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Quasi-isometric rigidity for the solvable Baumslag-Solitar groups, II.

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Introduction

Gromov's Polynomial Growth Theorem [Gro81] characterizes the class of virtually nilpotent groups by their asymptotic geometry. Since Gromov's theorem it has been a major open question (see, e.g. [GH91]) to find an appropriate generalization for solvable groups. This paper gives the first step in that direction.

One fundamental class of examples of finitely-generated solvable groups which are not virtually nilpotent are the *solvable Baumslag-Solitar groups*

$$\mathrm{BS}(1,n) = \langle a, b \mid bab^{-1} = a^n \rangle$$

where $n \geq 2$. Our main theorem characterizes the group $\mathrm{BS}(1,n)$ among all finitely-generated groups by its asymptotic geometry.

Theorem A (Quasi-isometric rigidity). *Let G be any finitely generated group. If G is quasi-isometric to $\mathrm{BS}(1,n)$ for some $n \geq 2$, then there is a short exact sequence*

$$1 \to N \to G \to \Gamma \to 1$$

where N is finite and Γ is abstractly commensurable to $\mathrm{BS}(1,n)$.

In fact we will describe the precise class of quotient groups Γ which can arise, and will classify all torsion-free G; see section 5 in the outline below.

Theorem A complements the main theorem of [FM97], where it is shown that $\mathrm{BS}(1,n)$ is quasi-isometric to $\mathrm{BS}(1,m)$ if and only if they are abstractly

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commensurable, which happens if and only if m, n are positive integer powers of the same positive integer.

Theorem A says that every finitely generated group quasi-isometric to $\mathrm{BS}(1,n)$ can be obtained from $\mathrm{BS}(1,n)$ by first passing to some abstractly commensurable group and then to some finite extension. We describe this phenomenon by saying that the group $\mathrm{BS}(1,n)$ is *quasi-isometrically rigid.* This property is even stronger than what we know for nilpotent groups, for while Gromov's theorem says that the class of nilpotent groups is a quasi-isometrically rigid class, outside of a few low-dimensional cases it is not known whether an individual nilpotent group must always be quasi-isometrically rigid.

Comparison with lattices Recent work on lattices in semisimple Lie groups has established the quasi-isometric classification of all such lattices.

In the case of a nonuniform lattice Λ in a semisimple Lie group $G \neq \mathrm{SL}(2, \mathbf{R})$, quasi-isometric rigidity of Λ follows from the deep fact that the quasi-isometry group $\mathrm{QI}(\Lambda)$ is the commensurator group of Λ in G, a countable group (see [Sch96b], [Sch96a], [FS96], [Esk96], or [Far96] for a survey).

In contrast, for uniform lattices Λ in the isometry group of $X = \mathbf{H}^n$ or $\mathbf{CH}^n$, the quasi-isometry group $\mathrm{QI}(\Lambda) \approx \mathrm{QI}(X)$ is $\mathrm{QC}(\partial X)$, the (Heisenberg) quasiconformal group of the sphere at infinity ∂X, an infinite dimensional group. In this situation it is the whole collection of lattices in $\mathrm{Isom}(X)$ which is quasi-isometrically rigid, not any individual lattice [Tuk86], [CC92], [Cho96].

For $\mathrm{BS}(1,n)$ something interesting happens. It exhibits both types of contrasting behavior just described: the group $\mathrm{BS}(1,n)$ is quasi-isometrically rigid *and yet* its quasi-isometry group is infinite-dimensional, as follows. Let $\mathbf{Q}_n$ be the metric space of "n-adic rational numbers". Let $\mathrm{Bilip}(X)$ denote the group of bilipschitz homeomorphisms of the metric space X. In [FM97], Theorem 8.1 we showed:

$$\mathrm{QI}(\mathrm{BS}(1,n)) \approx \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$$

In proving Theorem A, this formula plays a role similar to that played by the formula $\mathrm{QI}(X) \approx \mathrm{QC}(\partial X)$ when $X = \mathbf{H}^n$ or $\mathbf{CH}^n$.

Outline of the paper

As we shall see, the "mixed behavior" of $\mathrm{BS}(1,n)$ allows for some analogies with proof techniques developed in the case of lattices. Some fundamentally new phenomena occur, however, and these require new methods. We point out in particular:

- The notion of quasisimilarity (a delicate though crucial variant of quasisymmetric map) and the corresponding notion of dilatation.
- The theory of *biconvergence groups.*
- A method which applies these dynamical properties to proving quasi-isometric rigidity.

Section 1: Geometry and boundaries for $\mathbf{BS(1,n)}$. We review some results of [FM97]. We construct a metric 2-complex X_n on which $\mathrm{BS}(1,n)$ acts properly discontinuously and cocompactly by isometries, and we equip this complex with a boundary which is formed of two disjoint pieces: an *upper boundary* $\partial^u X_n \approx \mathbf{Q}_n$, which is the space of hyperbolic planes in X_n, and a *lower boundary* $\partial_\ell X_n \approx \mathbf{R}$. We also review the fact that a quasi-isometry of X_n induces bilipschitz homeomorphisms of the upper and lower boundaries (Proposition 1.2), giving the isomorphism

$$\mathrm{QI}(\mathrm{BS}(1,n)) \approx \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$$

Section 2: Representation into the quasi-isometry group. We recall a principle of Cannon and Cooper, that if a group G is quasi-isometric to a proper geodesic metric space X, then there is a "quasi-action" of G on X, and an induced representation $G \to \mathrm{QI}(X)$. Combining this with the results of §1 we obtain the fact that if G is quasi-isometric to $\mathrm{BS}(1,n)$ then there is an induced action $\rho = \rho_\ell \times \rho^u \colon G \to \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$.

Important point: At this point one might try to make an analogy with [Tuk86] (see also [CC92]), and attempt to prove that a "uniformly quasi-conformal" subgroup of $\mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$ is conjugate into some kind of "conformal" subgroup of $\mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$. However, there are serious difficulties with this approach (see comments in §6).

Section 3: Uniform quasisimilarity actions on R. Instead we consider the projected representation $\rho_l \colon G \to \mathrm{Bilip}(\mathbf{R})$. We show in in Proposition 2.2 that $\rho_l(G)$ is a group of *quasisimilarities* which is uniform with respect to a certain dilatation.

Inspired by Hinkkanen's Theorem [Hin85], we prove in Theorem 3.2 that a uniform group of quasisimilarities of the real line is bilipschitz conjugate to an affine group (Theorem 3.2). Applying this to the representation $\rho_\ell \colon G \to \mathrm{Bilip}(\mathbf{R})$ we obtain a bilipschitz conjugate representation $\theta \colon G \to \mathrm{Aff}(\mathbf{R})$. It is crucial that this conjugacy is bilipschitz as opposed to just quasisymmetric. This is why we cannot use Hinkkanen's original theorem.

Important point: For an arbitrary group quasi-isometric to a lattice in a semisimple Lie group, finding a representation into that Lie group usually finishes the proof of quasi-isometric rigidity. In the present case more work is required. One reason is that neither G nor even $\mathrm{BS}(1,n)$ is a lattice in $\mathrm{Aff}(\mathbf{R})$, although $\mathrm{BS}(1,n)$ is a nondiscrete subgroup of $\mathrm{Aff}(\mathbf{R})$. In fact $\mathrm{Aff}(\mathbf{R})$ is a nonunimodular Lie group and so does not admit any lattice. Hence we must find another way to prove that θ has finite kernel, and to analyze the image group $\theta(G)$.

Section 4: Biconvergence groups. We study the action of G on the boundary pair $(\partial_\ell X_n, \partial^u X_n)$. Exploiting analogies with convergence groups, the dynamical behavior of this pair of actions is encoded in what we call a *biconvergence group*. Using this boundary dynamics, we show that the representation $\theta \colon G \to \mathrm{Aff}(\mathbf{R})$ is virtually faithful (Proposition 4.4), and that the group of affine stretch factors, or *stretch group* of $\theta(G) \subset \mathrm{Aff}(\mathbf{R})$ is infinite cyclic (Proposition 4.5). The proof of Proposition 4.5 makes vital use of a *bilipschitz* conjugacy between ρ_ℓ and θ.

Section 5: Finishing the proof of Theorem A. We apply combinatorial group theory and quasi-isometry invariants, together with the results of §4, to identify the image group $\Gamma = \theta(G) \subset \mathrm{Aff}(\mathbf{R})$. We show that Γ is the mapping torus of some injective, nonsurjective endomorphism $\phi \colon A \to A$ where A is either the infinite cyclic group or the infinite dihedral group—that is, Γ has the presentation $\langle A, t \mid tat^{-1} = \phi(t), \forall a \in A \rangle$. In particular Γ contains a subgroup of index ≤ 2 isomorphic to $\mathrm{BS}(1,m)$ for some integer $m \geq 2$. Applying [FM97] it follows that $\mathrm{BS}(1,m)$ is abstractly commensurable to $\mathrm{BS}(1,n)$, and so the same is true of Γ. We also prove in Corollary 5.3 that if G is torsion free then G is isomorphic to $\mathrm{BS}(1,k)$ for some integer k with $|k| \geq 2$.

Section 6: Final comments. We discuss the possibility of strengthening Theorem A.

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1 Geometry and boundaries for $\mathrm{BS}(\mathbf{1}, \mathbf{n})$

In this section we briefly review the material from [FM97] which we will need.

1.1 Quasi-isometries

A (K, C)*quasi-isometry* between metric spaces is a map $f\colon X \to Y$ such that, for some constants $K, C > 0$:

- For all $x_1, x_2 \in X$ we have
$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$
- The C-neighborhood of $f(X)$ is all of Y.

A *coarse inverse* of a quasi-isometry $f\colon X \to Y$ is a quasi-isometry $g\colon Y \to X$ such that, for some constant $C' > 0$, we have $d(g \circ f(x), x) < C'$ and $d(f \circ g(y), y) < C'$ for all $x \in X$ and $y \in Y$. Every (K, C) quasi-isometry $f\colon X \to Y$ has a coarse inverse $g\colon Y \to X$, namely, for each $y \in Y$ define $g(y)$ to be any point $x \in X$ such that $d(f(x), y) \leq C$.

A fundamental observation due to Effremovich-Milnor-Švarc states that if a group G acts properly discontinuously and cocompactly by isometries on a proper geodesic metric space X, then G is finitely generated, and X is quasi-isometric to G equipped with the word metric. Two finitely generated groups G_1, G_2 are said to be *(abstractly) commensurable* if there are finite index subgroups $H_i < G_i$ such that H_1, H_2 are isomorphic to each other. It is easy to check that abstractly commensurable groups are quasi-isometric, and that if G is finitely generated and N is a finite normal subgroup then G is quasi-isometric to G/N.

Let $\mathrm{QIMap}(X)$ be the set of all quasi-isometries $f\colon X \to X$, equipped with the binary operation of composition. Given $f, g \in \mathrm{QIMap}(X)$ and

$C \geq 0$, we write $f \sim_C g$ if $d(f(x), g(x)) < C$ for all $x \in X$. We write $f \sim g$ if there exists $C \geq 0$ such that $f \sim_C g$; this is an equivalence relation on $\mathrm{QIMap}(X)$, known as *Hausdorff equivalence* or *coarse equivalence.* The set of equivalence classes is denoted $\mathrm{QI}(X)$. The operation of composition respects Hausdorff equivalence, in the sense that if $f_1 \sim f_2$ and $g_1 \sim g_2$ then $f_1 \circ g_1 \sim f_2 \circ g_2$. Composition therefore descends to a well-defined binary operation on $\mathrm{QI}(X)$. With respect to this operation, $\mathrm{QI}(X)$ is a group, whose identity element is the Hausdorff equivalence class of the identity map on X. Inverses exist in $\mathrm{QI}(X)$ because of the fact that every quasi-isometry has a coarse inverse.

1.2 The 2-complex X_n

Throughout this paper we use the upper half plane model of the hyperbolic plane

$$\mathbf{H}^2 = \{(x,y) \in \mathbf{R}^2 \bigm| y > 0\}$$

with metric $ds^2 = (dx^2 + dy^2)/y^2$. Also, let T_n denote the unique homogeneous, directed tree such that each vertex has one incoming directed edge and n outgoing directed edges, with the geodesic metric that makes each edge of T_n isometric to the interval $[0, \log(n)]$.

In [FM97] we constructed a metric 2-complex X_n on which $\mathrm{BS}(1,n)$ acts properly discontinuously and cocompactly by isometries. The 2-complex X_n is homeomorphic to $T_n \times \mathbf{R}$. There is a geodesic metric on X_n with the following properties:

- For each directed edge $E \subset T_n$, the subset of X_n corresponding to $E \times \mathbf{R}$ is isometric to the "horostrip of height $\log(n)$", namely

$$\{(x,y) \in \mathbf{H}^2 \bigm| 1 \leq y \leq n\}$$

- For each bi-infinite directed line $L \subset T_n$, the subset of X_n corresponding to $L \times \mathbf{R}$ is isometric to $\mathbf{H}^2$.

- For any two bi-infinite directed lines $L \neq L' \subset T_n$, the subset of X_n corresponding to $(L \cap L') \times \mathbf{R}$ is isometric to a "horodisc exterior" $\{(x,y) \in \mathbf{H}^2 \bigm| y \leq 1\}$.

A cocompact, properly discontinuous, free action of $\mathrm{BS}(1,n)$ on X_n is described in [FM97] by exhibiting X_n as the universal cover of a certain metric 2-complex C_n whose fundamental group is $\mathrm{BS}(1,n)$.

Various important features of the complex X_n, and the action of $\mathrm{BS}(1,n)$, are summarized in the following proposition. Given a continuous map between metric spaces $f\colon A \to B$, an *isometric section* is a subset $B' \subset A$ such that $f \mid B'$ is an isometry onto B. The isometric sections of the *height function* $h\colon \mathbf{H}^2 \to \mathbf{R}$, defined by $h(x,y) = \log(y)$, are precisely the *vertical lines* in $\mathbf{H}^2$, lines of the form $x = (\text{constant})$. Also define a height function $h\colon T_n \to \mathbf{R}$, by requiring $h(v_0) = 0$ for some chosen base vertex $v_0 \in T_n$, and requiring that h takes each edge of T_n onto some interval of length $\log(n)$ in $\mathbf{R}$ by an orientation preserving isometry. The isometric sections of $h\colon T_n \to \mathbf{R}$ are precisely the bi-infinite directed lines in T_n (called "coherent lines" in [FM97]).

