$  coarsely separates  $Z$  if and only if  $B$  does.

Here is an elementary consequence of the definitions:

**Lemma 7.4.** *Let  $f: X \rightarrow Y$  be a quasi-isometry between geodesic metric spaces. If  $A \subset X$  coarsely separates  $X$  then  $f(A)$  coarsely separates  $Y$ , with separation constant depending only on the quasi-isometry constants of  $f$  and the separation constant for  $A$ .  $\diamond$*

Here is the version of the Coarse Separation Theorem that we will use.

**Theorem 7.5 ([KK]).** *Let  $P$  be a contractible  $(n + 1)$ -manifold,  $Z$  a contractible  $(n + 2)$ -manifold, and suppose that  $P, Z$  are uniformly contractible, bounded geometry, metric simplicial complexes. Let  $\Phi: P \rightarrow Z$  be a uniformly proper map. Then  $\Phi(P)$  coarsely separates  $Z$ , with coarse separation constant  $D$  depending only on the uniform contractibility and bounded geometry data for  $P$  and  $Z$  and the uniform properness data for  $\Phi$ . Moreover if  $\Phi$  is continuous then we may take  $D = 0$ , that is,  $Z - \Phi(P)$  has at least two components which are deep in  $Z$ .  $\diamond$*

**Remark.** In fact there are exactly two components of  $Z - N_D(\Phi(P); Z)$  which are deep in  $Z$  (see [KK]).

Following [FS96] we have a corollary:

**Theorem 7.6 (Packing Theorem).** *Let  $Q, P$  be contractible  $(n+1)$ -manifolds, which are uniformly contractible, bounded geometry, metric simplicial complexes. Let  $\psi: Q \rightarrow P$  be a uniformly proper map. Then there exists  $R > 0$  such that  $N_R(\psi(Q); P) = P$ . The constant  $R$  depends only on the uniform contractibility data and bounded geometry data for  $Q, P$  and the uniform properness data for  $\psi$ .*

*Proof.* If no such  $R$  exists then the image of the map  $Q \xrightarrow{\psi} P \hookrightarrow P \times \mathbf{R}$  does not coarsely separate  $P \times \mathbf{R}$ , violating Theorem 7.5.  $\diamond$

Continuing with the proof of Theorem 7.3, compose the continuous, uniformly proper map  $\phi: P \rightarrow X$  with the isometric embedding  $X \rightarrow F \times D$  to obtain a continuous, uniformly proper map  $\Phi: P \rightarrow F \times D$ . By the Coarse Separation Theorem it follows that  $(F \times D) - \Phi(P)$  has at least two components which are deep in  $F \times D$ .

Now take the argument of [FM98], Step 1, pages 426–427, and apply it verbatim, to produce a hyperplane  $Q \subset X$  such that  $Q \subset \Phi(P)$ . Next take the argument of Step 2, pages 427–428, and apply it verbatim, replacing “quasi-isometric embeddings” with “uniformly proper maps” and using the Packing Theorem above, to show the existence of  $R'$  such that  $\phi(P) \subset N_{R'}(Q; X)$ , where  $R'$  depends only on the metric fibration data for  $\pi$ , the uniform contractibility and bounded geometry data for  $P$ , and the uniform properness data for  $\phi$ .

This finishes the proof of Theorem 7.3 and of Step 1.  $\diamond$

**Step 2. A quasi-isometry takes hyperplanes and horizontal leaves in  $X_M$  to hyperplanes and horizontal leaves in  $X_N$ :**

Consider integral matrices  $M, N \in \mathrm{GL}_\times(n, \mathbf{R})$  with  $\det M, \det N > 1$ , and let  $f: X_M \rightarrow X_N$  be a quasi-isometric embedding. Then there is a constant  $A \geq 0$ , depending only on  $X_M, X_N$  and the quasi-isometry constants of  $f$ , such that:

1. For each hyperplane  $P \subset X_M$  there exists a unique hyperplane  $Q \subset X_N$  such that  $d_{\mathcal{H}}(f(P), Q) \leq A$ .
2. For each horizontal leaf  $L$  of  $X_M$  there exists a horizontal leaf  $L'$  of  $X_N$  such that  $d_{\mathcal{H}}(f(L), L') \leq A$ .

The proof of this step is the first place in our arguments where the assumption that  $\det M, \det N > 1$  is crucial. Again we will investigate this step in the general setting of metric fibrations over trees.

Consider a metric fibration  $\pi: X \rightarrow T$ . The tree  $T$  is *bushy* if there exists a constant  $\beta$  such that each point of  $T$  is within distance  $\beta$  of some vertex  $v$  such that  $T - v$  has at least 3 unbounded components. Note that if  $M$  is an integer matrix in  $\mathrm{GL}_\times(n, \mathbf{R})$ , and if  $X_M \rightarrow T_M$  is the associated metric fibration over the Bass-Serre tree  $T_M$  of the group  $\Gamma_M$ , then  $T_M$  is bushy if and only if  $\det M > 1$ . In fact, for any graph of finitely generated groups, the Bass-Serre tree is either bounded, quasi-isometric to a line, or bushy,

and the question of which alternative holds is easily decided by inspection of the graph of groups.

Here is our generalization of Step 2:

**Theorem 7.7.** *Let  $\pi: X \rightarrow T$ ,  $\pi': X' \rightarrow T'$  be metric fibrations over  $\beta$ -bushy trees  $T, T'$ , such that the fibers of  $\pi$  and  $\pi'$  are contractible  $n$ -manifolds for some  $n$ . Let  $f: X \rightarrow X'$  be a quasi-isometry. Then there exists a constant  $A$ , depending only on the metric fibration data of  $\pi, \pi'$ , the quasi-isometry data for  $f$ , and the constant  $\beta$ , such that:*

- (1) *For each hyperplane  $P \subset X$  there exists a unique hyperplane  $Q \subset X'$  such that  $d_{\mathcal{H}}(f(P), Q) \leq A$ .*
- (2) *For each horizontal leaf  $L \subset X$  there is a horizontal leaf  $L' \subset X'$  such that  $d_{\mathcal{H}}(f(L), L') \leq A$ .*

**Remark.** This result is used in [MSW] to prove quasi-isometric rigidity for fundamental groups of geometrically homogeneous graphs of groups whose vertex groups are fundamental groups of manifolds in a category  $\mathcal{C}$  as above, as long as that class of groups is itself quasi-isometrically rigid. For example, quasi-isometric rigidity is proved for graphs of  $\mathbf{Z}$ 's,  $\mathbf{Z}^n$ 's, surface groups, lattices in semisimple Lie groups, nilpotent groups, etc.

