The lower central series and pseudo-Anosov dilatations

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Abstract

The theme of this paper is that algebraic complexity implies dynamical complexity for pseudo-Anosov homeomorphisms of a closed surface S_g of genus g. Penner proved that the logarithm of the minimal dilatation for a pseudo-Anosov homeomorphism of S_g tends to zero at the rate $1/g$. We consider here the smallest dilatation of any pseudo-Anosov homeomorphism of S_g acting trivially on Γ/Γ_k, the quotient of $\Gamma = \pi_1(S_g)$ by the k^{th} term of its lower central series, $k \geq 1$. In contrast to Penner's asymptotics, we prove that this minimal dilatation is bounded above and below, independently of g, with bounds tending to infinity with k. For example, in the case of the Torelli group $\mathcal{I}(S_g)$, we prove that $L(\mathcal{I}(S_g))$, the logarithm of the minimal dilatation in $\mathcal{I}(S_g)$, satisfies $.197 < L(\mathcal{I}(S_g)) < 4.127$. In contrast, we find pseudo-Anosov mapping classes acting trivially on Γ/Γ_k whose asymptotic translation lengths on the complex of curves tend to 0 as $g \to \infty$.

1 Introduction

Let $\text{Mod}(S)$ denote the mapping class group of a closed, orientable surface $S = S_g$ of genus $g \geq 2$; this is the group of homotopy classes of orientation preserving homeomorphisms of S. According to the Nielsen–Thurston classification, every mapping class $f \in \text{Mod}(S)$ which is not finite order and is not reducible (i.e. does not fix the isotopy class of any essential 1-submanifold) is *pseudo-Anosov*, i.e. it has a representative which is a pseudo-Anosov homeomorphism; see [FLP, Th].

Attached to each pseudo-Anosov $f \in \text{Mod}(S)$ is its *dilatation* $\lambda(f)$. This is an algebraic integer which records the exponential growth rate of lengths of curves under iteration of f, in any fixed metric on S; see [Th]. The number $\log(\lambda(f))$ equals the minimal topological entropy of any element in the homotopy class f; this minimum is realized by a pseudo-Anosov homeomorphism representing f (see [FLP, Exposé 10]). From another perspective, $\log(\lambda(f))$ is the translation length of f as an isometry of the *Teichmüller space of* S equipped with the Teichmüller metric.

Following Penner, we consider the set

$$\text{spec}(\text{Mod}(S)) = \{\log(\lambda(f)) : f \in \text{Mod}(S) \text{ is pseudo-Anosov}\} \subset \mathbb{R}$$

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which can be thought of as the *length spectrum* of the moduli space of genus g Riemann surfaces. We will also consider, for various subgroups $H < \text{Mod}(S)$, the subset $\text{spec}(H) \subset \text{spec}(\text{Mod}(S))$ obtained by restricting to pseudo-Anosov elements of H. Arnoux–Yoccoz [AY] and Ivanov [Iv1] proved that $\text{spec}(\text{Mod}(S))$ is discrete as a subset of $\mathbb{R}$. It follows that for any subgroup $H < \text{Mod}(S)$, the set $\text{spec}(H)$ has a least element $L(H)$.

If $F(x)$ and $G(x)$ are any real-valued functions, we write $F(x) \asymp G(x)$ if there exists a $C > 0$ so that $F(x)/G(x) \in [1/C, C]$ for all x. Penner [Pe] proved that

$$L(\text{Mod}(S_g)) \asymp \frac{1}{g}$$

In particular, as genus increases, there are pseudo-Anosov mapping classes with dilatations arbitrarily close to 1.

Torelli dilatations. The theme of this paper is that algebraic complexity implies dynamical complexity for pseudo-Anosov homeomorphisms. The following contrast to Penner's theorem is a first instance of this phenomenon. Below, $\mathcal{I}(S)$ denotes the *Torelli group*, which is defined to be the subgroup of $\text{Mod}(S)$ consisting of elements which act trivially on $H_1(S;\mathbb{Z})$.

Theorem 1.1. *For $g \geq 2$, we have*

$$.197 < L(\mathcal{I}(S_g)) < 4.127$$

The main point of Theorem 1.1 is that $L(\mathcal{I}(S_g)) \asymp 1$; in other words, the bounds given in Theorem 1.1 are universal with respect to g. In contrast, Theorem 1.6 below states that the minimal translation length in the complex of curves for pseudo-Anosov mapping classes in $\mathcal{I}(S_g)$ tends to 0 as $g \to \infty$.

We remark that every pseudo-Anosov element $f \in \mathcal{I}(S)$ has nonorientable stable and unstable foliations since otherwise $\lambda(f)$ would be a nontrivial eigenvalue for the action on homology; see [Th]. However, this condition alone is insufficient to guarantee uniform upper and lower bounds for $\log(\lambda(f))$. For example, a construction of McMullen [Mc] can be used to produce a sequence of pseudo-Anosov elements $f_n \in \text{Mod}(S_{g_n})$, where $g_n \to \infty$, each f_n has nonorientable foliations, and $\log(\lambda(f_n)) \asymp 1/g_n$.