Proposition 1.1. *There exist actions of* $\mathrm{BS}(1,n)$ *on* X_n, $\mathbf{H}^2$, T_n, *and* $\mathbf{R}$, *and equivariant maps between these spaces as summarized in the following commutative diagram:*

$$\begin{tikzcd}
& X_n \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "h"] & \\
\mathbf{H}^2 \arrow[dr, "h"'] & & T_n \arrow[dl, "h"] \\
& \mathbf{R} &
\end{tikzcd}$$

Moreover:

- *The action on X_n is properly discontinuous, cocompact, and free.*
- *The function $h\colon X_n \to \mathbf{R}$ is the fiber product of the height functions $h\colon \mathbf{H}^2 \to \mathbf{R}$ and $h\colon T_n \to \mathbf{R}$. That is, the map $X_n \to \mathbf{H}^2 \times T_n$ taking x to $(p(x), q(x))$ is an equivariant homeomorphism onto the subset of $\mathbf{H}^2 \times T_n$ consisting of all pairs (a,b) such that $h(a) = h(b)$.*
- *The map $q\colon X_n \to T_n$ induces a 1–1 correspondence between vertical lines in T_n and isometric sections of the map $p\colon X_n \to \mathbf{H}^2$.*
- *The map $p\colon X_n \to \mathbf{H}^2$ induces a 1–1 correspondence between vertical lines in $\mathbf{H}^2$ and isometric sections of the map $q\colon X_n \to T_n$.*

Isometric sections of $p\colon X_n \to \mathbf{H}^2$ are called *hyperbolic planes* in X_n, and isometric sections of $q\colon X_n \to T_n$ are called *trees in* X_n. Since $\mathrm{BS}(1,n)$ clearly acts on the set of vertical lines in T_n, it follows that $\mathrm{BS}(1,n)$ acts on the set of hyperbolic planes in X_n. Similarly, $\mathrm{BS}(1,n)$ acts on the set of trees in X_n.

The proof of the above proposition can be gleaned from the information in [FM97]. Here is an alternative proof, which gives an interesting new construction of X_n.

Proof. Recall the presentation $\mathrm{BS}(1,n) = \langle a, b \mid bab^{-1} = a^n \rangle$. Define the *height action* of $\mathrm{BS}(1,n)$ on $\mathbf{R}$ by

$$a \cdot t = t, \qquad b \cdot t = t + \log(n)$$

To describe the actions of $\mathrm{BS}(1,n)$ on $\mathbf{H}^2$ and T_n, we first define *affine actions* on $\mathbf{R}$ and on the n-adic rational numbers $\mathbf{Q}_n$, and then we describe how these induce the desired actions on $\mathbf{H}^2$ and T_n.

If R is a ring with unit, let $\mathrm{Aff}(R)$ be the group of all matrices of the form $\left(\begin{smallmatrix} r & s \\ 0 & 1 \end{smallmatrix}\right)$, where $r, s \in R$ and r is invertible; the group law is ordinary matrix multiplication. The group $\mathrm{Aff}(R)$ acts on R by fractional linear transformations: $\left(\begin{smallmatrix} r & s \\ 0 & 1 \end{smallmatrix}\right) \cdot x = rx + s$ for all $x \in R$. Note that if R is a commutative ring then $\mathrm{Aff}(R)$ is a solvable group.

If the integer n is invertible in the ring R, there is a representation $\mathrm{BS}(1,n) \mapsto \mathrm{Aff}(R)$ defined by

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad b \mapsto \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

$$a \cdot r = r + 1 \qquad\qquad b \cdot r = nr$$

for all $r \in R$. We call this the *affine action* of $\mathrm{BS}(1,n)$ on the ring R.

As a special case we obtain an affine action of $\mathrm{BS}(1,n)$ on the real numbers. The action of $\mathrm{Aff}(\mathbf{R})$ on $\mathbf{R}$ extends to an isometric action on $\mathbf{H}^2$, and by composition we obtain the desired action of $\mathrm{BS}(1,n)$ on $\mathbf{H}^2$.

For another special case, let $\mathbf{Q}_n$ be the ring of n-adic rational numbers. This is the ring of all formal series $\sum_{i \in \mathbf{Z}} \zeta_i n^i$ with $\zeta_i \in \{0, \ldots, n-1\}$ for all i, and $\zeta_i = 0$ for i sufficiently close to $-\infty$, with the obvious addition and multiplication. We write the series $\sum_{i \in \mathbf{Z}} \zeta_i n^i$ more succinctly as $(\zeta_i)_{i \in \mathbf{Z}}$ or just (ζ_i) when $i \in \mathbf{Z}$ is understood. The ring of n-adic integers $\mathbf{Z}_n$ is the subring of all $(\zeta_i) \in \mathbf{Q}_n$ with $\zeta_i = 0$ for all $i < 0$. Note that the integer n is invertible in $\mathbf{Q}_n$, and so the affine action of $\mathrm{BS}(1,n)$ on $\mathbf{Q}_n$ is defined.

The metric on $\mathbf{Q}_n$ is the usual *n-adic metric*, where the distance between (ζ_i) and (ζ'_i) in $\mathbf{Q}_n$ is n^{-k}, where $k \in \mathbf{Z}$ is the maximum integer such that $\zeta_i = \zeta'_i$ for all $i \leq k$.

The tree T_n may be identified with the Bruhat-Tits building of $\mathbf{Q}_n$, and so the action of $\mathrm{BS}(1,n)$ on $\mathbf{Q}_n$ induces an action on T_n. To be explicit, for each truncated series $\eta = (\eta_i)_{i\le k}$, where $\eta_i \in \{0,\dots,n-1\}$ for $i \le k$, and $\eta_i = 0$ for i sufficiently close to $-\infty$, define a set

$$C_\eta = \{(\zeta_i) \in \mathbf{Q}_n \mid \zeta_i = \eta_i,\ \forall i \le k\}$$

The set C_η is called a *clone of* $\mathbf{Z}_n$ *in* $\mathbf{Q}_n$. The integer k is called the *combinatorial height* of the clone C_η, denoted $h_c(C_\eta)$. The inclusion lattice on clones defines the directed tree T_n as follows. The vertices of T_n are the clones of $\mathbf{Z}_n$ in $\mathbf{Q}_n$. There is a directed edge $C_\eta \to C_{\eta'}$ if and only if the following hold:

- $C_\eta \supset C_{\eta'}$.
- For all $C_{\eta''}$, if $C_\eta \supset C_{\eta''} \supset C_{\eta'}$ then $C_{\eta''} = C_\eta$ or $C_{\eta''} = C_{\eta'}$.

Note that $h_c(C_{\eta'}) = h_c(C_\eta) + 1$. Note also that T_n *is* a tree, because any two clones are either disjoint or one contains the other, and it is easy to check that each vertex has one incoming and n outgoing edges.

The action of $\mathrm{Aff}(\mathbf{Q}_n)$ on $\mathbf{Q}_n$ takes clones to clones, preserving inclusion. The affine action of $\mathrm{BS}(1,n)$ on $\mathbf{Q}_n$ therefore induces a direction preserving action of $\mathrm{BS}(1,n)$ on the tree T_n.

Now we may define X_n, and the height function $h\colon X_n \to \mathbf{R}$, by applying the fiber product construction to the height functions $h\colon \mathbf{H}^2 \to \mathbf{R}$ and $h\colon T_n \to \mathbf{R}$. Since the latter two height functions are $\mathrm{BS}(1,n)$ equivariant, we obtain an action of $\mathrm{BS}(1,n)$ on X_n so that $h\colon X_n \to \mathbf{R}$ is equivariant.

To see that the action on X_n is properly discontinuous, note that the general element of $\mathrm{BS}(1,n)$ is a matrix of the form $\left(\begin{smallmatrix} n^i & k/n^j \\ 0 & 1 \end{smallmatrix}\right)$, and if a 1–1 sequence of such matrices is bounded in $\mathrm{Aff}(\mathbf{R})$ then it is unbounded in $\mathrm{Aff}(\mathbf{Q}_n)$. The rest of the proof follow easily. $\diamondsuit$

Remarks

1. This construction of X_n is equivalent to the construction given in [FM97].

2. The action of $\mathrm{BS}(1,n)$ on T_n is isomorphic to the action on the Bass-Serre tree of the HNN decomposition $\mathrm{BS}(1,n) \approx \mathbf{Z}*_\phi$ where $\phi\colon \mathbf{Z} \to \mathbf{Z}$ is given by $\phi(k) = nk$.

3. The action of $\mathrm{BS}(1,n)$ on $\mathbf{H}^2$ is a "laminable action" as described in [Mos96]. The action on T_n is also laminable: the decomposition of X_n into trees gives a T_n-lamination which resolves the indiscreteness of the action of $\mathrm{BS}(1,n)$ on T_n.

4. Let $\mathbf{Z}[1/n]$ be the ring of fractions obtained from $\mathbf{Z}$ by inverting n. Note that the affine representation $\mathrm{BS}(1,n) \to \mathrm{Aff}(\mathbf{Z}[1/n])$ is a group isomorphism. The natural inclusions from $\mathrm{Aff}(\mathbf{Z}[1/n])$ into $\mathrm{Aff}(\mathbf{R})$ and $\mathrm{Aff}(\mathbf{Q}_n)$ then give the affine representations $\mathrm{BS}(1,n) \to \mathrm{Aff}(\mathbf{R})$ and $\mathrm{BS}(1,n) \to \mathrm{Aff}(\mathbf{Q}_n)$.

1.3 The upper and lower boundaries of X_n

The boundary of the hyperbolic plane in the upper half plane model may be written as $\partial\mathbf{H}^2 = \mathbf{R} \cup \{+\infty\}$. Using the affine action of $\mathrm{BS}(1,n)$ on $\mathbf{R}$, Proposition 1.1 gives $\mathrm{BS}(1,n)$-equivariant bijections

$$\{\text{trees in } X_n\} \approx \{\text{vertical lines in } \mathbf{H}^2\} \approx \mathbf{R}$$

The *lower boundary* $\partial_\ell X_n$ is defined to be any of these objects. Note that any two hyperbolic planes $Q, Q' \subset X_n$ intersect in a common horodisc exterior, and so ∂Q and $\partial Q'$ share a line at infinity which may be identified with $\partial_\ell X_n$. This was how $\partial_\ell X_n$ was defined in [FM97].

As a dual picture we have the *upper boundary* $\partial^u X_n$, defined to be the set of isometrically embedded hyperbolic planes in X_n. The space of ends of the directed tree T_n can be written as $\mathrm{Ends}(T_n) = \mathbf{Q}_n \cup \{-\infty\}$ where $\mathbf{Q}_n$ is naturally identified with the set of positively asymptotic ends and $-\infty$ is the unique negatively asymptotic end. By Proposition 1.1 we have $\mathrm{BS}(1,n)$-equivariant bijections

$$\{\text{hyperbolic planes in } X_n\} \approx \{\text{vertical lines in } T_n\} \approx \mathbf{Q}_n$$

and so $\partial^u X_n$ may be identified with any of these.

Remark. The set $\overline{X}_n = X_n \cup \partial_\ell X_n \cup \partial^u X_n$ may be given a topology so that X_n is dense and so that the action of $\mathrm{BS}(1,n)$ on X_n extends to a continuous action on $\overline{X}_n$. However $\overline{X}_n$ is not compact, nor even locally compact—compare this with the compactifications of X_n described in [Bes95].

1.4 Metrics on $\partial_\ell X_n$ and $\partial^u X_n$

The metric on $\partial_\ell X_n = \mathbf{R}$ is the usual metric, and the metric on $\partial^u X_n = \mathbf{Q}_n$ is the n-adic metric discussed above.

There are a few other ways of visualizing this metric. If $(\zeta_i), (\zeta_i') \in \mathbf{Q}_n$ correspond to hyperbolic planes $Q, Q' \subset X_n$, and if S is the common horodisc exterior $Q \cap Q'$, then $d(Q,Q') = e^{-h(\partial S)}$. Equivalently, if L, L' are the vertical lines in T_n corresponding to ζ, ζ', and if $v \in T_n$ is the finite endpoint of the ray $L \cap L'$, then $d(L, L') = e^{-h(v)}$.

Note that $\partial^u X_n$ is a proper metric space, that is, closed balls are compact. The Hausdorff dimension of $\mathbf{Q}_n$ equals 1. The metric d is an *ultrametric*, also called a *nonarchimedean metric*, in other words a metric satisfying $d(x,z) \leq \mathrm{Max}\{d(x,y), d(y,z)\}$ for any $x, y, z \in \partial^u X_n$. Distance in $\mathbf{Q}_n$ takes values in the discrete set $\{n^k \bigm| k \in \mathbf{Z}\}$. For any $\zeta \in \mathbf{Q}_n$ and any $k \in \mathbf{Z}$, the closed ball around ζ of radius n^{-k} is precisely the clone of combinatorial height k that contains ζ.

1.5 The groups $\mathrm{Aff}(\mathbf{Q}_n)$ and $\mathrm{Sim}(\mathbf{Q}_n)$

Define a *similarity* of a metric space X to be a bijection $\phi \colon X \to X$ such that, for some constant $k > 0$ we have $d(\phi(x), \phi(y)) = k\, d(x,y)$ for all $x, y \in X$. The number k is called the *stretch factor* of ϕ, denoted $\mathrm{Stretch}(\phi)$; we also say that ϕ is a *k-similarity*. The set of all similarities of X forms a group under composition, denoted $\mathrm{Sim}(X)$.

The groups $\mathrm{Sim}(\mathbf{R})$ and $\mathrm{Aff}(\mathbf{R})$ acting on $\mathbf{R}$ are obviously identical, but the situation is different in $\mathbf{Q}_n$.

First note that there is a natural isomorphism between $\mathrm{Sim}(\mathbf{Q}_n)$ and the group $\mathrm{Aut}(T_n)$ of direction preserving automorphisms of T_n. Since the closed balls in $\mathbf{Q}_n$ are precisely the clones, each similarity of $\mathbf{Q}_n$ takes clones to clones preserving inclusion. Every element of $\mathrm{Aut}(T_n)$ clearly arises in this manner.

The group $\mathrm{Aff}(\mathbf{Q}_n)$ acts by similarities on $\mathbf{Q}_n$: given $M = \left(\begin{smallmatrix} \zeta & \zeta' \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Aff}(\mathbf{Q}_n)$, if k is the least integer such that $\zeta_k \neq 0$, then $\mathrm{Stretch}(M) = n^{-k}$. We therefore have a natural monomorphism $\mathrm{Aff}(\mathbf{Q}_n) \hookrightarrow \mathrm{Sim}(\mathbf{Q}_n)$, but it is not surjective. For example, given $x, y \in \mathbf{Q}_n$ such that $x - y$ is invertible in $\mathbf{Q}_n$, the only affine transformation fixing x and y is the identity, but for any clone C not containing x or y the subgroup of $\mathrm{Sim}(\mathbf{Q}_n)$ fixing x, y and preserving C acts transitively on C, as can be seen by constructing appropriate automorphisms of T_n.