*Proof.* To prove (1), by Lemma 7.2 the inclusion map  $P \hookrightarrow X$  is uniformly proper and  $P$  is uniformly contractible, and clearly  $P$  is a contractible  $(n+1)$ -manifold. Composing with  $f$  we obtain a uniformly proper map  $P \rightarrow X'$ . Now apply Theorem 7.3.

The idea of the proof of (2) is that bushiness of the tree allows one to gain quasi-isometric control over horizontal leaves by considering them as “coarse intersections” of hyperplanes.

**Definition (Coarse intersection).** A subset  $W$  of a metric space  $X$  is a *coarse intersection* of subsets  $U, V \subset X$ , denoted  $W = U \cap_{\mathcal{C}} V$ , if there exists  $K_0$  such that for every  $K \geq K_0$  there exists  $K' \geq 0$  so that

$$d_{\mathcal{H}}(\text{Nbhd}_K(U) \cap \text{Nbhd}_K(V), W) \leq K'$$

Note that although such a set  $W$  may not exist, when it does exist then any two such sets are a bounded Hausdorff distance from each other.

The following fact is an elementary consequence of the definitions.



Figure 1: Possible coarse intersections of distinct hyperplanes in  $X$ , projected to  $T$ . In (a),  $P_1 \cap_C P_2 = P_1 \cap P_2$  is a half-plane. In (b–d),  $P_1 \cap_C P_2$  is a horizontal leaf;  $P_1 \cap P_2$  can be empty (b), a horizontal leaf (c), or a finite strip of horizontal leaves (d).

**Lemma 7.8.** *For any quasi-isometry  $f: X \rightarrow Y$  of metric spaces, and  $U, V \subset X$ , if  $U \cap_C V$  exists then  $f(U \cap_C V)$  is a coarse intersection of  $f(U)$ ,  $f(V)$ , with constants depending only on the quasi-isometry constants for  $f$  and the coarse intersection constants for  $U$  and  $V$ .  $\diamond$*

Consider now a metric fibration  $\pi: X \rightarrow T$ . A subset of  $X$  of the form  $X_\sigma = \pi^{-1}(\sigma)$ , where  $\sigma$  is an infinite ray in  $T$ , will be called a *half-plane* in  $X$ . The next lemma is an easy observation—see Figure 1.

**Lemma 7.9.** *Let  $\pi: X \rightarrow T$  be a metric fibration over a tree  $T$ . Let  $P_1$  and  $P_2$  be distinct hyperplanes in  $X$ . Then  $P_1 \cap_C P_2$  exists and is a bounded Hausdorff distance from either a half-plane or a horizontal leaf in  $X$ . Moreover,  $P_1 \cap_C P_2$  is a bounded Hausdorff distance from a half-plane if and only if  $P_1 \cap P_2$  is a half-plane.  $\diamond$*

We remark that  $P_1 \cap_C P_2$  can be an arbitrarily large finite Hausdorff distance from a horizontal leaf; see Figure 1(b,d).

**Lemma 7.10.** *Let  $\pi: X \rightarrow T$ ,  $\pi': X' \rightarrow T'$  be metric fibrations. Let  $f: X \rightarrow X'$  be a quasi-isometry. Suppose  $P_1$  and  $P_2$  are distinct hyperplanes in  $X$  which intersect in a half-plane. Then  $f(P_1)$  and  $f(P_2)$  are a*

uniformly bounded Hausdorff distance from distinct hyperplanes  $Q_1, Q_2$  in  $X'$  which intersect in a half-plane in  $X'$ .

*Proof.* By Theorem 7.3, there exists a constant  $A$  so that  $f(P_i)$  is within Hausdorff distance  $A$  of a unique hyperplane  $Q_i$  in  $X'$ . Since  $P_1, P_2$  are distinct they have infinite Hausdorff distance, so  $Q_1$  and  $Q_2$  have infinite Hausdorff distance and hence  $Q_1 \neq Q_2$ .

By Lemma 7.9, it is enough to prove that  $Q_1 \cap_C Q_2$  is not a bounded Hausdorff distance from a horizontal leaf in  $X'$ . If  $Q_1 \cap_C Q_2$  is a bounded Hausdorff distance from a horizontal leaf, then since any horizontal leaf in  $Q_1$  coarsely separates  $Q_1$  it must be that  $Q_1 \cap_C Q_2$  coarsely separates  $Q_1$ . But  $P_1 \cap_C P_2$  does not coarsely separate  $P_1$ . This contradicts Lemma 7.4.  $\diamond$

We now prove Theorem 7.7. Consider the quasi-isometry  $f: X \rightarrow X'$ . Since  $T$  is bushy, any horizontal leaf  $L$  in  $X$  can be realized as a coarse intersection of three hyperplanes  $P_1, P_2, P_3$ , such that the pairwise intersections  $P_1 \cap P_2, P_2 \cap P_3, P_3 \cap P_1$  form three half-planes, any two of which have infinite Hausdorff distance. Moreover,  $d_{\mathcal{H}}(L, P_1 \cap P_2 \cap P_3) \leq \beta$  where  $\beta$  is a bushiness constant for  $T$  (see Figure 2).