For the *Johnson kernel*, which is the subgroup $\mathcal{K}(S)$ of $\mathcal{I}(S)$ generated by Dehn twists about separating curves, we obtain slightly better bounds, $.693 < L(\mathcal{K}(S)) < 4.127$; see Proposition 3.4 below.

The Johnson filtration. The groups $\mathcal{I}(S)$ and $\mathcal{K}(S)$ are the first terms of the *Johnson filtration* of $\text{Mod}(S)$, which is the sequence of groups

$$\mathcal{N}_k(S) = \text{kernel}(\text{Mod}(S) \to \text{Out}(\Gamma/\Gamma_k))$$

where Γ_k is the k^{th} term of the lower central series for $\Gamma = \pi_1(S)$, defined inductively by $\Gamma_0 = \Gamma$ and $\Gamma_{k+1} = [\Gamma_k, \Gamma]$. It is a classical theorem of Magnus that $\{\Gamma_k\}$ is a *filtration* of Γ, which means that $\Gamma_{k+1} < \Gamma_k$ and $\bigcap_{k=1}^{\infty} \Gamma_k = \emptyset$; it follows that $\{\mathcal{N}_k(S)\}$ is a filtration of

Mod(S). By definition, $\mathcal{N}_0(S) = \text{Mod}(S)$ and $\mathcal{N}_1(S) = \mathcal{I}(S)$. It is a theorem of Johnson [Jo2] that $\mathcal{N}_2(S)$ is isomorphic to $\mathcal{K}(S)$. It is a fact that $\{\mathcal{N}_k(S)\}$ is a *central* filtration of Mod(S) (i.e. successive quotients are abelian) [BL], and so $\mathcal{N}_{k+1}(S)$ contains the k^{th} term of the lower central series of the Torelli group $\mathcal{I}(S)$ (the lower central series descends faster than any central series).

For a fixed surface S, a compactness argument (see Proposition 4.1 below) readily gives that $L(\mathcal{N}_k(S)) \to \infty$ as $k \to \infty$; that is, as one specifies more and more algebraic conditions by considering pseudo-Anosov homeomorphisms fixing deeper quotients Γ/Γ_k, the corresponding dynamical complexity (measured as the dilatation) must diverge to infinity. Our main result is that this divergence is uniform over all surfaces.

Theorem 1.2. *Given $k \geq 1$, there exist $M(k)$ and $m(k)$, where $m(k) \to \infty$ as $k \to \infty$, so that*

$$m(k) < L(\mathcal{N}_k(S_g)) < M(k)$$

for every $g \geq 2$.

Again, we compare Theorem 1.2 with Theorem 1.6 below.

We do not have good control over the constants $m(k)$ and $M(k)$ in this theorem, and we are interested in more precise asymptotics for $L(\mathcal{N}_k(S_g))$ as $g \to \infty$. We pose the following.

Question 1.3. *Let k be fixed. As we increase the genus g, what is* $\inf L(\mathcal{N}_k(S_g))$*? What is* $\sup L(\mathcal{N}_k(S_g))$*? Does* $\lim L(\mathcal{N}_k(S_g))$ *exist? Are any of these quantities realized for some g? Of particular interest is $L(\mathcal{I}(S_g))$.*

We consider Theorem 1.1 (and Proposition 3.4 below) as warmups for Theorem 1.2, as their proofs contain many of the main ideas. Moreover, in these cases we compute explicit bounds, whereas for arbitrary values of k we do not.

The upper bound for $L(\mathcal{I}(S))$ and $L(\mathcal{K}(S))$ in Theorem 1.1 and Proposition 3.4 is given by explicit construction; see §2.3 below. In addition, this construction is used to derive the upper bound in Theorem 1.2, using the relationship between the lower central series of $\mathcal{I}(S)$ and $\{\mathcal{N}_k(S)\}$. One easily checks that this upper bound grows at most exponentially with k; see §4.2. The proof of the lower bound begins with the following.

Proposition 1.4. *Suppose $f \in \mathcal{I}(S)$ is pseudo-Anosov. If c is a separating curve, then $i(c, f(c)) \geq 4$. If c is a nonseparating curve, then $i(c, f^j(c)) \geq 2$ for $j = 1$ or $j = 2$.*

The idea is to use this proposition, combined with a surgery argument, to find a curve whose length in a certain metric is stretched by a definite amount under a pseudo-Anosov mapping class. The metric comes from a quadratic differential with vertical and horizontal foliations given by the stable and unstable foliations for the mapping class. The relationship between the metric and the foliations implies that the amount of stretching bounds the dilatation from below; see Lemma 2.4.

The proof of the lower bound in Theorem 1.2 follows a similar line of reasoning and requires an asymptotic version of Proposition 1.4. We show that, for f lying in a deep term of the Johnson filtration, $i(c, f(c))$ is large for every curve c with $f(c) \neq c$ (Lemma 4.6).