1.6 Boundary maps induced by a quasi-isometry

In [FM97] we showed that any quasi-isometry $f \colon X_m \to X_n$ induces maps $f^u \colon \partial^u X_m \to \partial^u X_n$ and $f_\ell \colon \partial_\ell X_m \to \partial_\ell X_n$ characterized as follows. For each hyperbolic plane $Q \subset X_m$, there is a unique hyperbolic plane $f^u(Q) \subset X_n$ such that

$$d_H(f(Q), f^u(Q)) < C < \infty$$

where d_H denotes Haudorff distance, and C depends only on the quasi-isometry constants of f. Also, for each hyperbolic plane $Q \subset X_m$, the closest point projection from $f(Q)$ to $f^u(Q)$ induces a map from the line at infinity of Q to the line at infinity of $f^u(Q)$, which induces the desired map f_ℓ. One could also prove that for any tree $\tau \subset X_m$, there is a unique tree $f_\ell(\tau) \subset X_n$ such that

$$d_H(f(\tau), f_\ell(\tau)) < C < \infty$$

It is proved in [FM97] that the maps f^u and f_ℓ are bilipschitz homeomorphisms. In fact what we proved is a little stronger, as seen in the next proposition.

Given a homeomorphism $h \colon X \to Y$ between metric spaces, and given $k \geq 1$, we say that h is *k-bilipschitz* if

$$\frac{1}{k}\, d(x,y) \leq d(f(x), f(y)) \leq k\, d(x,y) \quad \text{for all} \quad x, y \in X$$

We find it useful to introduce a more precise notion, as follows. Given $0 < a < b$, we say that h is *$[a,b]$-bilipschitz* if

$$a \cdot d(x,y) \leq d(h(x), h(y)) \leq b \cdot d(x,y) \quad \text{for all} \quad x, y \in X$$

We use the notation $[a(h), b(h)]$ to denote the *stretch interval* of h, the smallest subinterval of $\mathbf{R}_+$ such that h is $[a(h), b(h)]$-bilipschitz.

For example, "k-bilipschitz" is equivalent to "$[1/k, k]$-bilipschitz", and "k-similarity" is equivalent to "$[k, k]$-bilipschitz".

The next proposition comes from [FM97] Proposition 5.3 and the following remark.

Proposition 1.2. *For each $K \geq 1$, $C > 0$ there exists $L \geq 1$ such that if $f \colon X_m \to X_n$ is a (K, C)-quasi-isometry, then there exist $0 < a \leq b$ such that $b/a < L$, the map $f_\ell \colon \partial_\ell X_m \to X_n$ is $[a,b]$ bilipschitz, and the map $f^u \colon \partial^u X_m \to \partial^u X_n$ is $[1/b, 1/a]$-bilipschitz.* $\diamondsuit$

Warning. This does *not* say that the bilipschitz constants of f^u, f_ℓ depend only on K, C. Even the natural *isometric* action of $\mathrm{BS}(1,n)$ on X_n does not induce uniformly bilipschitz actions on $\partial_\ell X_n, \partial^u X_n$. For example the element $b \in \mathrm{BS}(1,n)$ acts on $\partial_\ell X_n \approx \mathbf{R}$ as a similarity with stretch factor n, and so the best bilipschitz constant of b^i acting on $\partial_\ell X_n$ is n^i, although b^i acts isometrically on X_n.

2 Representation into the quasi-isometry group

In this short section we use a (now standard) principle due essentially to Cannon-Cooper [CC92], also indicated by Gromov [Gro83], to prove that if a finitely generated group G is quasi-isometric to $\mathrm{BS}(1,n)$ then there is an induced representation $G \to \mathrm{QI}(\mathrm{BS}(1,n)) \approx \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$. We also study the quality of this representation.

2.1 Quasi-isometries and quasi-actions

Given a proper, geodesic metric space X and a group G, a *quasi-action* of G on X is a map $\psi\colon G \to \mathrm{QIMap}(X)$ such that, for some constants $K \geq 1$ and $C \geq 0$, we have:

- $\psi(g)$ is a (K,C)-quasi-isometry, for all $g \in G$.
- $\psi(\mathrm{Id}) \sim_C \mathrm{Id}_X$.
- $\psi(g) \circ \psi(h) \sim_C \psi(gh)$, for all $g, h \in G$.

A quasi-action $\psi\colon G \to \mathrm{QIMap}(X)$ evidently induces a homomorphism $\Psi\colon G \to \mathrm{QI}(X)$, where $\Psi(g)$ is the Hausdorff equivalence class of $\psi(g)$.

A quasi-action $\psi\colon G \to \mathrm{QIMap}(X)$ is *cocompact* if for some $x \in X$ and $C \geq 0$ the C-neighborhood of the set $\{\psi(g)(x) \mid g \in G\}$ is X. Also, ψ is *properly discontinuous* if for each $x, y \in X$ and $C \geq 0$ the set $\{g \in G \mid d(\psi(g)(x), y) \leq C)\}$ is finite.

Proposition 2.1 (QI rigidity condition). *Let X be a proper geodesic metric space, and Γ a finitely generated group. If $f\colon \Gamma \to X$ is a quasi-isometry with coarse inverse $\bar f\colon X \to \Gamma$, and if $L_g\colon \Gamma \to \Gamma$ denotes left multiplication by $g \in \Gamma$, then $\psi(g) = f \circ L_g \circ \bar f$ defines a properly discontinuous, cocompact quasi-action $\psi\colon \Gamma \to \mathrm{QIMap}(X)$.*

Proof. It is evident that ψ is a quasi-action. To see that ψ is cocompact, for any $x \in X$ we have $\{L_g(\bar f(x)) \mid g \in G\} = G$, and so $\{\psi(g)(x) \mid g \in G\} = f(G)$, whose C neighborhood equals X.

To see that ψ is properly discontinuous, fix $x, y \in X$ and $C \geq 0$. Given $g \in G$, if $d(\psi(g)(x), y) \leq C$ then $d(\bar f \circ \psi(g)(x), \bar f(y)) \leq KC + C^2$, and since $\bar f \circ \psi(g) = \bar f \circ f \circ L_g \circ \bar f$ we also have $d(\bar f \circ \psi(g)(x), L_g(\bar f(y))) \leq C$ and so $d(\bar f(y), g \cdot \bar f(y)) \leq KC + C^2 + C$. Since Γ is finitely generated, there are only finitely many g which satisfy this inequality. $\diamondsuit$

Remarks

1. If $\psi\colon G \to \mathrm{QIMap}(X)$ is a properly discontinuous, cocompact quasi-action on a proper geodesic metric space X, does G have a true action on X by quasi-isometries?

2. The passage from a quasi-action $\psi\colon G \to \mathrm{QIMap}(X)$ to its associated representation $\Psi\colon G \to \mathrm{QI}(X)$ seems to involve a loss of information. The multiplicative constant can be recovered, because if $f \sim g$ and if f is a (K, C)-quasi-isometry then g is a (K, C')-quasi-isometry for some C'. In particular, one can define the *dilatation* of an element of $\mathrm{QI}(X)$ as the infimum of K such that each representative $f \in \mathrm{QIMap}(X)$ is a (K, C)-quasi-isometry for some C.

It is unclear how to recover the additive constant in general, but for example if $\psi\colon G \to \mathrm{QI}(\mathbf{H}^3) = \mathrm{QC}(S^2)$ has bounded dilatation then $\psi(G)$ is induced by a uniformly quasi-isometric action of G on $\mathbf{H}^3$; this follows from [Tuk86].

2.2 Consequences for $\mathbf{BS(1, n)}$

By Proposition 2.1, if G is a finitely generated group quasi-isometric to $\mathrm{BS}(1, n)$ we obtain a quasi-action $\psi\colon G \to \mathrm{QIMap}(X_n)$. Let $\rho\colon G \to \mathrm{QI}(X_n)$ be the induced representation. Applying Proposition 1.2 to $\psi(g)$ for each $g \in G$, we obtain:

Proposition 2.2. *Let G be a finitely generated group quasi-isometric to $\mathrm{BS}(1, n)$. Then there is an induced representation $\rho\colon G \to \mathrm{QI}(\mathrm{BS}(1, n)) \approx \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$. Furthermore, let $\rho_\ell\colon G \to \mathrm{Bilip}(\mathbf{R})$ and $\rho^u\colon G \to \mathrm{Bilip}(\mathbf{Q}_n)$ be the projected actions, and let $[a_\ell(g), b_\ell(g)]$ and $[a^u(g), b^u(g)]$ be the stretch intervals of $\rho_\ell(g)$, $\rho^u(g)$ respectively. There exists $L \geq 1$ such that for all $g \in G$ we have*

- $b_\ell(g)/a_\ell(g) \leq L$.
- $b^u(g)/a^u(g) \leq L$.
- $[a_\ell(g), b_\ell(g)] \cdot [a^u(g), b^u(g)] \subset [1/L, L]$.

The multiplication of intervals used in this proposition is defined as follows: if $[a, b]$, $[c, d]$ are subintervals of $\mathbf{R}_+$, then

$$[a, b] \cdot [c, d] = \{xy \mid x \in [a, b] \text{ and } y \in [c, d]\}$$

3 Uniform quasisimilarity actions on R

An important part of our proof of Theorem A will be to understand uniform groups of quasisimilarities of the real line. This entire section is devoted to such a study, and is based on the following theorem of Hinkkanen (the terms are defined below). Let $\mathrm{Aff}_+(\mathbf{R})$ be the index 2 subgroup of $\mathrm{Aff}(\mathbf{R})$ which preserves orientation of $\mathbf{R}$.

Theorem 3.1 (Hinkkanen's Theorem). *A uniform group of orientation preserving quasisymmetric homeomorphisms of* $\mathbf{R}$ *is quasisymmetrically conjugate to a subgroup of* $\mathrm{Aff}_+(\mathbf{R})$.

We cannot use Hinkkanen's Theorem directly because we will need to make use of a bilipschitz conjugacy, which is not generally produced by a uniformly quasisymmetric group. In Theorem 3.2 we recast Hinkkanen's Theorem in the quasisimilarity setting. This setting has a pedagogical advantage as well—the technical details of the proof are simpler, and we believe that it is easier to see the geometric ideas underlying Hinkkanen's proof.

3.1 A Hinkkanen Theorem for uniform quasisimilarity groups

Let X be a metric space.

Definition (quasisimilarity). A function $f\colon X \to X$ is a K*-quasisimilarity* if for each distinct triple $x, y, z \in X$ we have

$$\frac{1}{K} \leq \frac{d(f(z), f(x))/d(z,x)}{d(f(y), f(x))/d(y,x)} \leq K \qquad (*)$$

An action of a group G on X is a *uniform quasisimilarity action* if there exists $K \geq 1$ such that the action of each $g \in G$ is a K-quasisimilarity of X.

Remarks

1. Note that a similarity is the same as a 1-quasisimilarity.

2. This property is called "quasiconformal" in the appendix to [FM97], because it implies that if S is a metric sphere in X then $f(S)$ is nested between two metric spheres S_1, S_2 such that $\mathrm{radius}(S_2)/\mathrm{radius}(S_1)$ is bounded. However, even more is true: the ratios

$$\frac{\mathrm{radius}(S_2)}{\mathrm{radius}(S)} \quad \text{and} \quad \frac{\mathrm{radius}(S)}{\mathrm{radius}(S_1)}$$

lie in a fixed interval in $\mathbf{R}_+$, independent of the original sphere S. For this reason it now seems more appropriate to us to refer to this property as "quasisimilarity".

3. An orientation preserving homeomorphism $f\colon \mathbf{R} \to \mathbf{R}$ is said to be K-*quasisymmetric* if the above inequality $(*)$ holds whenever $z - y = y - x$. An orientation preserving K-quasisimilarity is therefore K-quasisymmetric. The converse is not true: K-quasisimilarities are bilipschitz and hence absolutely continuous; whereas there exist K-quasisymmetric maps which are not absolutely continuous.

As Cooper notes in [FM97], if f is a K-quasisimilarity then, fixing two points $z, w \in X$, we have

$$\frac{1}{K^2} \le \frac{d(f(x), f(y))/d(x,y)}{d(f(z), f(w))/d(z,w)} \le K^2 \quad \text{for all } x, y \in X$$

With $L = d(f(z), f(w))/d(z,w)$ it follows that f is $K^2 \cdot \mathrm{Max}\{L, 1/L\}$-bilipschitz. However, this conclusion throws some information away. What one really obtains from this argument is that

$$K^{-2} L \le \frac{d(f(x), f(y))}{d(x,y)} \le K^2 L$$

and so f is $[a,b]$-bilipschitz with $a = K^{-2}L$ and $b = K^2 L$. Thus, if f is a K-quasisimilarity then f is $[a,b]$-bilipschitz for some a, b such that $b/a \le K^4$. Conversely, it is easy to see that if $f\colon X \to X$ is $[a,b]$-bilipschitz with $b/a \le K$, then f is a K-quasisimilarity. Note that there does *not* exist a constant C depending only on K such that every K-quasisimilarity is C-bilipschitz, however a map $f\colon \mathbf{R} \to \mathbf{R}$ is a K-quasisimilarity if and only if there exists a similarity $g\colon \mathbf{R} \to \mathbf{R}$ such that $f \circ g$ is $\sqrt{(K)}$-bilipschitz.

Combining these observations with Proposition 2.2, it follows that any group quasi-isometric to $\mathrm{BS}(1,n)$ has a uniform quasisimilarity action on $\partial_\ell X_n = \mathbf{R}$ and on $\partial^u X_n = \mathbf{Q}_n$.

Here is our quasisimilarity version of Hinkkanen's Theorem:

Theorem 3.2 (Quasisimilarity Hinkkanen's Theorem). *Let $\rho\colon G \to \mathrm{Bilip}(\mathbf{R})$ be a uniform quasisimilarity action of a group G on $\mathbf{R}$. Then there exists $\phi \in \mathrm{Bilip}(\mathbf{R})$ such that $\phi \circ \rho(g) \circ \phi^{-1} \in \mathrm{Aff}(\mathbf{R})$ for all $g \in G$. Moreover the bilipschitz constant for ϕ depends only on the quasisimilarity constant for ρ.*

Remark. Hinkkanen's paper considers only groups that preserve orientation on $\mathbf{R}$, but his methods indicate an easy way to reduce the general case to the orientation preserving case. We use these methods at the end of our proof.

3.2 Proof of Theorem 3.2

The proof of this theorem is an adaptation of Hinkkanen's proof for quasi-symmetric maps. The first part is the following proposition which proves Theorem 3.2 in the special case that $G = \mathbf{Z}$ and G preserves orientation. This is analogous to §3 of [Hin85], but the details are quite different in the current quasisimilarity setting. After this proposition, the remainder of the proof is an almost verbatim quotation of Hinkkanen's proof, and we will mention only the highlights.

Proposition 3.3. *Let $\{f^n \mid n \in \mathbf{Z}\}$ be a nontrivial, orientation preserving, uniform quasisimilarity action of $\mathbf{Z}$ on $\mathbf{R}$. Then one of two things happens:*

- *f has no fixed points, the action is uniformly bilipschitz, and f is bilipschitz conjugate to a translation.*
- *f has a single fixed point p, the action is not uniformly bilipschitz, and f is bilipschitz conjugate to multiplication $M_s(x) = sx$, for some $s \in (0, \infty) - \{1\}$.*

In either case, there is a conjugating map whose bilipschitz constant depends only on the quasisimilarity constant for $\{f^n\}$.