Consider the unique hyperplane  $Q_i$  which lies a Hausdorff distance of at most  $A$  from  $f(P_i)$ ,  $i = 1, 2, 3$ . By Lemma 7.10, the pairwise intersections  $Q_1 \cap Q_2, Q_2 \cap Q_3, Q_3 \cap Q_1$  are all half-planes, any two of which have infinite Hausdorff distance. The following elementary fact about trees, applied to  $T'$ , now shows that  $Q_1 \cap Q_2 \cap Q_3$  is a horizontal leaf  $L'$  in  $X'$ :

**Fact about trees:** Let  $\ell_1, \ell_2, \ell_3$  be bi-infinite lines in a simplicial tree  $T'$ , such that the pairwise intersections  $\ell_1 \cap \ell_2, \ell_2 \cap \ell_3, \ell_3 \cap \ell_1$  are all infinite rays in  $T'$ , any two of which have infinite Hausdorff distance. Then  $\ell_1 \cap \ell_2 \cap \ell_3$  is a vertex of  $T'$ .

Since  $L \subset N_{\beta}(P_i)$  it follows that

$$f(L) \subset N_{K\beta+C}(f(P_i)) \subset N_{K\beta+C+A}(Q_i), \quad i = 1, 2, 3$$

But clearly we have  $\bigcap_{i=1}^3 N_{K\beta+C+A}(Q_i) = N_{K\beta+C+A}(L')$ .

To summarize, given a horizontal leaf  $L$  of  $X$ , we have found a horizontal leaf  $L'$  of  $X'$  such that  $L \subset N_{A'}(L')$  where  $A' = K\beta + C + A$ . A similar argument using a coarse inverse for  $f$  provides the desired bound for  $d_{\mathcal{H}}(f(L), L')$ . This completes the proofs of Theorem 7.7 and of Step 2.  $\diamond$

Figure 2: Any point  $x \in T$  is a bounded distance  $\beta$  from a vertex  $v \in T$  that separates  $T$  into at least three unbounded components. The vertex  $v$  is the (coarse) intersection of three proper lines  $\ell_1, \ell_2, \ell_3$ , such that the pairwise intersections  $\ell_1 \cap \ell_2, \ell_2 \cap \ell_3, \ell_3 \cap \ell_1$  are rays in  $T$ , any two of which have infinite Hausdorff distance. Moreover,  $d(x, \ell_1 \cap \ell_2 \cap \ell_3) \leq \beta$ .

**Step 3. A quasi-isometry takes coherent hyperplanes in  $X_M$  to coherent hyperplanes in  $X_N$ .**

Let  $M, N$  be as in the statement of Proposition 7.1, and fix a quasi-isometry  $f: X_M \rightarrow X_N$ .

Let  $P$  be any coherent hyperplane in  $X_M$ . By Step 2 it follows that  $f(P)$  is within a Hausdorff distance  $A$  from a unique hyperplane  $Q$  in  $X_N$ . By composing  $f|_P$  with vertical projection  $X_N \rightarrow Q$  we obtain a map  $\phi: P \rightarrow Q$ . The inclusion maps  $P \hookrightarrow X_M$  and  $Q \hookrightarrow X_N$  are coarsely Lipschitz and uniformly proper; indeed they are isometric embeddings with respect to the induced path metrics on  $P, Q$ . By Lemma 2.1,  $\phi$  is a quasi-isometry, with quasi-isometry constants depending only on those for  $f$ . By Step 2,  $f$  coarsely respects the horizontal foliations of  $X_M$  and  $X_N$ ; vertical projection  $X_N \rightarrow Q$  takes horizontal leaves to horizontal leaves, and so  $\phi$  coarsely respects the horizontal foliations of  $P$  and  $Q$ , with a coarseness constant depending only on the quasi-isometry constants of  $f$ .

Since  $P$  is a coherent hyperplane it is isometric to  $G_M$ . Since  $Q$  is a hyperplane it is isometric to either  $G_N$  or  $H_N$ , and we now show that the second possibility cannot occur.

**Proposition 7.11.** *Given matrices  $M, N \in \text{GL}_\times(n, \mathbf{R})$  with  $\det M, \det N > 1$ , there is no quasi-isometry  $\phi: G_M \rightarrow H_N$  which coarsely respects horizon-*

*tal foliations.*

*Proof.* The idea of the proof is to compare the growth types of the filling area functions for “quasivertical bigons” in  $G_M$  and in  $H_N$ . In  $G_M$  this growth type will be quadratic, while in  $H_N$  it will be exponential.

Let  $H = G_M, H_M, G_N,$  or  $H_N$ . There is a quotient map  $H \rightarrow \mathbf{R}$  whose point pre-images give the horizontal foliation of  $H$ , and such that the Hausdorff distance between two horizontal leaves equals the distance between the corresponding points in  $\mathbf{R}$ . A path  $\gamma$  in  $H$  is said to be  $(K, C)$ -*quasivertical* if its projection to  $\mathbf{R}$  is a  $(K, C)$ -quasigeodesic. Define a  $(K, C)$ -*quasivertical bigon* in  $H$  to be a pair of  $(K, C)$ -quasivertical paths  $\gamma, \gamma'$  which begin and end at the same point.

If  $K, C$  are fixed, we define a filling area function  $A(L)$  for  $(K, C)$ -quasivertical bigons in  $H$ . Given a  $(K, C)$ -quasivertical bigon  $\gamma, \gamma'$ , its *filling area* is the infimal area of a Lipschitz map  $D^2 \rightarrow H$  whose boundary is a reparameterization of the closed curve  $\gamma^{-1} * \gamma'$ ; such a map  $D^2 \rightarrow H$  is called a *filling disc* for  $\gamma^{-1} * \gamma'$ . For each  $L \geq 0$  define  $\mathcal{A}(L)$  to be the supremal filling area over all  $(K, C)$ -quasivertical bigons  $\gamma, \gamma'$  in  $H$  such that  $\text{Length}(\gamma) + \text{Length}(\gamma') \leq L$ .