Proof. Let K' be a quasisimilarity constant for each f^n, and so the stretch interval $[a(f^n), b(f^n)]$ satisfies $b(f^n)/a(f^n) \le (K')^4 = K$.

Case 1: f has no fixed points. It follows that f^n has no fixed points for $n \neq 0$. The stretch interval $[a(f^n), b(f^n)]$ must contain 1, for otherwise f^n or f^{-n} would be a contraction mapping of $\mathbf{R}$ which always has a fixed point. Since $b(f^n)/a(f^n) \le K^4$ it follows that $[a(f^n), b(f^n)] \subset [1/K^4, K^4]$ and so f^n is K^4-bilipschitz for all n.

By replacing f with f^{-1}, if necessary, we may assume that $f(x) > x$ for all $x \in \mathbf{R}$. Let $x_n = f^n(x_0)$ be the orbit of some point x_0, and so

$$\dots x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \dots$$

Let $\alpha = x_1 - x_0$. Define $\phi \mid [x_0, x_1]$ to be the translation $x \mapsto x - x_0$. Extend ϕ to a homeomorphism of $\mathbf{R}$ as follows: for each $n \in \mathbf{Z}$ define

$\phi \mid [x_n, x_{n+1}]$ by $\phi(x) = \phi(f^{-n}(x)) + n\alpha$. Clearly ϕ is a homeomorphism of $\mathbf{R}$ conjugating f to the map $x \mapsto x + \alpha$. The maps $\phi \mid [x_0, x_1]$ and $x \mapsto x + \alpha$ are 1-bilipschitz. Since f^{-n} is K-bilipschitz it follows that $\phi \mid [x_n, x_{n+1}]$ is K-bilipschitz.

The proof is completed by applying the following:

Rubber band principle. *Given metric spaces X, Y and a collection of subsets $\mathcal{A}$ covering X, suppose that $\phi\colon X \to Y$ is a homeomorphism such that:*

1. *For each $A \in \mathcal{A}$ the map $\phi \mid A\colon A \to \phi(A)$ is K-bilipschitz.*
2. *For any pair $x, y \in X$ there is a sequence $x = x_0, x_1, \ldots, x_m = y$ such that $d(x,y) = \sum_{i=1}^m d(x_{i-1}, x_i)$, and for each $i = 1, \ldots, m$ there exists $A \in \mathcal{A}$ with $x_{i-1}, x_i \in A$.*
3. *For any pair $\xi, \eta \in Y$, there is a sequence $\xi = \xi_0, \xi_1, \ldots, \xi_m = \eta$ such that $d(\xi, \eta) = \sum_{i=1}^m d(\xi_{i-1}, \xi_i)$, and for each $i = 1, \ldots, m$ there exists $A \in \mathcal{A}$ with $\xi_{i-1}, \xi_i \in \phi(A)$.*

Then ϕ is K-bilipschitz. ◊

Applying this principle to ϕ using $\mathcal{A} = \{[x_n, x_{n+1}] \mid n \in \mathbf{Z}\}$, it follows that ϕ is K-bilipshitz.

Case 2: f has at least one fixed point. In this case we will prove that f is bilipschitz conjugate to the multiplication map $M_s(x) = sx$, for some $s \in (0,\infty) - \{1\}$. Note that if $s \neq r$ then M_s and M_r are not bilipschitz conjugate, so we have to search carefully to find the correct value of s.

Step 1: f has a unique fixed point. Suppose f has more than one fixed point. Since f is not the identity map, the fixed point set is a proper, closed subset of $\mathbf{R}$, and so the non-fixed point set is an open subset of $\mathbf{R}$ one of whose components is a finite interval (x,y). We have $f(x) = x$ and $f(y) = y$, and replacing f by f^{-1} if necessary we have $f(z) > z$ for all $z \in (x,y)$. Since no point in (x,y) is fixed it follows that $f^n(z) \to y$ as $n \to \infty$. Choose $z = (x+y)/2$, and so $d(x,z)/d(z,y) = 1$. By choosing n sufficiently large, the ratio

$$\frac{d(z,y) \cdot d\big(f^n(x), f^n(z)\big)}{d(x,z) \cdot d\big(f^n(z), f^n(y)\big)} = \frac{d\big(x, f^n(z)\big)}{d\big(f^n(z), y\big)}$$

may be made larger than K', violating the fact that f^n is a K'-quasisimilarity.

Let p be the unique fixed point of f.

Step 2: p is either attracting or repelling. Suppose not. Then either $f(x) > x$ for all $x \neq p$, or $f(x) < x$ for all $x \neq p$. We assume the former and derive a contradiction; the latter case is similar.

If $x < p$ then $f^n(x) \to p$ as $n \to \infty$, and if $y > p$ then $f^n(y) \to +\infty$ as $n \to \infty$. Taking $x = p - 1$ and $y = p + 1$, if n is sufficiently large we may make the ratio

$$\frac{d(x,p) \cdot d\big(f^n(p), f^n(y)\big)}{d\big(f^n(x), f^n(p)\big) \cdot d(p,y)} = \frac{d\big(p, f^n(y)\big)}{d\big(f^n(x), p\big)}$$

as large as we like, contradicting that $\{f^n\}$ is a uniform quasisimilarity action.

Replacing f by f^{-1} if necessary, we will assume that p is repelling, so $f^n(x) \to +\infty$ if $x > p$, and $f^n(x) \to -\infty$ if $x < p$. Under this assumption the stretch factor s will be > 1.

Step 3: f^n is *not* uniformly bilipschitz. If f^n is L-bilipschitz for all n, then for any $y > p$ consider the sequence $f^n(y)$ as $n \to +\infty$. Then $d(p, f^n(y)) = d(f^n(p), f^n(y)) \leq L \cdot d(p,y)$ and so $f^n(y)$ is a bounded sequence. Since $y < f(y) < f^2(y) < \cdots$ it follows that $f^n(y)$ converges to a fixed point of f distinct from p, a contradiction.

Step 4: Finding the expansion constant s. Using the assumptions that f preserves orientation and p is repelling, we show that there is a unique real number $s > 1$ such that $s^n \in [a(f^n), b(f^n)]$ for all n.

First we note an interesting property of any uniform quasisimilarity action of a group G on a metric space X. Let $[a(g), b(g)]$ be the stretch interval for the action of g on X. Note that the map $g \mapsto [a(g), b(g)]$ satisfies the following properties:

- $b(g)/a(g) \leq K$ for all $g \in G$.
- $a(\mathrm{Id}) = b(\mathrm{Id}) = 1$.
- $[a(gh), b(gh)] \subset [a(g)a(h), b(g)b(h)]$ for all $g, h \in G$.
- $[a(g^{-1}), b(g^{-1})] = [b(g)^{-1}, a(g)^{-1}]$ for all $g \in G$.

Definition (uniform quasihomomorphism). Given a group G, a *uniform quasihomomorphism* from G to $\mathbf{R}_+$ is a map which associates to each $g \in G$ an interval $[a(g), b(g)]$ satisfying the properties listed above, for some constant $K \geq 1$ independent of g.

Hence to each uniform quasisimilarity action of a group G on a metric space, there is an associated uniform quasihomomorphism from G to $\mathbf{R}_+$.

The following lemma, applied to the uniform quasihomomorphism $n \mapsto [a(f^n), b(f^n)]$, produces the expansion factor s:

Lemma 3.4. *For any uniform quasihomomorphism $n \mapsto [a_n, b_n]$ from $\mathbf{Z}$ to $\mathbf{R}_+$, there exists a unique $s > 0$ such that $s^n \in [a_n, b_n]$.*

Proof. Multiplying $\mathbf{Z}$ by -1 if necessary, and assuming that $[a_n, b_n]$ is not uniformly bounded, we may assume that if n is sufficiently large then $[a_n, b_n] \subset (1, \infty)$. In this case the s that we find will be larger than 1. Let K be a quasihomomorphism constant for $n \mapsto [a_n, b_n]$.

For all $n \geq 1$ define the following subinterval of $\mathbf{R}_+$:

$$I_n = \left[\sqrt[n]{a_n},\ \sqrt[n]{b_n}\right].$$

Given $m, n \geq 1$ we have $I_{mn} \subset I_n$, because

$$[a_{mn}, b_{mn}] = [a_{\underbrace{n + \cdots + n}_{m \text{ times}}}, b_{\underbrace{n + \cdots + n}_{m \text{ times}}}] \subset [a_n^m, b_n^m]$$

In particular, $I_n \subset I_1 = [a_1, b_1]$ for all $n \geq 1$. Also, the ratio of the upper and lower endpoints of I_n is $\sqrt[n]{b_n/a_n} \leq \sqrt[n]{K}$, which approaches 1 as $n \to +\infty$. Since I_n is a subinterval of the fixed interval $[a_1, b_1]$ it follows that $\text{Length}(I_n)$ approaches 0 as $n \to +\infty$.

We have shown that the intervals I_n are nested with respect to the divisor lattice on the natural numbers. Since $\text{Length}(I_n) \to 0$ it follows that $\cap_n I_n$ is a singleton $\{s\}$. This proves the existence of a unique $s > 1$ such that $s^n \in [a_n, b_n]$ for all $n \in \mathbf{Z}_+$.

Since $[a_{-n}, b_{-n}] = [b_n^{-1}, a_n^{-1}]$, it follows that $s^n \in [a_n, b_n]$ for all $n \in \mathbf{Z}$. $\diamondsuit$

Step 5: f is bilipschitz conjugate to M_s. Conjugating by a translation if necessary, we may assume that the fixed point of f is $p = 0$. We'll define the conjugacy ϕ on $[0, \infty)$, and prove it is bilipschitz there. The extension to $(-\infty, 0]$ is defined similarly, and the rubber band principle proves that ϕ is bilipschitz on $\mathbf{R}$.

Let $x_0 = 1$, $x_n = f^n(x_0)$. Define $\phi \mid [x_0, x_1]$ to be the unique orientation preserving affine homeomorphism from $[x_0, x_1]$ to $[1, s]$. This map has constant derivative $(s-1)/(x_1 - x_0) = (s-1)/(x_1 - 1)$, and we must get a bound on this derivative depending only on K.

Lemma 3.5. *With the notation above,*

$$\frac{s-1}{x_1 - 1} \in \left[\frac{1}{K^2}, K^2\right]$$

Accepting this lemma for the moment, let us define the conjugacy ϕ on all of $[0, \infty)$ and prove that it is bilipschitz there, with a constant depending only on K.

Define $\phi(0) = 0$, and define $\phi \mid [x_n, x_{n+1}]$ by

$$\phi(x) = \phi(f^{-n}(x)) \cdot s^n$$

Obviously ϕ is a homeomorphism of $[0, +\infty)$ conjugating f to M_s.

We prove that $\phi\colon [0, +\infty) \to [0, +\infty)$ is bilipschitz by applying the rubber band principle to the family of sets

$$S_n = \{0, x_n\}, \quad I_n = [x_n, x_{n+1}], \quad n \in \mathbf{Z}$$

Clearly this family of sets covers $[0, +\infty)$, and it satisfies properties (2) and (3) in the rubber band principle.

Note that

$$\begin{aligned} d(0, x_n) &= d(f^n(0), f^n(1)) \\ &\in [a(f^n), b(f^n)] \cdot d(0, 1) \\ &= [a(f^n), b(f^n)] \end{aligned}$$

According to Step 4 this interval is a subset of $[s^n/K, s^n \cdot K]$. Since

$$d(\phi(0), \phi(x_n)) = d(0, s^n) = s^n$$

it follows that ϕ is K-bilipschitz on the set S_n.

To prove that ϕ is bilipschitz on I_n, we have the following decomposition of $\phi \mid I_n$:

$$\begin{tikzcd} I_n \arrow[r, "f^{-n}"'] \arrow[rrr, bend left, "\phi"] & {[x_0, x_1]} \arrow[r, "\phi"'] & {[1, s]} \arrow[r, "y \to y s^n"'] & {[s^n, s^{n+1}]} \end{tikzcd}$$

The first map is $[a(f^{-n}), b(f^{-n})]$-bilipschitz, and this interval is contained in $[s^{-n}/K, s^{-n}K]$. The second map is $[1/K^2, K^2]$ bilipschitz by Lemma 3.5. The third map is $[s^n, s^n]$-bilipschitz. The composition is therefore $[1/K^3, K^3]$-bilipschitz. $\diamondsuit$

Proof of Lemma 3.5. The idea of the proof is that $x_1/s = d(f(1), f(0))/s \in [a(f), b(f)] \cdot d(0,1)/s \subset [s/K, K] \cdot 1/s = [1/K, K]$, and so x_1/s is bounded by a constant depending only on K. If s were large it would follow that $(x_1 - 1)/(s - 1)$ is bounded. However, since s may not be large, we must work a little harder to make use of the fact that x_n/s^n is also bounded, by a constant depending only on K.

It follows from Step 4 that for all $n \in \mathbf{Z}$ the map f^n is $[s^n/K, s^nK]$-bilipschitz. In particular,

$$d(x_{n+1}, x_n) = d(f^n(x_1), f^n(x_0)) \in [1/K, K] \cdot s^n \cdot d(x_1, x_0)$$

and so

$$\begin{aligned}\frac{d(x_{n+1}, x_n)}{s^{n+1} - s^n} &\in [1/K, K] \cdot \frac{s^n}{s^{n+1} - s^n} \cdot d(x_1, x_0) \\ &= [1/K, K] \cdot \frac{d(x_1, x_0)}{s - 1}\end{aligned}$$

We also have

$$\begin{aligned}d(x_n, x_0) &= d(x_n, x_{n-1}) + \cdots + d(x_1, x_0) \\ &\in [1/K, K] \cdot \frac{d(x_1, x_0)}{s-1} \cdot (s^n - s^{n-1}) + \cdots \\ &\qquad\qquad + [1/K, K] \cdot \frac{d(x_1, x_0)}{s-1} \cdot (s^1 - s^0) \\ &= [1/K, K] \cdot \frac{d(x_1, x_0)}{s-1} \cdot (s^n - 1)\end{aligned}$$

and so

$$\frac{d(x_n, x_0)}{s^n - 1} \in [1/K, K] \cdot \frac{d(x_1, x_0)}{s - 1}$$

On the other hand, we have

$$x_n = x_n - 0 = f^n(x_0) - f^n(0) \in [1/K, K] \cdot s^n \cdot (x_0 - 0)$$

and, since $x_0 = 1$ it follows that

$$\frac{x_n}{s^n} \in [1/K, K]$$

As $n \to \infty$, the ratio

$$\frac{d(x_n, x_0)}{s^n - 1} \Big/ \frac{x_n}{s^n} = \frac{x_n - 1}{s^n - 1} \Big/ \frac{x_n}{s^n}$$

approaches 1. Hence, for any $\epsilon > 0$, if n is sufficiently large we have

$$\frac{d(x_n, x_0)}{s^n - 1} \in [1/(K + \epsilon), K + \epsilon]$$

We have shown that the two intervals

$$[1/K, K] \cdot \frac{d(x_1, x_0)}{s - 1} \qquad \text{and} \qquad [1/(K + \epsilon), K + \epsilon]$$

have nonempty intersection, both containing $d(x_n, x_0)/(s^n - 1)$, and it follows that

$$\frac{1}{K + \epsilon} \le K \frac{d(x_1, x_0)}{s - 1} \qquad \text{and} \qquad \frac{1}{K} \cdot \frac{d(x_1, x_0)}{s - 1} \le K + \epsilon$$

and so

$$\frac{1}{K(K + \epsilon)} \le \frac{d(x_1, x_0)}{s - 1} \le K(K + \epsilon)$$

Since $\epsilon > 0$ is arbitrary, the lemma follows. $\diamondsuit$

Now we prove Theorem 3.2 in the special case that G preserves orientation of $\mathbf{R}$, in which case we conjugate G into $\mathrm{Aff}_+(\mathbf{R})$. The proof proceeds along the lines laid out on p. 332 of [Hin85], with a few comments needed to translate the quasisymmetric setting to the quasisimilarity setting.