Suppose there is a quasi-isometry  $\phi: G_M \rightarrow H_N$  which coarsely respects horizontal foliations. Let  $\bar{\phi}: H_N \rightarrow G_M$  be a coarse inverse for  $\phi$ , also coarsely respecting horizontal foliations. Clearly  $\bar{\phi}$  takes any  $K, C$ -quasivertical bigon in  $H_N$  to a  $(K', C')$ -quasivertical bigon in  $G_M$ , distorting lengths by at worst an affine function; this affine function, and the constants  $K', C'$ , depend only on  $K, C$ , the quasi-isometry constants for  $\phi$ , and the Hausdorff constant for the induced height function. Fill the resulting bigon in  $G_M$  as efficiently as possible, and map back to  $H_N$  via  $\phi$ , distorting area by at worst an affine function which again has the same dependencies. We thereby obtain a filling of the original bigon in  $H_N$ . If  $\mathcal{A}_1(L)$  denotes the filling area function for  $(K', C')$ -quasivertical bigons in  $G_M$ , and if  $\mathcal{A}_2(L)$  denotes the filling area function for  $(K, C)$ -quasivertical bigons in  $H_N$ , it follows that the growth type of  $\mathcal{A}_2(L)$  is dominated by the growth type of  $\mathcal{A}_1(L)$ , that is,

$$\mathcal{A}_2(L) \leq \alpha \cdot \mathcal{A}_1(\beta L + \delta) + \zeta$$

for some positive constants  $\alpha, \beta, \delta, \zeta$  independent of  $L$ .

However, we shall now show that  $\mathcal{A}_1(L)$  has a quadratic upper bound while  $\mathcal{A}_2(L)$  has an exponential lower bound, contradicting the above inequality.

Consider a  $K', C'$ -quasivertical bigon  $\gamma, \gamma'$  in  $G_M$ . Applying the argument of Claim 5.7, there are center leaves  $\tau, \tau'$  in  $G_M$  and quasivertical

paths  $\rho \subset \tau, \rho' \subset \tau'$  which stay uniformly close to  $\gamma, \gamma'$ , respectively. The initial points of  $\rho, \rho'$  are at a uniformly bounded distance, as are the terminal points, and it follows that  $\rho'$  stays uniformly close to a quasivertical path  $\rho'' \subset \tau$ . Connecting initial and terminal endpoints with short paths  $\eta, \eta'$  we thus obtain a closed curve  $\rho^{-1} * \eta * \rho'' * \eta'$ , contained in a center leaf of  $G_M$ , which stays uniformly close to  $\gamma^{-1} * \gamma'$ . Since center leaves of  $G_M$  are isometric to Euclidean space, in which the filling function is quadratic, it follows that  $\mathcal{A}_1(L)$  has a quadratic upper bound.

To show that  $\mathcal{A}_2(L)$  has an exponential lower bound, we now construct quasivertical bigons in  $H_N$  which can be filled only by discs of exponential area. In the case where  $N$  is a  $1 \times 1$  matrix such loops are given explicitly in [ECH<sup>+</sup>92], Chapter 7.4; examples for general  $N$  are simple modifications of this example. To be explicit, choose an eigenvalue of  $N$  of absolute value  $\alpha > 1$ ; such an eigenvalue exists because  $\det N > 1$ . Choose an affine subspace  $A \subset \mathbf{R}^n$  parallel to the  $\alpha$ -eigenspace of  $N$ . Consider the subspace  $A \times \mathbf{R} \subset \mathbf{R}^n \times \mathbf{R} \approx G_N$ .

For each fixed  $L \geq 0$ , choose two vertical segments  $g, g'$  in  $A \times [0, \infty)$  whose upper endpoints are in  $A \times L$  and whose lower endpoints are in  $A \times 0$ , and so that the distance in  $A \times L$  between the upper endpoints, measured using the Riemannian metric on  $G_N$ , is equal to 1; it follows that the distance in  $A \times 0$  between the lower endpoints, measured using the Riemannian metric on  $G_N$ , is within a constant multiple of  $\alpha^L$ .

Now double this picture, in the doubled  $G_N$  horoball  $H_N$ , to get a closed loop in  $H_N$ , that is: in one horoball go up  $g$ , across 1 unit, and down  $g'$ , and then in the other horoball go up  $g'$ , across 1 unit, and down  $g$ ; let  $\rho$  be the resulting closed curve in  $H_N$ . We have  $\text{Length}(\rho) = 4L + 2$ . To see that the filling area of  $\rho$  is exponential in  $L$ , note that any filling disc for  $\rho$  must contain a path in  $A \times 0$  connecting the lower endpoints of  $g, g'$ , because  $A \times 0$  separates the two halves of  $\rho$  in  $H_N$ . This path has length exponential in  $L$ ; and a neighborhood of this path in the filling disc has area exponential in  $L$ .  $\diamond$

#### Step 4. A horizontal respecting quasi-isometry preserves transverse orientation

Let  $M, N$ , and  $f: \Gamma_M \rightarrow \Gamma_N$  be as in the statement of Proposition 7.1. By Step 3 there is a quasi-isometry  $\phi: G_M \rightarrow G_N$ , and by Step 2,  $\phi$  coarsely respects the horizontal foliations of  $G_M$  and  $G_N$ . Suppose that  $\phi$  reverses the transverse orientation. There is a quasi-isometry  $G_N \rightarrow G_{N-1}$  which coarsely respects horizontal foliations, *reversing* transverse orienta-

tions. Precomposing with  $\phi: G_M \rightarrow G_N$  and applying Steps 1–3, we obtain a quasi-isometry  $G_M \rightarrow G_{N^{-1}}$  which coarsely respects the transversely oriented horizontal foliations. Applying Theorem 5.2, it follows that  $M$  and  $N^{-1}$  have positive real powers with the same absolute Jordan form, and so these powers also have the same determinant. But each positive power of  $M$  has determinant  $> 1$ , whereas every positive power of  $N^{-1}$  has determinant  $< 1$ , a contradiction showing that  $\phi$  must preserve the transverse orientation.