Let $\mathrm{Homeo}_+(\mathbf{R})$ be the topological group of orientation preserving homeomorphisms of $\mathbf{R}$ in the topology of uniform convergence on compact sets. Given a sequence $g_i \in \mathrm{Homeo}_+(\mathbf{R})$ of K-quasisimilarities of $\mathbf{R}$, the following two statements are equivalent:

- The sequence g_i has a convergent subsequence in $\mathrm{Homeo}_+(\mathbf{R})$.
- There exist $x_1, x_2 \in \mathbf{R}$ such that $g_i(x_1)$ is bounded and $|g_i(x_2) - g_i(x_1)|$ is bounded away from 0 and ∞.

In this case, the limiting homeomorphism is also a K-quasisimilarity. To prove that the second statement above implies the first, one uses the Ascoli-Arzela Theorem, observing that the second statement is equivalent to saying that $\{g_i\}$ is uniformly bilipschitz and $\{g_i(x_1)\}$ is bounded for some x_1.

If $G \subset \mathrm{Homeo}_+(\mathbf{R})$ is a group of K-quasisimilarities of $\mathbf{R}$, it follows that the closure $\overline{G}$ in $\mathrm{Homeo}_+(X)$ is also a group of K-quasisimilarities. If $\overline{G}$ is bilipschitz conjugate to a subgroup of $\mathrm{Aff}_+(\mathbf{R})$ then so is G. We may therefore assume for the rest of the proof that G is closed in $\mathrm{Homeo}_+(\mathbf{R})$.

Lemma 6 of [Hin85] says that if $g, h \in G - \{\mathrm{Id}\}$ have distinct fixed points then $H = g \circ h \circ g^{-1} \circ h^{-1}$ has no fixed points. Lemma 7 of [Hin85] says that if $g, h \in G$ each have no fixed points and $g \neq h$ then $g(x) \neq h(x)$ for all $x \in \mathbf{R}$. We need not reprove these lemmas, because in each case the hypothesis requires only that the group generated by g, h be uniformly quasisymmetric, which is true in the present situation where g, h generate a uniform quasisimilarity group. If one desires, the proofs of Lemmas 6 and 7 may by improved by using the quasisimilarity conjugacies of Proposition 3.3, instead of the quasisymmetric conjugacies of Hinkkanen's paper.

Define G_T to be the set of all elements of G without fixed points. By the previous paragraph, G_T is a subgroup of G. By Proposition 3.3, G_T consists of those elements which are bilipschitz conjugate to translations.

Lemma 8 of [Hin85] proves that if G is not cyclic and $G \neq G_T \neq \{Id\}$ then G_T is not cyclic. Lemma 8 also proves, from the fact that G is closed in $\mathrm{Homeo}(\mathbf{R})$, that G_T is also closed. Lemma 9 of [Hin85] proves that if $G_T = \mathrm{Transl}(\mathbf{R})$, the group of all translations of $\mathbf{R}$, then $G \subset \mathrm{Aff}_+(\mathbf{R})$.

Lemma 10 of [Hin85] says, assuming G is closed in $\mathrm{Homeo}(\mathbf{R})$, that the theorem is true in two special cases: $G_T = G$; and $G_T = \{\mathrm{Id}\}$. Hinkkanen's proof gives a quasisymmetric conjugacy, but if one follows the proof through verbatim one sees in the present setting that the conjugating map is bilipschitz. The point is this. Suppose $G = G_T$. One constructs a strictly increasing sequence of cyclic subgroups $G_1 \supset G_2 \supset \cdots$ whose union is dense in G_T, and then one conjugates each G_i to a translation group by applying the cyclic version of the theorem, using a conjugating map f_i. Then one shows that there is a limiting map $f = \lim f_i$ which conjugates all of G_T to a translation group. In our present case, the maps f_i are provided by Proposition 3.3, and f_i has a bilipschitz constant depending only on the quasisimilarity constant of the subgroups G_i. Since each G_i is a subgroup of G_T, the quasisimilarity constants of G_i are uniformly bounded, and so the f_i are uniformly bilipschitz. The limiting map f is therefore bilipschitz.

The only remaining case is when $G \neq G_T \neq \{\mathrm{Id}\}$, and in this case one applies Lemmas 8, 9, and 10 as on page 332 of [Hin85] to finish the proof in the case that G preserves orientation.

To complete the proof we consider the case that G does not preserve orientation. Let G_0 be the orientation preserving subgroup of G, a normal subgroup of index 2. Applying the orientation preserving case of the theorem, we may do a bilipschitz conjugacy of G_0 into $\mathrm{Aff}_+(\mathbf{R})$. Applying this conjugacy to all of G, we may assume that $G_0 \subset \mathrm{Aff}_+(\mathbf{R})$. Choose a single element $g \in G - G_0$.

The most interesting subcase (and the only case we really care about) is when $(G_0)_T$ is a dense subgroup of $\mathrm{Transl}(\mathbf{R})$. We follow the proof of Lemma 9 of [Hin85]. There is a dense additive subgroup $S \subset \mathbf{R}$, and an isomorphism $\phi\colon S \to S$, such that $g(x) + s = g(x + \phi(s))$ for all $x \in \mathbf{R}$ and $s \in S$. Since g reverses orientation on $\mathbf{R}$, it follows that ϕ reverses order on S, and so ϕ is continuous on S. The map ϕ therefore extends to a continuous isomorphism of the additive group $\mathbf{R}$. It follows that $\phi(s) = cs$ for some $c \neq 0$. We therefore have $g(x) + s = g(x + cs)$ for all $x, s \in \mathbf{R}$. Taking $x = 0$ and $r = cs$ we have $g(r) = c^{-1}r + g(0)$, and so g is affine.

If $(G_0)_T$ is not a dense subgroup of $\mathrm{Transl}(\mathbf{R})$, then as we have seen there are two further subcases. In one subcase G_0 is a cyclic group of translations, and in the other case $(G_0)_T$ is trivial and G_0 is a group of dilations with a global fixed point. In particular G_0 is abelian and so G is virtually abelian (these subcases will therefore not arise in applications to groups quasi-isometric to $\mathrm{BS}(1, n)$, because of Proposition 4.4). The interested reader can show in either of these cases that one can do a further conjugation keeping G_0 in $\mathrm{Aff}_+(\mathbf{R})$ and taking g to an orientation reversing affine transformation.

4 A virtually faithful affine action on R

Recall the similarity groups $\mathrm{Sim}(\mathbf{R}) = \mathrm{Aff}(\mathbf{R})$ and $\mathrm{Sim}(\mathbf{Q}_n)$. There is an obvious inclusion $\mathrm{Isom}(X_n) \hookrightarrow \mathrm{Aff}(\mathbf{R}) \times \mathrm{Sim}(\mathbf{Q}_n)$, whose image is the subgroup of pairs $(f, g) \in \mathrm{Aff}(\mathbf{R}) \times \mathrm{Sim}(\mathbf{Q}_n)$ such that $\mathrm{Stretch}(f) \cdot \mathrm{Stretch}(g) = 1$. We have natural inclusions

$$\begin{aligned} \mathrm{BS}(1, n) &\hookrightarrow \mathrm{Isom}(X_n) \\ &\hookrightarrow \mathrm{Aff}(\mathbf{R}) \times \mathrm{Sim}(\mathbf{Q}_n) \\ &\hookrightarrow \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n) \\ &\approx \mathrm{QI}(\mathrm{BS}(1, n)) \end{aligned}$$

the final isomorphism being Theorem 7.1 in [FM97].

There is a split short exact sequence

$$1 \to \mathrm{Isom}(\mathbf{R}) \to \mathrm{Aff}(\mathbf{R}) \to \mathrm{Stretch}(\mathbf{R}) \to 1$$

where $\mathrm{Isom}(\mathbf{R})$ is the group of isometries of $\mathbf{R}$, and $\mathrm{Stretch}(\mathbf{R})$ is the group of orientation preserving stretch homeomorphisms of $\mathbf{R}$, otherwise known as $\mathrm{SL}(1,\mathbf{R})$. Note that there are canonical isomorphisms

$$\mathrm{Isom}(\mathbf{R}) \approx \mathbf{R} \rtimes (\mathbf{Z}/2\mathbf{Z}) \quad \text{and} \quad \mathrm{Stretch}(\mathbf{R}) \approx \mathbf{R}_+ \xrightarrow{\log} \mathbf{R}$$

where $\mathbf{Z}/2\mathbf{Z}$ acts on $\mathbf{R}$ by a reflection.

Given a subgroup $\Gamma < \mathrm{Aff}(\mathbf{R})$ we therefore obtain a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & A(\Gamma) \arrow[r, hook] \arrow[d, hook] & \Gamma \arrow[r] \arrow[d, hook] & \mathrm{Stretch}(\Gamma) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{Isom}(\mathbf{R}) \arrow[r, hook] & \mathrm{Aff}(\mathbf{R}) \arrow[r] & \mathrm{Stretch}(\mathbf{R}) \arrow[r] & 1
\end{tikzcd}$$

The purpose of this section is to prove:

Theorem 4.1. *Let G be a finitely generated group quasi-isometric to* $\mathrm{BS}(1,n)$. *There exists a representation $\theta\colon G \to \mathrm{Aff}(\mathbf{R})$ such that $\ker(\theta)$ is finite. Moreover, setting $\Gamma = \theta(G)$, the group $\mathrm{Stretch}(\Gamma)$ is infinite cyclic.*

To begin the proof, by Proposition 2.2 we obtain a properly discontinuous, cocompact quasi-action $\psi\colon G \to \mathrm{QIMap}(X_n)$, inducing a representation $\rho\colon G \to \mathrm{QI}(X_n) \approx \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$, with projected representations $\rho_\ell\colon G \to \mathrm{Bilip}(\mathbf{R})$ and $\rho^u\colon G \to \mathrm{Bilip}(\mathbf{Q}_n)$. The representations ρ_ℓ and ρ^u are uniform quasi-similarity actions.

Applying Theorem 3.2 to the representation ρ_ℓ, there exists a bilipschitz homeomorphism $\phi\colon \mathbf{R} \to \mathbf{R}$ such that $\theta(g) = \phi \circ \rho_\ell(g) \circ \phi^{-1} \in \mathrm{Aff}(\mathbf{R})$ for all $g \in G$, and so we obtain a representation

$$\theta\colon G \to \mathrm{Aff}(\mathbf{R})$$

We now need to use further properties of the situation to show that the representation θ has finite kernel and that the group $\theta(G) \subset \mathrm{Aff}(\mathbf{R})$ has infinite cyclic stretch group.

4.1 Biconvergence groups

Recall that if Γ is a word hyperbolic group then the action of Γ on its boundary $\partial\Gamma$ is a *uniform convergence group* action, which means that Γ acts properly discontinuously and cocompactly on the triple space

$$T(\partial\Gamma, \partial\Gamma, \partial\Gamma) = \{(x,y,z) \in \partial\Gamma \mid x \neq y, y \neq z, z \neq x\}$$

Furthermore if G is quasi-isometric to Γ then the induced action of G on $\partial\Gamma$ is also a uniform convergence group action.

Despite the fact that the groups $\mathrm{BS}(1,n)$ are far from being word hyperbolic, the upper and lower boundaries interact in a way reminiscent of convergence group actions, motivating the following definition.

Definition (Biconvergence group). Suppose that X, Y are topological spaces. Define two triple spaces

$$T(X,X,Y) = \{(x,y,\zeta) \in X \times X \times Y \mid x \neq y\}$$

and

$$T(X,Y,Y) = \{(x,\eta,\zeta) \in X \times Y \times Y \mid \eta \neq \zeta\}$$

A *biconvergence action* of a group G on the pair (X,Y) consists of an action of G on X and an action on Y, such that the induced diagonal actions of G on $T(X,X,Y)$ and on $T(X,Y,Y)$ are properly discontinuous. If furthermore the actions of G on $T(X,X,Y)$ and on $T(X,Y,Y)$ are cocompact, then we call this a *uniform biconvergence action* of G on (X,Y).

Example. $\mathrm{BS}(1,n)$ is a uniform biconvergence group on $(\mathbf{R},\mathbf{Q}_n)$. Note that $\mathrm{BS}(1,n)$ does *not* act properly discontinuously on the triple space $T(\mathbf{R},\mathbf{R},\mathbf{R}) = \{(x,y,z) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R} \mid x \neq y, y \neq z, z \neq x\}$, nor on $T(\mathbf{Q}_n,\mathbf{Q}_n,\mathbf{Q}_n)$. For example the action on $\mathbf{R}$ contains translations of $\mathbf{R}$ by any rational number of the form i/n^j.

Example. Let Sol be the unique 3-dimensional, solvable, non-nilpotent, connected Lie group which has a cocompact lattice. Sol can be identified with $\mathbf{R}^3$ in such a way that the left invariant metric is

$$ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2$$

There is an "upper boundary" $\partial^u\mathrm{Sol}$ consisting of the set of all "right side up" hyperbolic planes of the form $y =$(constant), and we have an identification $\partial^u\mathrm{Sol} = \mathbf{R}$. There is also a "lower boundary" $\partial_\ell\mathrm{Sol}$ consisting of all "upside down" hyperbolic planes $x =$(constant), also identified with $\mathbf{R}$. If Γ is the fundamental group of a compact 3-manifold M fibering over S^1 with torus fiber T^2, so that the monodromy map $T^2 \to T^2$ is an Anosov homeomorphism, then M has a Riemannian metric locally modelled on Sol, and so Γ may be identified with a cocompact, discrete subgroup of Sol. Under these conditions, one can show that the induced pair of actions of Γ on $(\partial_\ell\mathrm{Sol},\partial^u\mathrm{Sol}) \approx (\mathbf{R},\mathbf{R})$ is a biconvergence action.

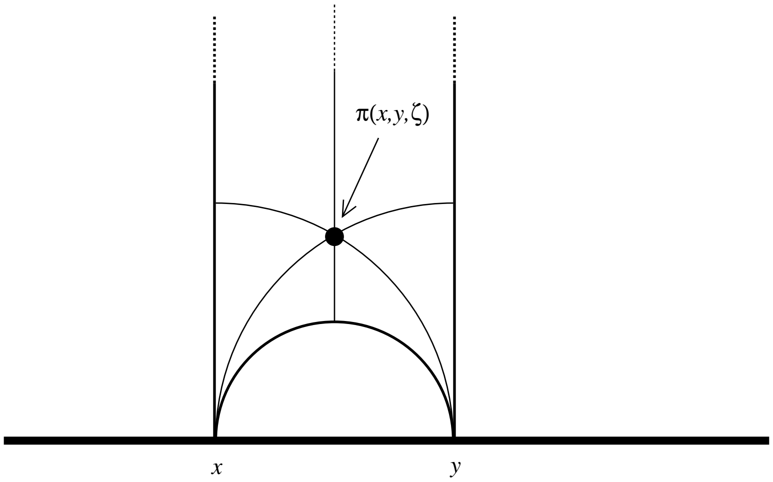

Figure 1: The point $\pi(x, y, \zeta)$ is the barycenter of the triangle $\Delta(x, y, +\infty)$ in the hyperbolic plane $H \subset X_n$ correponding to ζ.

Proposition 4.2 **(G is a biconvergence group).** *Let G be a finitely generated group which is quasi-isometric to* $\mathrm{BS}(1, n)$*. The induced action of G on $(\partial_\ell X_n, \partial^u X_n) \approx (\mathbf{R}, \mathbf{Q}_n)$ is a uniform biconvergence action.*

Proof. We use the symbols ∂_ℓ, ∂^u as shorthand for $\partial_\ell X_n$, $\partial^u X_n$. First we give a detailed proof that the action of G on $T(\partial_\ell, \partial_\ell, \partial^u)$ is properly discontinuous and cocompact. We then indicate the changes needed for $T(\partial_\ell, \partial^u, \partial^u)$.

By Proposition 2.1 we have a (K, C)-quasi-action $\psi \colon G \to \mathrm{QIMap}(X_n)$ which induces the given action of G on $T(\partial_\ell, \partial_\ell, \partial^u)$.

There is a map $\pi \colon T(\partial_\ell, \partial_\ell, \partial^u) \to X_n$ defined as follows (see Figure 1). Given $(x, y, \zeta) \in T(\partial_\ell, \partial_\ell, \partial^u)$, let $H \subset X_n$ be the hyperbolic plane corresponding to $\zeta \in \partial^u$. The boundary ∂H is identified with ∂_ℓ plus a point denoted $+\infty$. Consider the ideal triangle $\Delta = \Delta(x, y, +\infty) \subset H$. Define $\pi(x, y, \zeta)$ to be the barycenter of Δ, that is, the intersection of the perpendiculars from each vertex of Δ to the opposite side.

The map π is obviously continuous, proper, and equivariant with respect to each isometry ϕ of X_n, that is

$$\pi(\phi(x, y, \zeta)) = \phi(\pi(x, y, \zeta))$$

We claim that π is almost equivariant with respect to a quasi-isometry. That is, there exists C_1 depending on K, C such that if f is a (K, C)-quasi-isometry of X_n, with induced boundary maps $f_\ell \colon \mathbf{R} \to \mathbf{R}$ and $f^u \colon \mathbf{Q}_n \to \mathbf{Q}_n$, then for all $x, y \in \mathbf{R}, \zeta \in \mathbf{Q}_n$ we have

$$d\big(\pi(f_\ell(x), f_\ell(y), f^u(\zeta)), f(\pi(x, y, \zeta))\big) \le C_1$$

To prove this claim, note first that $d_H(f(\zeta), f^u(\zeta)) \le C_2$ for some constant C_2 depending only on K, C. Composing $f \mid \zeta$ with the closest point projection from $f(\zeta)$ to $f^u(\zeta)$, we obtain a quasi-isometry from the hyperbolic plane ζ to the hyperbolic plane $f^u(\zeta)$. This quasi-isometry takes x, y, regarded as elements of the line at infinity of ζ, to $f_\ell(x), f_\ell(y)$, regarded as elements of the line at infinity of $f^u(\zeta)$, and it takes the point at infinity of ζ to the point at infinity of $f^u(\zeta)$. The claim now follows from the fact that the barycenter map of $\mathbf{H}^2$ is almost equivariant with respect to quasi-isometries of $\mathbf{H}^2$.

Suppose that the action of G on $T(\partial_\ell, \partial_\ell, \partial^u)$ is not properly discontinuous. Then there is a sequence of distinct elements $g_i \in G$ and a point $(x_0, y_0, \zeta_0) \in T(\partial_\ell, \partial_\ell, \partial^u)$ such that, setting $(x_i, y_i, \zeta_i) = g_i(x_0, y_0, \zeta_0)$, the sequence (x_i, y_i, ζ_i) converges to some $(x, y, \zeta) \in T(\partial_\ell, \partial_\ell, \partial^u)$. After chopping off an initial subsequence, it follows that $\psi_{g_i}(\pi(x_0, y_0, \zeta_0))$ stays in a bounded neighborhood of $\pi(x, y, \zeta)$ in X_n. However, this contradicts the fact that the quasi-action ψ is properly discontinuous.

Next we prove that the action of G on $T(\partial_\ell, \partial_\ell, \partial^u)$ is cocompact. Note that the map $\pi \colon T(\partial_\ell, \partial_\ell, \partial^u) \to X_n$ is continuous and proper. By cocompactness of the quasi-action $\psi \colon G \to \mathrm{QIMap}(X_n)$, there exists $x_0 \in X_n$ and $C_2 > 0$ such that for each $y \in X_n$ there exists $g \in G$ with $d(y, \psi_g(x_0)) \le C_2$.

We claim that the compact set $\pi^{-1}\big(\overline{B}(x_0, KC_2 + 2C + C_1)\big)$ is a fundamental domain for the action of G on $T(\partial_\ell, \partial_\ell, \partial^u)$.

To prove this claim, consider any $t \in T(\partial_\ell, \partial_\ell, \partial^u)$, and choose g so that $d(\pi(t), \psi_g(x_0)) \le C_2$. Applying $\psi_{g^{-1}}$ we obtain

$$\begin{aligned} d\big(\psi_{g^{-1}}(\pi(t)), \psi_{g^{-1}}\psi_g(x_0)\big) &\le KC_2 + C \\ d\big(\psi_{g^{-1}}(\pi(t)), x_0\big) &\le KC_2 + C + C \end{aligned}$$

Also,

$$\begin{gathered} d\big(\pi(g^{-1} \cdot t), \psi_{g^{-1}}(\pi(t))\big) \le C_1 \\ d(\pi(g^{-1} \cdot t), x_0) \le KC_2 + 2C + C_1 \\ g^{-1} \cdot t \in \pi^{-1}\big(\overline{B}(x_0, KC_2 + 2C + C_1)\big) \end{gathered}$$

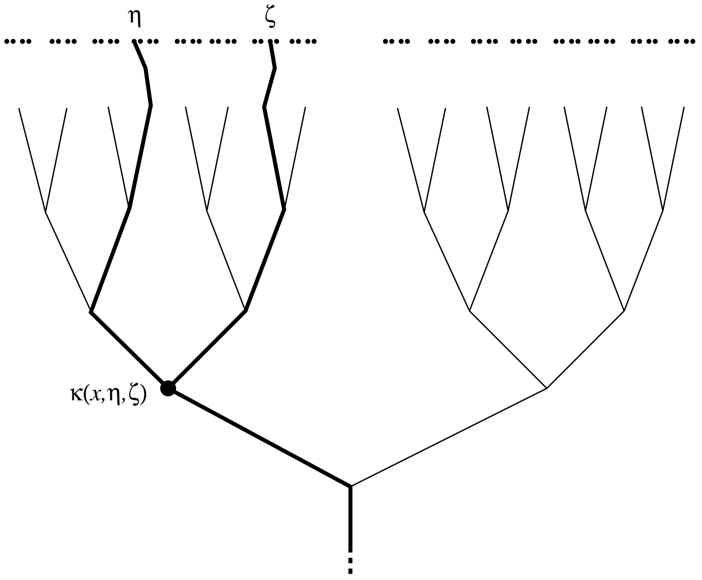

Figure 2: The point $\kappa(x,\eta,\zeta)$ is the intersection of the three sides of the triangle $\Delta(-\infty,\eta,\zeta)$ in the tree $\tau\subset X_n$ correponding to x.

completing the proof of cocompactness.

To prove that the action of G on $T(\partial_\ell,\partial^u,\partial^u)$ is properly discontinuous and cocompact, it suffices to describe a continuous, proper, almost G-equivariant map $\kappa\colon T(\partial_\ell,\partial^u,\partial^u)\to X_n$ (see Figure 2), and mimic the above proof. Given $(x,\eta,\zeta)\in T(\partial_\ell,\partial^u,\partial^u)$, the point x corresponds to some tree $\tau\subset X_n$, the image of an isometric section of the projection map $q\colon X_n\to T_n$. The set $\mathrm{Ends}(\tau)$ is identified with ∂^u plus a point denoted $-\infty$. The three points $-\infty,\eta,\zeta\in\mathrm{Ends}(\tau)$ determine an ideal triangle $\Delta(-\infty,\eta,\zeta)\subset\tau$ whose three sides intersect in a unique point which we take to be $\kappa(x,\eta,\zeta)$. $\Diamond$

We now give a property of uniform biconvergence groups which is similar in spirit to what holds for uniform convergence groups.

Proposition 4.3 (Contraction property). *Let G be a finitely generated group quasi-isometric to X_n. Let $(\rho_\ell,\rho^u)\colon G\to(\mathrm{Bilip}(\mathbf{R}),\mathrm{Bilip}(\mathbf{Q}_n))$ be the uniform biconvergence action given by Proposition 4.2. For every compact subset $K\subset\mathbf{Q}_n$ and every open subset $U\subset\mathbf{Q}_n$, there exists $g\in G$ such that $g\cdot K\subset U$.*

Proof. The appropriate generalization of this proposition should hold for any uniform biconvergence group, but here we will make use of the fact that ρ^u is a uniform quasisimilarity action.

Choose $x \in \mathbf{R}$ and $\zeta \in \mathbf{Q}_n$, and choose a one-to-one convergent sequence $\zeta_1, \zeta_2, \ldots \to \zeta$ in $\mathbf{Q}_n$ such that $\zeta_i \neq \zeta$ for all $i \in \mathbf{N}$. Choose a compact fundamental domain $K \subset T(\mathbf{R}, \mathbf{Q}_n, \mathbf{Q}_n)$ for the action of G. Choose $g_i \in G$ so that $g_i^{-1} \cdot (x, \zeta_i, \zeta) \in K$. Pass to a subsequence so that $g_i^{-1} \cdot (x, \zeta_i, \zeta)$ converges to $(y, \eta, \omega) \in K$. Of course $\eta \neq \omega$, because $(y, \eta, \omega) \in T(\mathbf{R}, \mathbf{Q}_n, \mathbf{Q}_n)$. Note that $d(g_i^{-1} \cdot \zeta_i, g_i^{-1} \cdot \zeta) \to d(\eta, \omega) \neq 0$, while $d(\zeta_i, \zeta) \to 0$. It follows that the stretch interval $[a_i, b_i]$ for the action of g_i on $\mathbf{Q}_n$ converges to 0, and so for sufficiently large i the map g_i contracts distance uniformly in $\mathbf{Q}_n$, with a contraction factor b_i that goes to 0 as $i \to \infty$. On the other hand, we also know that $d(\omega, g_i^{-1} \cdot \zeta)$ converges to 0, and so

$$\begin{aligned} d(g_i \cdot \omega, \zeta) &= d(g_i \cdot \omega, g_i \cdot (g_i^{-1} \cdot \zeta)) \\ &\leq b_i \, d(\omega, g_i^{-1} \cdot \zeta) \\ &\to 0 \quad \text{as} \quad i \to \infty \end{aligned}$$

It follows that the action of the sequence (g_i) on $\mathbf{Q}_n$ converges uniformly on compact sets to the constant map with value ζ. Choosing ζ to be any point in U, and choosing $K \subset \mathbf{Q}_n$ to be any compact set, the proposition follows. ♢

Remark. In the above proof, we know by Proposition 2.2 that the stretch intervals for the action of g_i on $\mathbf{R}$ and on $\mathbf{Q}_n$ are inversely related, and so the stretch interval for g_i^{-1} on $\mathbf{R}$ converges to 0. Since $d(g_i^{-1}(x), y) \to 0$ it follows that g_i^{-1} converges uniformly on compact sets to the constant map with value y. In some sense the sequence (g_i) therefore has "source-sink" dynamics, with source $y \in \mathbf{R}$ and sink $\zeta \in \mathbf{Q}_n$. This gives another analogy between convergence groups and biconvergence groups.

4.2 Two applications of the biconvergence property

Our first application shows that the representation $\theta\colon G \to \mathrm{Aff}(\mathbf{R})$ is virtually faithful.

Proposition 4.4 (θ is virtually faithful). *The homomorphisms $\rho_\ell\colon G \to \mathrm{Bilip}(\mathbf{R})$ and $\theta\colon G \to \mathrm{Aff}(\mathbf{R})$ each have finite kernel.*

Proof. Since ρ_ℓ and θ are conjugate in $\mathrm{Bilip}(\mathbf{R})$, they have the same kernel. So we need only prove that ρ_ℓ has finite kernel.

Pick a fixed clone $C \subset \mathbf{Q}_n$. Let $A = 0 \times 1 \times C$, a compact subset of $T(\mathbf{R}, \mathbf{R}, \mathbf{Q}_n)$. Note that C is also an open subset of $\mathbf{Q}_n$.

Suppose the proposition is false. Then there exists an arbitrarily large subset $\{g_1, \ldots, g_k\} \subset G$ acting trivially on $\mathbf{R}$. We shall construct $g \in G$, depending on $g_1, \ldots, g_k$, such that $(gg_ig^{-1}) \cdot A \cap A \neq \emptyset$ in $T(\mathbf{R}, \mathbf{R}, \mathbf{Q}_n)$, for $i = 1, \ldots, k$. Since k is arbitrarily large, this contradicts proper discontinuity of the action of G on $T(\mathbf{R}, \mathbf{R}, \mathbf{Q}_n)$ (Proposition 4.2).