This completes the proof of Proposition 7.1.  $\diamond$

**Remark.** Note in the proof of Proposition 7.1 that different choices of coherent hyperplanes in  $X_M$  yield different quasi-isometries  $\phi$ . In some cases  $\phi$  is well-defined up to some constant  $A$ , that is, for any two choices of coherent hyperplane in  $X_M$ , the induced maps  $\phi_1, \phi_2: G_M \rightarrow G_N$  satisfy  $\sup_x d(\phi_1(x), \phi_2(x)) \leq A$ . This is true, for example, in the “centerless” case where  $M, N$  have no eigenvalues on the unit circle. In the general case, the best that can be said is that the map induced by  $\phi$  from the center leaf space of  $G_M$  to the center leaf space of  $G_N$  is well defined up to a constant, with respect to the Hausdorff metrics on the center leaf spaces.

## 8 Finding the Integers

In this section we prove Theorem 1.1. Let  $M, N$  be  $n \times n$  integral matrices with  $|\det M|, |\det N| > 1$ . We must prove that  $\Gamma_M$  is quasi-isometric to  $\Gamma_N$  if and only if there exist positive integers  $a, b$  such that  $M^a$  and  $N^b$  have the same absolute Jordan form.

First we show that the groups  $\Gamma_{M^a}$  and  $\Gamma_M$  are quasi-isometric, for any positive integer  $a$ , by showing that  $\Gamma_{M^a}$  is a subgroup of finite index in  $\Gamma_M$ , specifically of index  $a$ . To see why, consider the presentations

$$\begin{aligned}\Gamma_M &= \langle \mathbf{Z}^n, t \mid t^{-1}xt = M(x), x \in \mathbf{Z}^n \rangle \\ \Gamma_{M^a} &= \langle \mathbf{Z}^n, s \mid s^{-1}xs = M^a(x), x \in \mathbf{Z}^n \rangle\end{aligned}$$

Define a homomorphism  $\Gamma_M \mapsto \mathbf{Z}/a\mathbf{Z}$  by  $\mathbf{Z}^n \mapsto 0, t \mapsto 1$ . This homomorphism is onto, and its kernel is generated by  $\mathbf{Z}^n, t^a$ . This kernel is isomorphic to  $\Gamma_{M^a}$  under the injection  $\Gamma_{M^a} \hookrightarrow \Gamma_M$  given by  $x \mapsto x, s \mapsto t^a$ .

Similarly,  $\Gamma_{N^b}$  is quasi-isometric to  $\Gamma_N$ , for any positive integer  $b$ .

By squaring  $M, N$  if necessary, we may therefore assume that  $\det M, \det N > 1$ , and that  $M$  and  $N$  lie on 1-parameter subgroups; we continue with this assumption up through the end of the proof in §8.2. Choose 1-parameter

subgroups  $M^t, N^t$  of  $\mathrm{GL}(n, \mathbf{R})$  with  $M = M^1, N = N^1$ , let  $G_M, G_N$  be the associated Lie groups constructed in §4, and let  $X_M, X_N$  be the associated geodesic metric spaces constructed in §7. The group  $\Gamma_M$  is quasi-isometric to  $X_M$ , and  $\Gamma_N$  is quasi-isometric to  $X_N$ .

### 8.1 The first half of the classification

Assuming that  $M^a$  and  $N^b$  have the same absolute Jordan form, where  $a, b$  are positive integers, we must prove that  $\Gamma_M$  and  $\Gamma_N$  are quasi-isometric. We have shown above that  $\Gamma_{M^a}$  and  $\Gamma_M$  are quasi-isometric, and that  $\Gamma_{N^b}$  and  $\Gamma_N$  are quasi-isometric. Replacing  $M$  by  $M^a$  and  $N$  by  $N^b$ , we may therefore assume that  $M, N$  have the same absolute Jordan form. We shall prove that  $\Gamma_M, \Gamma_N$  are quasi-isometric by constructing a bilipschitz homeomorphism between  $X_M$  and  $X_N$ .

Since the absolute Jordan forms of  $M, N$  are equal it follows that  $\det M = \det N$ ; let  $d$  be the common value. Applying Proposition 4.1, there is a bilipschitz homeomorphism from  $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$  to  $G_N = \mathbf{R}^n \rtimes_N \mathbf{R}$  of the form  $(x, t) \mapsto (Ax, t)$  for some  $A \in \mathrm{GL}(n, \mathbf{R})$ . In the fiber product description of  $X_M, X_N$ , the trees  $T_M$  and  $T_N$  may both be identified with the homogeneous, oriented tree  $T_d$  with one incoming and  $d$  outgoing edges at each vertex. The bilipschitz homeomorphism  $G_M \rightarrow G_N$  and the identity homeomorphism  $T_d \rightarrow T_d$  both respect the height functions, and so these two homeomorphisms combine to give the desired bilipschitz homeomorphism  $X_M \rightarrow X_N$ .

### 8.2 Quasi-isometric implies integral powers have same absolute Jordan forms

Assuming  $\Gamma_M, \Gamma_N$  are quasi-isometric, there is a quasi-isometry  $f: X_M \rightarrow X_N$ . Combining Proposition 7.1 and Theorem 5.2 gives  $r \in \mathbf{R}_+$  such that  $M^r$  and  $N$  have the same absolute Jordan form. We must show that there exist  $a, b \in \mathbf{Z}_+$  so that  $M^a$  and  $N^b$  have the same absolute Jordan form.