Pick any point $\zeta_0 \in \mathbf{Q}_n$. Let $\zeta_i = g_i \cdot \zeta_0$. Let K be the smallest clone containing $\zeta_0, \zeta_1, \ldots, \zeta_k$, in particular K is compact. Furthermore, $(g_i \cdot K) \cap K \neq \emptyset$ for $i = 1, \ldots, k$.

By Proposition 4.3 applied to the compact set K and the open set C, there exists $g \in G$ such that $g \cdot K \subset C$ in $\mathbf{Q}_n$. Thus, $(gg_ig^{-1}) \cdot C \cap C \neq \emptyset$ in $\mathbf{Q}_n$, for each $i = 1, \ldots, k$. Since gg_ig^{-1} acts as the identity on $\mathbf{R}$ it follows that $(gg_ig^{-1}) \cdot A \cap A \neq \emptyset$ in $T(\mathbf{R}, \mathbf{R}, \mathbf{Q}_n)$, contradicting proper discontinuity as indicated above. $\diamondsuit$

Our next application of the biconvergence property shows that the stretch group of $\theta(G) \subset \mathrm{Aff}(\mathbf{R})$ is infinite cyclic.

Proposition 4.5 (stretch group is cyclic). *Let* $\Gamma = \theta(G)$. *The group* $\mathrm{Stretch}(\Gamma)$ *is infinite cyclic.*

Proof. Recall that the representation $\theta\colon G \to \mathrm{Aff}(\mathbf{R})$ is obtained from $\rho_\ell\colon G \to \mathrm{Bilip}(\mathbf{R})$ by a bilipschitz conjugacy: there exists a bilipschitz map $\phi\colon \mathbf{R} \to \mathbf{R}$ such that

$$\theta(g) = \phi \circ \rho_\ell(g) \circ \phi^{-1}$$

for all $g \in G$.

Let $s(g) = \mathrm{Stretch}(\theta(g))$. Let $[a_\ell(g), b_\ell(g)]$ be the stretch interval of $\rho_\ell(g)$ acting on $\mathbf{R}$, and let $[a^u(g), b^u(g)]$ be the stretch interval of $\rho^u(g)$ acting on $\mathbf{Q}_n$. We know by Proposition 2.2 that there exists $L_1 \geq 1$ such that

$$[a^u(g), b^u(g)] \cdot [a_\ell(g), b_\ell(g)] \subset [\frac{1}{L_1}, L_1]$$

We also know since the conjugating map ϕ is bilipschitz that

$$\frac{[a_\ell(g), b_\ell(g)]}{s(g)} \subset [\frac{1}{L_2}, L_2] \tag{$*$}$$

where L_2 is the square of the bilipschitz constant of ϕ. Therefore, setting $L_3 = L_1 L_2$ we have

$$[a^u(g), b^u(g)] \cdot s(g) \subset [\frac{1}{L_3}, L_3] \tag{$**$}$$

Suppose the proposition is false. Then there exists a sequence $g_i \in G$ such that $s(g_i) \neq 1$ but $s(g_i) \to 1$ as $i \to \infty$. Since $s(g_i) \neq 1$ it follows that $\theta(g_i)$ has a unique fixed point in $\mathbf{R}$, and so $\rho_\ell(g_i)$ has a unique fixed point in $\mathbf{R}$.

We claim also that $\rho^u(g_i)$ has a unique fixed point in $\mathbf{Q}_n$. To see this, since $s(g_i) \neq 1$ we may choose a sufficiently high power g_i^k so that $s(g_i^k) \notin \left[\frac{1}{L_3}, L_3\right]$. It follows that

$$1 \notin [a^u(g_i^k), b^u(g_i^k)]$$

and therefore either $\rho^u(g_i^k)$ or $\rho^u(g_i^{-k})$ is a contraction mapping of $\mathbf{Q}_n$ and so has a unique fixed point, and so $\rho^u(g_i)$ has a unique fixed point.

Let $x_i \in \mathbf{R}$ be the unique fixed point of $\rho_\ell(g_i)$, and let $\zeta_i \in \mathbf{Q}_n$ be the unique fixed point of $\rho^u(g_i)$. For the rest of the proof we use only the actions $\rho_\ell \colon G \to \mathrm{Bilip}(\mathbf{R})$ and $\rho^u \colon G \to \mathrm{Bilip}(\mathbf{Q}_n)$, and so we may unambiguously use the "dot" notation for these group actions.

Since the action of G on $T(\mathbf{R}, \mathbf{R}, \mathbf{Q}_n)$ is cocompact, we may pick a compact fundamental domain A.

Pick any point $w \in \mathbf{R}$ distinct from all the points x_i. Choose $h_i \in G$ so that $h_i(x_i, w, \zeta_i) \in A$. Replacing g_i by its conjugate $h_i g_i h_i^{-1}$, and replacing x_i, w, ζ_i by their images under h_i, we may assume that $(x_i, w_i, \zeta_i) \in A$ for some sequence w_i; since s is invariant under conjugation we still have $s(g_i) \to 1$. By compactness of A we may pass to a subsequence so that $x_i \to x$ in $\mathbf{R}$ and $\zeta_i \to \zeta$ in $\mathbf{Q}_n$.

Since $s(g_i) \to 1$, using $(*)$ and $(**)$ it follows that for all sufficiently large i we have

$$[a_\ell(g_i), b_\ell(g_i)] \subset [\frac{1}{L}, L]$$

and

$$[a^u(g_i), b^u(g_i)] \subset [\frac{1}{L}, L]$$

where $L = 2\,\mathrm{Max}(L_2, L_3)$.

Choose a point $y \neq x$ in $\mathbf{R}$. We will contradict proper discontinuity of the action of G on $T(\mathbf{R}, \mathbf{R}, \mathbf{Q}_n)$ by showing that $(g_i \cdot x, g_i \cdot y, g_i \cdot \zeta)$ stays in a compact subset of $T(\mathbf{R}, \mathbf{R}, \mathbf{Q}_n)$.

We know that $g_i \cdot x \to x$ in $\mathbf{R}$, because

$$\begin{aligned} d(x, g_i \cdot x) &\leq d(x, x_i) + d(x_i, g_i \cdot x) \\ &= d(x, x_i) + d(g_i \cdot x_i, g_i \cdot x) \\ &\in d(x, x_i) + d(x, x_i) \cdot \left[\frac{1}{L}, L\right] \\ &= d(x, x_i)\left[1 + \frac{1}{L}, 1 + L\right] \\ &\to 0 \quad \text{as} \quad i \to \infty \end{aligned}$$

because $d(x, x_i) \to 0$. The same argument shows that $g_i \cdot \zeta \to \zeta$ in $\mathbf{Q}_n$.

To complete the proof, we will show that $g_i \cdot y$ stays in a compact subset of $\mathbf{R}$ disjoint from x, from which it follows that $(g_i \cdot x, g_i \cdot y, g_i \cdot \zeta)$ stays in a compact subset of triple space, a contradiction as noted above.

We have

$$\begin{aligned} d(x_i, g_i \cdot y) &= d(g_i \cdot x_i, g_i \cdot y) \\ &\subset \left[\frac{1}{L}, L\right] \cdot d(x_i, y) \\ &\subset \left[\frac{1}{2L}, 2L\right] \cdot d(x, y) \end{aligned}$$

for sufficiently large i, because of the fact that $d(x_i, y) \to d(x, y)$. Since $d(x_i, x) \to 0$ it follows that

$$d(x, g_i \cdot y) \in \left[\frac{1}{3L}, 3L\right] \cdot d(x, y)$$

for sufficiently large i. The set of points in $\mathbf{R}$ whose distance from x is in the interval $[1/3L, 3L] \cdot d(x, y)$ is a compact subset of $\mathbf{R}$ disjoint from x, and this compact subset contains $g_i \cdot y$ for all sufficiently large i, as desired. $\diamondsuit$

Remark. The inequality $(*)$ is *precisely* where we need Theorem 3.2, which gives a bilipschitz conjugacy between ρ_ℓ and θ, instead of the quasisymmetric conjugacy provided by Hinkkanen's original theorem.

5 Finishing the proof of Theorem A

Suppose G is quasi-isometric to $\mathrm{BS}(1,n)$. By Theorem 4.1 we have a quotient group $\Gamma = G/N$, where N is a finite normal subgroup of G, and we have a diagram of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & A = A(\Gamma) \arrow[r, hook] \arrow[d, hook] & \Gamma \arrow[r] \arrow[d, hook] & \mathrm{Stretch}(\Gamma) = \langle t \rangle \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{Isom}(\mathbf{R}) \arrow[r, hook] & \mathrm{Aff}(\mathbf{R}) \arrow[r] & \mathrm{Stretch}(\mathbf{R}) \arrow[r] & 1
\end{tikzcd}$$

where $\langle t \rangle$ is infinite cyclic. Choosing a splitting, we may regard t as an element of $\mathrm{Aff}(\mathbf{R})$, and replacing t with t^{-1} if necessary we may assume $\mathrm{Stretch}(t) > 1$.

We have the following additional properties of Γ:

- Γ is solvable, because $\mathrm{Aff}(\mathbf{R})$ is solvable.
- Γ has a torsion free subgroup of index at most 2, namely

$$\Gamma_+ = \Gamma \cap \mathrm{Aff}_+(\mathbf{R})$$

- Γ is quasi-isometric to $\mathrm{BS}(1,n)$.

Since finite presentability is a quasi-isometry invariant ([GH91], Proposition 10.18) we also have:

- Γ is finitely presented.

Note that there are finitely generated subgroups of $\mathrm{Aff}(\mathbf{R})$ which are not finitely presented; once such group is described in [Str84].

We may now quote the following theorem of Bieri-Strebel, taken from [Str84] (see also [BS78]):

Theorem 5.1 (Bieri-Strebel). *Let Γ be a finitely presented solvable group, and suppose that Γ has an HNN presentation of the form*

$$\Gamma = A *_{\mathbf{Z}} = \left\langle A, t \;\middle|\; tA_1t^{-1} = A_2 \right\rangle$$

where A_1, A_2 are subgroups of A. Then there is another HNN presentation

$$\Gamma = B *_{\mathbf{Z}} = \left\langle B, t \;\middle|\; tBt^{-1} = B' \right\rangle$$

where B is a finitely generated subgroup of A and B' is a subgroup of B.

For completeness, here is a quick proof suggested to us by T. Delzant.

Proof. Let K be a finite complex with fundamental group Γ and universal covering map $p\colon \widetilde{K} \to K$. Let T be a Bass-Serre tree for the given HNN decomposition of Γ. Let $f\colon \widetilde{K} \to T$ be a Γ-equivariant map, transverse to the midpoint of each edge of T. Pulling back those midpoints gives a Γ-equivariant 1-complex $\widetilde{L} \subset \widetilde{K}$. Let $L = p(\widetilde{L})$, a 1-complex in K. Construct a new Γ-tree T' whose edges correspond to components of $\widetilde{L}$ and whose vertices correspond to components of $\widetilde{K} - \widetilde{L}$. The vertex and edge stabilizers of T' are subgroups of vertex and edge stabilizers of T. Moreover, if v is a vertex of T' corresponding to a component $\widetilde{C}$ of $\widetilde{K} - \widetilde{L}$, and if $C = p(\widetilde{C})$, then $\mathrm{Stab}(v)$ is isomorphic to the image of the inclusion induced map $\pi_1(C) \to \pi_1(K)$, and so $\mathrm{Stab}(v)$ is finitely generated. Passing to the quotient graph of groups obtained from the action of Γ on T', and collapsing subgraphs in the appropriate manner, one obtains an HNN decomposition $\Gamma = \langle B, t' \mid t' B_1 t'^{-1} = B_2 \rangle$ where B is a finitely generated subgroup of A, $B_1, B_2 < B$, and t' is conjugate into the infinite cyclic subgroup $\langle t \rangle$. If both B_1 and B_2 are proper subgroups of B, then Γ has a free subgroup of rank ≥ 2; this follows from the normal forms theorem for HNN decompositions, or from a ping-pong argument on the Bass-Serre tree. Since Γ is solvable it has no free subgroups of rank ≥ 2, and so one of B_1, B_2 equals B. $\diamondsuit$

Applying the above theorem to Γ, we have a finitely generated subgroup $B \subset A(\Gamma)$, and an HNN decomposition

$$\Gamma = \langle B, t \mid tbt^{-1} = \phi(b), \forall b \in B \rangle$$

where $\phi\colon B \to B$ is an injective endomorphism, and B is a finitely generated subgroup of $\mathrm{Isom}(\mathbf{R})$.

Recall that $\Gamma_+ = \Gamma \cap \mathrm{Aff}_+(\mathbf{R})$, and let $B_+ = B \cap \mathrm{Isom}_+(\mathbf{R})$. The group B_+ is a finitely generated subgroup of $\mathrm{Isom}_+(\mathbf{R}) = \mathrm{Transl}(\mathbf{R})$, and so B_+ is free abelian of some finite rank $r \geq 1$. The indices $[\Gamma : \Gamma_+]$, $[B : B_+]$ are both ≤ 2, and so we fall into three cases:

Case 1. $[\Gamma : \Gamma_+] = [B : B_+] = 1$.

Case 2. $[\Gamma : \Gamma_+] = 2$ and $[B : B_+] = 1$.

Case 3. $[\Gamma : \Gamma_+] = [B : B_+] = 2$.

In case (1) we obviously have $t \in \mathrm{Aff}_+(\mathbf{R})$, that is, t preserves orientation of $\mathbf{R}$. In case (3), if $t \in \mathrm{Aff}(\mathbf{R}) - \mathrm{Aff}_+(\mathbf{R})$, in other words if t reverses orientation of $\mathbf{R}$, then we can replace t by its product with an element of $B - B_+$, a reflection of $\mathbf{R}$; and hence we may assume in case (3) that $t \in \mathrm{Aff}_+(\mathbf{R})$. In case (2) we necessarily have $t \in \mathrm{Aff}(\mathbf{R}) - \mathrm{Aff}_+(\mathbf{R})$. Setting $s = t^2$ and $\psi = \phi^2$ in case (2), or $s = t$ and $\psi = \phi$ in cases (1) and (3), we have an HNN decomposition

$$\Gamma_+ = \langle B_+, s \mid sbs^{-1} = \psi(b), \forall b \in B_+ \rangle$$

where $s \in \mathrm{Aff}_+(\mathbf{R})$. Since $B_+ \approx \mathbf{Z}^r$ and $\psi(B_+) \approx B_+$, it follows that $\psi(B_+)$ has finite index in B_+. Let $I = [B_+ : \psi(B_+)]$.

Next we extract some homological information about Γ_+ (see [Bro82]). Recall that if K is a finitely generated, virtually torsion free group, the *virtual cohomological dimension* of K, denoted $\mathrm{vcd}(K)$, is the cohomological dimension of any finite index, torsion free subgroup $K' < K$; by Serre's Theorem ([Bro82] Theorem VIII.3.1) this number is independent of the choice of K'.

Lemma 5.2. *For any injective homomorphism $\psi\colon \mathbf{Z}^r \to \mathbf{Z}^r$, the HNN group $K = \langle \mathbf{Z}^r, t \mid tbt^{-1} = \psi(b), \forall b \in \mathbf{Z}^r \rangle$ has virtual cohomological dimension equal to $r+1$.*

Proof. We first construct a compact Eilenberg-Maclane space for K by mimicking the construction in [FM97] of an Eilenberg-Maclane space for $\mathrm{BS}(1,n)$.

Extend the endomorphism $\psi\colon \mathbf{Z}^r \to \mathbf{Z}^r$ to a linear isomorphism $\psi\colon \mathbf{R}^r \to \mathbf{R}^r$ which commutes with the action of $\mathbf{Z}^r$. Let $T^r = \mathbf{R}^r/\mathbf{Z}^r$ be the r-dimensional torus. The map ψ descends to a self covering map $\Psi\colon T^r \to T^r$. Let I be the index $[\mathbf{Z}^r : \psi(\mathbf{Z}^r)]$, and so I is the determinant of ψ acting on $\mathbf{R}^r$, and I is the degree of the covering map Ψ. Let C be the mapping torus of Ψ,

$$C = T^r \times [0,1] \,/\, (x,0) \sim (\Psi(x), 1)$$

Note that $\pi_1(C) \approx K$. Let X be the universal covering space of C. Let T_I be the tree constructed in §1, a homogeneous directed tree with one incoming edge and I outgoing edges at each vertex. By construction, the space X fibers over T_I with fiber $\mathbf{R}^r$. Since T_I is contractible we have a homeomorphism

$$X \approx \mathbf{R}^r \times T_I$$

In particular X is contractible, and so C is a compact Eilenberg-Maclane space for the group K.

Since K is torsion free we have $\mathrm{vcd}(K) = \mathrm{cd}(K)$. By [Bro82] corollary VIII.7.6, the number $\mathrm{cd}(K)$ is the maximum dimension k such that

$H^k_c(X;\mathbf{Z}) \neq 0$, where H^k_c denotes cohomology with compact supports. By [Spa81] (p. 341, corollary 9 and p. 360, exercise 6), there is a Künneth formula

$$H^k_c(X;\mathbf{Z}) \approx \sum_{i=0}^{k} H^i_c(\mathbf{R}^r;\mathbf{Z}) \otimes H^{k-i}_c(T_I;\mathbf{Z})$$

from which it follows that

$$H^k_c(X;\mathbf{Z}) \approx \begin{cases} 0 & \text{if } k \neq r+1 \\ H^1_c(T_I;\mathbf{Z}) & \text{if } k = r+1 \end{cases}$$

Since T_I has infinitely many ends, it follows that $H^1_c(T_I;\mathbf{Z})$ is an infinite rank abelian group, and so $\mathrm{vcd}(K) = r+1$. $\diamondsuit$

Remark. This proof gives extra information about the group K, namely a complete computation of the cohomology groups $H^k_c(X;\mathbf{Z}) \approx H^k(K;\mathbf{Z}K)$, which by [Ger93] are quasi-isometry invariants of K. Note in particular that K is a duality group in the sense of Bieri and Eckmann (see [Bro82] §VIII), which means that, setting $D = H^{r+1}(K;\mathbf{Z}K) \approx H^{r+1}_c(X;\mathbf{Z})$, we have

$$H^i(K, M) \approx H_{r+1-i}(K, D \otimes M) \text{ for all } K\text{-modules } M.$$

Now we complete the proof of Theorem A.

Applying Lemma 5.2 we have the following computations of virtual cohomological dimensions:

$$\mathrm{vcd}(\Gamma_+) = r+1 \text{ and } \mathrm{vcd}(\mathrm{BS}(1,n)) = 2$$

By a theorem of Gersten [Ger93] and Block-Weinberger [BW97], if two virtually torsion free groups have finite Eilenberg-Maclane spaces, and if the groups are quasi-isometric, then they have the same vcd. It follows that $\mathrm{vcd}(\Gamma_+) = \mathrm{vcd}(\mathrm{BS}(1,n))$, and so $r = 1$ and B_+ is an infinite cyclic group $B_+ = \langle b \rangle$. The endomorphism $\psi\colon \mathbf{Z} \to \mathbf{Z}$ preserves orientation, and so $\psi(b) = b^m$ for some $m \geq 1$. If $m = 1$ then $\Gamma_+ \approx \mathbf{Z}^2$, but $\mathbf{Z}^2$ is not quasi-isometric to $\mathrm{BS}(1,n)$—for example, $\mathbf{Z}^2$ has quadratic growth but $\mathrm{BS}(1,n)$ has exponential growth. It follows that $m \geq 2$, and Γ_+ is isomorphic to $\mathrm{BS}(1,m)$. Applying the main result of [FM97] it follows that Γ_+ is abstractly commensurable to $\mathrm{BS}(1,n)$, in fact m, n are positive powers of the same positive integer.

As for the group Γ itself, we have already shown that Γ has a subgroup Γ_+ isomorphic to $\mathrm{BS}(1,m)$ of index ≤ 2, and $\mathrm{BS}(1,m)$ is abstractly commensurable to $\mathrm{BS}(1,n)$; therefore Γ is abstractly commensurable with $\mathrm{BS}(1,n)$. This completes the proof of Theorem A. $\diamondsuit$

Here is a corollary:

Corollary 5.3. *Let G be a finitely generated, torsion free group which is quasi-isometric to* $\mathrm{BS}(1,n)$ *for some* $n \geq 2$. *Then* $G \approx \mathrm{BS}(1,k)$ *for some integer* k *with* $|k| \geq 2$, *such that* $\mathrm{BS}(1,n)$ *and* $\mathrm{BS}(1,k)$ *are abstractly commensurable.*

Proof. Apply Theorem A to get a short exact sequence

$$1 \to N \to G \to \Gamma \to 1$$

with N finite. Since G is torsion free it follows that $N = \{\mathrm{Id}\}$ and $G \approx \Gamma$. In the proof of Theorem A, it also follows that $B = B_+$ is infinite cyclic, $B = \langle b \rangle$. Thus we fall into case 1 or 2 in the proof of Theorem A. The injective endomorphism $\phi\colon B \to B$ must have the form $\phi(b) = b^k$ for some integer $k \neq 0$. If $k = \pm 1$ then Γ has polynomial growth, a contradiction as above. Therefore, $|k| \geq 2$ and $G \approx \mathrm{BS}(1,k)$. Since $\mathrm{BS}(1,k)$ contains $\mathrm{BS}(1,k^2)$ with index 2, and since $k^2 \geq 2$, the main result of [FM97] shows that $\mathrm{BS}(1,k^2)$ is abstractly commensurable to $\mathrm{BS}(1,n)$, and so $G \approx \mathrm{BS}(1,k)$ is abstractly commensurable to $\mathrm{BS}(1,n)$. $\diamondsuit$

In the proof of Theorem A, recall the three cases in the analysis of the group Γ. As the proof of the corollary shows, in case 1 we have $\Gamma \approx \mathrm{BS}(1,k)$ for some $k \geq 2$, and in case 2 we have $\Gamma \approx \mathrm{BS}(1,k)$ for some $k \leq -2$.

In case 3, B is the infinite dihedral group $B = \langle a, r \mid r^2 = 1, rar = a^{-1} \rangle$. To enumerate the possibilities for Γ, it suffices to enumerate the injective, nonsurjective, orientation preserving endomorphisms $\phi\colon B \to B$ (orientation preserving means that ϕ fixes each of the two ends of the group B). Fix the representation $B \to \mathrm{Aff}(\mathbf{R})$ given by $a \cdot x = x + 2$, $r \cdot x = -x$. The reflections of B may be enumerated as $r_i = a^{-i} r$, $i \in \mathbf{Z}$, and r_i is the reflection about the point $i \in \mathbf{R}$. Up to conjugation by an automorphism of B, there are two infinite sequences of endomorphisms ϕ, depending on whether ϕ fixes some reflection, as follows.

Case 3.i. If ϕ fixes a reflection then, replacing r by the fixed reflection, there exists an integer $m \geq 2$ such that $\phi(a) = a^m$, $\phi(r) = r$. We may represent ϕ in $\mathrm{Aff}_+(\mathbf{R})$ as the expansion with fixed point 0 and stretch m, that is, $\phi(x) = mx$. We therefore have

$$\Gamma = B*_\phi = \langle a, r, t \mid r^2 = 1, rar^{-1} = a^{-1}, tat^{-1} = a^m, trt^{-1} = r \rangle$$

Case 3.ii. If ϕ does not fix a reflection then, up to a replacement of r, there is an odd integer $m = 2k+1 \geq 3$ such that $\phi(a) = a^m$ and $\phi(r) = a^{-k}r$. We may represent ϕ in $\mathrm{Aff}_+(\mathbf{R})$ as the expansion with fixed point $-1/2$ and stretch m, that is $\phi(x) = m(x+1/2) - 1/2$. It is clear from this description that ϕ indeed fixes no reflections; one can also compute directly that $\phi(r_i) = r_{2ki+k+i}$, and clearly there is no $i \in \mathbf{Z}$ such that $2ki + k + i = i$. We have:

$$\Gamma = B*_\phi = \left\langle a, r, t \;\middle|\; r^2 = 1, rar^{-1} = a^{-1}, tat^{-1} = a^m, trt^{-1} = a^{-k}r \right\rangle$$

6 Final comments: The Sullivan-Tukia Theorem

The Sullivan-Tukia Theorem says that a uniformly quasiconformal subgroup of $\mathrm{QC}(S^2) \approx \mathrm{QI}(\mathbf{H}^3)$ is quasiconformally conjugate into the conformal group $\mathrm{Conf}(S^2) \approx \mathrm{Isom}(\mathbf{H}^3)$.

Here is one possible way to formulate an analogue of this theorem for $\mathrm{BS}(1,n)$. The analogue of the group $\mathrm{QC}(S^2) \approx \mathrm{QI}(\mathbf{H}^3)$ is $\mathrm{QI}(\mathrm{BS}(1,n)) \approx \mathrm{QI}(X_n) \approx \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$. The analogue of a uniformly quasiconformal subgroup of $\mathrm{QC}(S^2)$ is a "bounded" subgroup $G < \mathrm{QI}(X_n)$, which means that for some $K \geq 1$, each $g \in G$ is represented by a (K, C)-quasi-isometry of X_n (where C may depend on g). Equivalently, there is a constant $K' \geq 1$ such that for each $g \in G$, letting $\rho_\ell(g) \in \mathrm{Bilip}(\mathbf{R})$, $\rho^u(g) \in \mathrm{Bilip}(\mathbf{Q}_n)$ be the projections, we have

$$[a(\rho_\ell(g)), b(\rho_\ell(g))] \cdot [a(\rho^u(g)), b(\rho^u(g))] \subset [1/K', K']$$

What is the analogue of $\mathrm{Conf}(S^2)$? One guess is $\mathrm{Isom}(X_n)$. However, the Sullivan-Tukia Theorem is false in this case, because if $n = m^2$ then $\mathrm{BS}(1,m)$ is a bounded subgroup of $\mathrm{QI}(X_n)$, but there does not exist an isometric action of $\mathrm{BS}(1,m)$ on X_n.

A more reaonable guess is that if $n \geq 2$ is an integer which is not a proper power, then every bounded subgroup of $\mathrm{QI}(X_n)$ is conjugate to a subgroup of $\mathrm{Isom}(X_n)$. One possible approach is to prove a version of Theorem 3.2 for $\mathbf{Q}_n$, namely that any uniform quasisimilarity group action on $\mathbf{Q}_n$ is bilipschitz conjugate to a similarity action; for this to be true it is necessary that n not be a proper power. However, we do not know how to prove this.

References

[Bes95] M. Bestvina, *Local homology properties of boundaries of groups*, preprint, 1995.

[Bro82] K. Brown, *Cohomology of groups*, Graduate Texts in Math., vol. 87, Springer, 1982.

[BS78] R. Bieri and K. Strebel, *Almost finitely presented soluble groups*, Comm. Math. Helv. **53** (1978), 258–278.

[BW97] J. Block and S. Weinberger, *Large scale homology theories and geometry*, AMS/IP Stud. in Adv. Math. **2** (1997).

[CC92] J. Cannon and D. Cooper, *A charactrization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three*, Trans. AMS **330** (1992), 419–431.

[Cho96] R. Chow, *Groups coarse quasi-isometric to complex hyperbolic space*, Trans. AMS (1996), no. 5, 1757–1769.

[Esk96] A. Eskin, *Quasi-isometric rigidity for nonuniform lattices in higher rank*, Jour. AMS (1997), to appear.

[Far96] B. Farb, *The quasi-isometry classification of lattices in semisimple Lie groups*, Math. Res. Letters (1997), to appear.

[FM97] B. Farb and L. Mosher, appendix by D. Cooper, *A rigidity theorem for the solvable Baumslag-Solitar groups*, Invent. Math. (1997), to appear.

[FS96] B. Farb and R. Schwartz, *The large-scale geometry of Hilbert modular groups*, J. Diff. Geom. **44** (1996), no. 3, 435–478.

[GH91] E. Ghys and P. de la Harpe, *Infinite groups as geometric objects (after Gromov)*, Ergodic theory, symbolic dynamics and hyperbolic spaces (T. Bedford, M. Keane, and C. Series, eds.), Oxford Univ. Press, 1991.

[Ger93] S. M. Gersten, *Quasi-isometry invariance of cohomological dimension*, Comptes Rendues Acad. Sci. Paris Série 1 Math. **316** (1993), 411–416.

[Gro81] M. Gromov, *Groups of polynomial growth and expanding maps*, IHES Sci. Publ. Math. **53** (1981), 53–73.

[Gro83] M. Gromov, *Infinite groups as geometric objects*, Proceedings of the International Congress of Mathematicians, Warsaw, Amer. Math. Soc., 1983, pp. 385–392.

[Hin85] A. Hinkkanen, *Uniformly quasisymmetric groups*, Proc. London Math. Soc. **51** (1985), 318–338.

[KL95] M. Kapovich and B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds*, Invent. Math. **128** (1997), no. 2, 393–416.

[Mos96] L. Mosher, *Indiscrete representations, laminations, and tilings*, Proc. of Geometry Group Theory Year, Melbourne 1996, to appear.

[Sch96a] R. Schwartz, *Quasi-isometric rigidity and Diophantine approximation*, Acta Math. **177** (1996), 75–112.

[Sch96b] R. Schwartz, *The quasi-isometry classification of rank one lattices*, IHES Sci. Publ. Math. **82** (1996).

[Spa81] E. Spanier, *Algebraic topology*, Springer, New York—Berlin, 1981.

[Str84] K. Strebel, *Finitely presented soluble groups*, Group theory: Essays for Philip Hall (Gruenberg and Roseblade, eds.), 1984, pp. 257–314.

[Tuk86] P. Tukia, *On quasi-conformal groups*, J. D'Anal. Math. **46** (1986), 318–346.

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