Since  $M^r$  and  $N$  have the same absolute Jordan form, listing the absolute values of the eigenvalues of  $M$  and  $N$  in increasing order we obtain

$$\begin{aligned} \mu_{-a} \leq \cdots \leq \mu_0 \leq 1 < \mu_1 =: \alpha_M \leq \cdots \leq \mu_b \\ \nu_{-a} \leq \cdots \leq \nu_0 \leq 1 < \nu_1 =: \alpha_N \leq \cdots \leq \nu_b \end{aligned}$$

with  $\mu_i^r = \nu_i$ ,  $-a \leq i \leq b$ . From this it follows that

$$\frac{\log \alpha_N}{\log \alpha_M} = r = \frac{\log \det N}{\log \det M}$$

Let  $\mathbf{Q}_M$  denote the set of coherent hyperplanes in  $X_M$ , and let  $h_M$  denote the height function on  $M$ . We define a metric on  $\mathbf{Q}_M$  as follows: given coherent hyperplanes  $P_1, P_2$ , let  $L$  denote the horizontal leaf  $L = \partial(P_1 \cap P_2)$ . Then we set

$$d_{\mathbf{Q}_M}(P_1, P_2) = (\det M)^{-h_M(L)}$$

It is easy to check that this defines a metric on  $\mathbf{Q}_M$ , and since the tree  $T_M$  branches  $m = \det M$  times as  $h_M$  increases by 1, the metric space  $(\mathbf{Q}_M, d_{\mathbf{Q}_M})$  is isometric to the  $m$ -adic rational numbers in their usual metric of Hausdorff dimension 1. Similarly, attached to  $X_N$  is a metric space  $(\mathbf{Q}_N, d_{\mathbf{Q}_N})$  isometric to the  $n$ -adic rational numbers, with  $n = \det N$ .

From Step 3 in the proof of Proposition 7.1 (see §7.2), the quasi-isometry  $f : X_M \rightarrow X_N$  takes each coherent hyperplane in  $X_M$  to within a uniform Hausdorff distance of a unique coherent hyperplane in  $X_N$ , hence induces a bijection  $\psi : \mathbf{Q}_M \rightarrow \mathbf{Q}_N$ . For each  $\ell \in \mathbf{Q}_M$ , setting  $\ell' = \psi(\ell)$ , there is an induced horizontal-respecting quasi-isometry  $P_\ell \rightarrow P_{\ell'}$ , and by Time Rigidity (Proposition 5.8) this quasi-isometry has an induced time change of the form  $t \mapsto mt + b$  where

$$m = \frac{\log \alpha_M}{\log \alpha_N} = 1/r$$

and where  $b$  depends ostensibly on  $\ell$ . However, for another  $\ell_1, P_\ell$  and  $P_{\ell_1}$  coincide below some value of  $t$ , and so  $t \mapsto mt + b$  is an induced time change for both  $P_\ell \mapsto P_{\ell'}$  and  $P_{\ell_1} \mapsto P_{\ell'_1}$ , possibly with a larger coarseness constant (this argument is taken from Claim 6.3 on p. 436 of [FM98]). Therefore, there is a uniform induced time change  $t \mapsto mt + b$  with  $b$  independent of  $\ell$ , and with a uniform Hausdorff constant  $A$ .

We now claim that  $\psi$  is a bilipschitz homeomorphism. To this end, let  $P_1, P_2 \in \mathbf{Q}_M$  be given. Let  $L = \partial(P_1 \cap P_2)$  and let  $L' = \partial(\psi(P_1) \cap \psi(P_2))$ . Then

$$h_N(L') \geq m \cdot h_M(L) + b - A$$

Hence

$$\begin{aligned} \frac{d_{\mathbf{Q}_N}(\psi(P_1), \psi(P_2))}{d_{\mathbf{Q}_M}(P_1, P_2)} &= \frac{(\det N)^{-h_N(L')}}{(\det M)^{-h_M(L)}} \\ &\leq \frac{(\det N)^{-mh_M(L) - b + A}}{(\det M)^{-h_M(L)}} \\ &= \frac{((\det N)^{\log \det M / \log \det N})^{-h_M(L)} (\det N)^{-b + A}}{(\det M)^{-h_M(L)}} \\ &= (\det N)^{-b + A} \end{aligned}$$



which is a constant not depending on  $P_1$  or  $P_2$ . Hence  $\psi$  is Lipschitz. The same argument applied to  $\psi^{-1}$  shows that  $\psi$  is bilipschitz.

Applying Cooper's Theorem (appendix to [FM98], Corollary 10.11) on bilipschitz homeomorphisms of Cantor sets, we obtain that there exist integers  $a, b > 0$  such that  $(\det M)^a = (\det N)^b$ . Since  $M^r$  and  $N$  have the same absolute Jordan form, we have

$$\frac{b}{a} = \frac{\log \det M}{\log \det N} = r$$

and so  $(M^r)^a = M^b$  and  $N^a$  have the same absolute Jordan form.

## 9 Quasi-isometric rigidity

In this section we prove Theorem 1.2 in a series of steps. Recall the hypotheses:  $M$  is an integer matrix in  $\mathrm{GL}(n, \mathbf{R})$  with  $|\det M| > 1$ , and  $G$  is a finitely generated group quasi-isometric to  $\Gamma_M$ . By squaring  $M$  if necessary we may assume that  $M \in \mathrm{GL}_{\times}(n, \mathbf{R})$  and  $\det M > 1$ , and therefore  $\Gamma_M$  is quasi-isometric to  $X_M$ . It follows that  $G$  is quasi-isometric to  $X_M$ .

**Step 1.** The action of  $G$  on itself by left multiplication can be conjugated by the quasi-isometry  $G \rightarrow X_M$  to give a proper, cobounded quasi-action of  $G$  on  $X_M$  (see [FM99b], Proposition 2.1). Since  $\det M > 1$  we may apply Theorem 7.7, concluding that the quasi-action of  $G$  on  $X_M$  coarsely respects the fibers of the uniform metric fibration  $X_M \rightarrow T_M$ .

**Step 2.** Now we use the following result of [MSW]. Suppose  $\pi: X \rightarrow T$  is a uniform metric fibration over a bushy tree  $T$ . If  $G$  is a finitely presented group with a cobounded, proper quasi-action on  $X$ , and if the quasi-action coarsely respects the fibers, then  $G$  is the fundamental group of a graph of groups whose vertex and edge groups are quasi-isometric to a fiber  $X_t = \pi^{-1}(t)$ .

By Step 1, this result applies to the quasi-action of  $G$  on  $X_M$ , because  $G$  is quasi-isometric to the finitely presented group  $\Gamma_M$  and so  $G$  is finitely presented. The fibers of the map  $X_M \rightarrow T_M$  are isometric to  $\mathbf{R}^n$ , and it follows that  $G$  is the fundamental group of a graph of groups with each vertex and edge group quasi-isometric to  $\mathbf{R}^n$ .

**Step 3.** Any finitely-generated group quasi-isometric to  $\mathbf{R}^n$  is virtually  $\mathbf{Z}^n$  (see [Ger95]), and so  $G$  is the fundamental group of a graph of groups whose vertex and edge groups are virtually  $\mathbf{Z}^n$ .

**Step 4.** Applying the argument in Section 5 of [FM99b] to  $G$  gives that either  $G$  contains a noncyclic free group or  $G$  is an ascending HNN extension of the form

$$G = A_\phi = \langle A, t \mid tat^{-1} = \phi(a), \forall a \in A \rangle$$

where  $A$  is virtually  $\mathbf{Z}^n$  and  $\phi: A \rightarrow A$  is an injective endomorphism. Since  $\Gamma_M$  is amenable, and since  $G$  is quasi-isometric to  $\Gamma_M$ , then  $G$  is amenable, and so  $G$  cannot contain a noncyclic free group. The second possibility must therefore occur:  $G = A_\phi$  as above.

**Step 5.** Now we turn to an analysis of injective endomorphisms of virtually abelian groups. Suppose  $A$  is a finitely generated, virtually abelian group. Any injective endomorphism of  $A$  has finite index image.

A subgroup  $B \subset A$  is *characteristic for endomorphisms* if, for any injective endomorphism  $\phi: A \rightarrow A$ , we have  $\phi(B) \subset B$ .

Given a group  $A$  and  $g \in A$ , the centralizer of  $g$  in  $A$  is denoted  $C_A(g)$ . The *virtual center* of  $A$ , denoted  $V(A)$ , is the set of all  $g \in A$  such that  $[A : C_A(g)] < \infty$ . This is a subgroup, because if  $g, h \in V(A)$  then the subgroup  $C_A(gh)$ , which contains  $C_A(g) \cap C_A(h)$ , has finite index.

**Lemma 9.1 (Some characteristic subgroups).** *Let  $A$  be a finitely generated, virtually abelian group. Then the virtual center  $V(A)$ , its center  $ZV(A)$ , and its torsion subgroup  $TVZ(A)$ , are all characteristic for endomorphisms of  $A$ . Moreover,  $V(A)$  and  $ZV(A)$  both have finite index in  $A$ , whereas  $TVZ(A)$  is finite.*

Lemma 9.1 is proved below.

**Step 6.** Consider the HNN extension  $G = A_\phi$  above. Let  $V(A)$ ,  $ZV(A)$ ,  $TVZ(A)$  be as in Lemma 9.1, so all these subgroups are taken into themselves by  $\phi$ . Since  $TVZ(A)$  is finite we in fact have  $\phi(TVZ(A)) = TVZ(A)$ , and so  $K = TVZ(A)$  is a finite, normal subgroup of  $G$ .

Replacing  $G$  by  $G/K$ , we may assume that  $TVZ(A)$  is trivial, and it follows that  $ZV(A)$  is torsion-free abelian, and so is isomorphic to  $\mathbf{Z}^n$ . Since  $\phi(ZV(A)) \subset ZV(A)$ , the action of  $\phi$  on  $ZV(A)$  is given by some  $n \times n$  matrix of integers  $N$ . Thus,  $G/K$  has a finite-index subgroup isomorphic to  $\Gamma_N$ , finishing the proof of Theorem 1.2.

**Proof of Lemma 9.1.** To see  $[A : V(A)] < \infty$ , if  $B$  is any finite-index abelian subgroup of  $A$  then obviously  $B \subset V(A)$ .

Consider an endomorphism  $\phi: A \rightarrow A$ . We now show that  $\phi(V(A)) \subset V(A)$ . Consider  $g \in V(A)$ , so  $[A : C_A(g)] < \infty$ . It follows that  $[\phi(A) : C_{\phi(A)}(\phi(g))] < \infty$ , and so  $[A : C_{\phi(A)}(\phi(g))] < \infty$ . But  $C_{\phi(A)}(\phi(g)) \subset C_A(\phi(g))$ , and so  $\phi(g) \in V(A)$ .

Next we claim that  $V(V(A)) = V(A)$ . To see why, if  $g \in V(A)$  then  $[A : C_G(g)] < \infty$ , and so  $[V(A) : C_G(g) \cap V(A)] < \infty$ . But  $C_G(g) \cap V(A) \subset C_{V(A)}(g)$ , and so  $[V(A) : C_{V(A)}(g)] < \infty$ , i.e.  $g \in V(V(A))$ .

Next we claim that  $[V(A) : ZV(A)] < \infty$ . In fact if  $V$  is any finitely generated group which is its own virtual center, then  $[V : ZV] < \infty$  (the converse is also true, trivially). To see why, let  $g_1, \dots, g_k$  be a generating set for  $V$ . Since  $V(V) = V$ , each of the groups  $C_V(g_1), \dots, C_V(g_k)$  has finite index in  $V$ . It follows that their intersection has finite index in  $V$ ; but their intersection is precisely  $ZV$ .

Now we claim that  $ZV(A)$  is characteristic for endomorphisms of  $V(A)$  (and so is also characteristic for endomorphisms of  $A$ ). In fact, if  $V$  is any finitely generated group whose center  $ZV$  has finite index, then  $ZV$  is characteristic for any injective endomorphism  $\phi: V \rightarrow V$  whose image has finite index. To see why, we have  $Z(\phi(V)) = \phi(ZV)$ , and so

$$[\phi(V) : Z(\phi(V))] = [\phi(V) : \phi(ZV)] = [V : ZV] < \infty$$

Clearly  $\phi(V) \cap ZV \subset Z(\phi(V))$ , and so

$$[\phi(V) : Z(\phi(V))] \leq [\phi(V) : \phi(V) \cap ZV]$$

The quotient group  $V/ZV$  is finite, and the quotient homomorphism  $V \rightarrow V/ZV$ , when restricted to the subgroup  $\phi(V)$ , has kernel  $\phi(V) \cap ZV$ . It follows that

$$[\phi(V) : \phi(V) \cap ZV] \leq |V/ZV| = [V : ZV] = [\phi(V) : Z(\phi(V))]$$

All of the above inequalities are therefore equalities, and so

$$\phi(ZV) = Z(\phi(V)) = \phi(V) \cap ZV$$

which implies  $\phi(ZV) \subset ZV$ .

Finally, it is clear that for any finitely generated abelian group, the torsion subgroup is characteristic for injective endomorphisms.

## 10 Questions

### 10.1 Remarks on the polycyclic case

Given an integer matrix  $M \in \mathrm{GL}(n, \mathbf{R})$ , the group  $\Gamma_M$  is polycyclic if and only if  $|\det M| = 1$ , and if  $M \in \mathrm{GL}_{\times}(n, \mathbf{R})$  this occurs if and only if  $\Gamma_M$  is a cocompact *discrete* subgroup of  $G_M$ . In this case it follows that  $\Gamma_M$  is quasi-isometric to  $G_M$ , and the notion of horizontal-respecting quasi-isometry clearly transfers to  $\Gamma_M$ . The techniques of this paper do not provide a quasi-isometric classification in this case, however they do yield the following partial result:

**Theorem 10.1.** *Given  $M, N \in \mathrm{SL}(n, \mathbf{Z})$  lying on 1-parameter subgroups of  $\mathrm{GL}(n, \mathbf{R})$ , there is a horizontal respecting quasi-isometry  $\Gamma_M \rightarrow \Gamma_N$  if and only if there is a horizontal respecting quasi-isometry  $G_M \rightarrow G_N$ , and this occurs if and only if there are real numbers  $a, b \neq 0$  such that  $M^a, N^b$  have the same absolute Jordan form.  $\diamond$*

This raises the question: Is every quasi-isometry  $\Gamma_M \rightarrow \Gamma_N$  horizontal respecting? Equivalently, is every quasi-isometry  $G_M \rightarrow G_N$  horizontal respecting? The answer is obviously no, for example when  $M, N$  are identity matrices and  $G_M, G_N$  are Euclidean spaces. However, we conjecture:

**Conjecture 10.2.** *If  $M, N \in \mathrm{SL}(n, \mathbf{Z})$  lie on 1-parameter subgroups of  $\mathrm{GL}(n, \mathbf{R})$ , and if  $M, N$  have no eigenvalues on the unit circle, then any quasi-isometry  $G_M \rightarrow G_N$  is horizontal respecting.*

Moreover, Theorem 10.1 and Conjecture 10.2 together would imply the following (see [FM99c]):

**Conjecture 10.3.** *Suppose  $M \in \mathrm{SL}(n, \mathbf{Z})$  has no eigenvalues on the unit circle. If  $G$  is any finitely generated group quasi-isometric to  $\Gamma_M$ , then there is a finite normal subgroup  $F$  of  $G$  so that  $G/F$  is abstractly commensurable to  $\Gamma_N$ , for some  $N \in \mathrm{SL}(n, \mathbf{Z})$  with no eigenvalues on the unit circle.*

### 10.2 The quasi-isometry group of $\Gamma_M$

Given a finitely generated group  $G$ , the set of quasi-isometries from  $G$  to itself, modulo the identification of quasi-isometries which differ by a bounded amount, forms a group called the *quasi-isometry group* of  $G$ , denoted  $\mathrm{QI}(G)$ . Given a  $1 \times 1$  matrix  $M = (m)$  with  $m \geq 2$ , the quasi-isometry group of the solvable Baumslag-Solitar group  $\Gamma_M \approx \mathrm{BS}(1, m)$  was computed in [FM98]:

$$\mathrm{QI}(\mathrm{BS}(1, m)) \approx \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_m)$$

where  $\mathbf{Q}_m$  is the metric space of  $m$ -adic rational numbers, and  $\text{Bilip}(X)$  denotes the group of bilipschitz self maps of a metric space  $X$ .

**Problem 10.4.** *Compute the quasi-isometry group of  $\Gamma_M$  in general.*

The strongest result we have on this problem so far is Proposition 6.3, but see the remarks after that proposition.

In [FM99b] the computation of  $\text{QI}(\text{BS}(1, m))$  was applied to prove quasi-isometric rigidity of  $\text{BS}(1, m)$ , using techniques of Hinkkanen [Hin85] and Tukia [Tuk86]. While quasi-isometric rigidity of  $\text{BS}(1, m)$  now has a completely different proof [MSW], which we have here generalized to  $\Gamma_M$ , one might still pursue:

**Problem 10.5.** *Give a proof of quasi-isometric rigidity of  $\Gamma_M$ , generalizing the results of [FM99b].*

This should lead to a deeper understanding of the geometry of  $\Gamma_M$ . For example, Tukia [Tuk86] characterizes subgroups of the quasiconformal group of a sphere which are conjugate into the Möbius group. We have analogous results for lattices in three-dimensional SOLV-geometry [FM99c], and there should be generalizations to solvable Baumslag-Solitar groups and to  $\Gamma_M$ .

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