Translation lengths on the complex of curves. One can also consider the global topological complexity of a pseudo-Anosov homeomorphism given by the translation lengths on the (1-skeleton of the) *complex of curves* $\mathcal{C} = \mathcal{C}(S)$. This complex, defined by Harvey [H], has a vertex for each isotopy class of essential simple closed curves in S and an edge for each pair of vertices with disjoint representatives. We endow $\mathcal{C}$ with the path metric $d_{\mathcal{C}}$ (after declaring each edge to have length 1) and define the *asymptotic translation length* for the action of f on $\mathcal{C}$ by

$$\tau_{\mathcal{C}}(f) = \liminf_{j\to\infty} \frac{d_{\mathcal{C}}(c, f^j(c))}{j}$$

for any curve c (this is independent of the choice of curve c). For any subgroup $H < \mathrm{Mod}(S)$, we denote by $L_{\mathcal{C}}(H)$ the infimum of $\tau_{\mathcal{C}}(f)$ over all pseudo-Anosov elements $f \in H$. Masur–Minsky [MM, Prop 4.6] proved that for any fixed g, $L_{\mathcal{C}}(\mathrm{Mod}(S_g)) > 0$.

Our first result in this direction shows that $L_{\mathcal{C}}(\mathrm{Mod}(S_g))$ tends to 0 strictly faster than $L(\mathrm{Mod}(S_g)) \asymp 1/g$.

Theorem 1.5. *For any $g \geq 2$, we have*

$$L_{\mathcal{C}}(\mathrm{Mod}(S_g)) < \frac{4\log(2+\sqrt{3})}{g\log\left(g-\frac{1}{2}\right)}$$

The following result provides a contrast to Theorem 1.1 and Theorem 1.2.

Theorem 1.6. *For any k, we have*

$$L_{\mathcal{C}}(\mathcal{N}_k(S_g)) \to 0$$

as $g \to \infty$.

Congruence subgroups. The ideas involved in the proof of Theorem 1.2 provide bounds for a different sequence of subgroups of $\mathrm{Mod}(S)$. Let $\mathrm{Mod}(S)[r]$ denote the *principal level r congruence subgroup* of $\mathrm{Mod}(S)$, which is defined to be the finite index subgroup of $\mathrm{Mod}(S)$ consisting of those elements acting trivially on $H_1(S;\mathbb{Z}/r\mathbb{Z})$. We prove the following in §2.4.

Theorem 1.7. *If $g \geq 2$ and $r \geq 3$, then*

$$.197 < L(\mathrm{Mod}(S_g)[r]) < 4.127$$

Theorem 1.7 puts strong constraints on the possibilities for pseudo-Anosov elements of least dilatation in $\mathrm{Mod}(S)$.

Brunnian subgroups. In §6 we provide a different illustration of our theme by considering pseudo-Anosov mapping classes in the *Brunnian subgroup* $\mathrm{Brun}(S_{g,p})$ of the mapping class group of the orientable surface $S_{g,p}$ of genus g with $p > 0$ punctures. This is the subgroup consisting of those mapping classes which are isotopic to the identity once any puncture is filled in (see §6 for details).

Theorem 1.8. *For any $g \geq 0$ and any $p \geq 5$, we have*

$$L(\mathrm{Brun}(S_{g,p})) > \log\left(\frac{p}{4}\right)$$

Related results in the literature. As we noted above, Penner [Pe] gave the first proof that $L(\mathrm{Mod}(S_g)) \asymp 1/g$. His upper bound was improved upon by Bauer [Ba1, Ba2], who gave new examples with small dilatation. McMullen [Mc] gave a different construction for the upper bound of Penner's asymptotics using fibered 3-manifolds with infinitely many fibrations. Brinkmann [Br], Hironaka–Kin [HK], and Minakawa [Mk] also gave examples proving the same upper bound for the asymptotics. The best known general upper bound is $\log(2+\sqrt{3})/g$ given by Hironaka–Kin [HK] and Minakawa [Mk]. The precise value of $L(\mathrm{Mod}(S_g))$ is not known for any $g > 1$; some related values have been calculated [Zh, SKL, HS].

The second author [Le] investigated the question of the minimal dilatation for the class of subgroups $\langle T_A, T_B \rangle$ generated by two positive multitwists T_A and T_B. In this case the infimum of $L(\langle T_A, T_B \rangle)$ over all genus and all such subgroups is the logarithm of Lehmer's number $\log(\lambda_L) \approx .162$, and is realized on a genus 5 surface. For pure braid groups PB_n, Song [So] proved that $\log(2+\sqrt{5}) \leq L(PB_n)$. In fact, one has $L(PB_n) \asymp 1$; see §2.3 for an upper bound. Finally, for the hyperelliptic subgroups, Hironaka–Kin [HK] proved that the asymptotics are the same as those of $\mathrm{Mod}(S_g)$, giving an explicit upper bound of $\log(2+\sqrt{3})/g$. Moreover, their examples descend to braids that cyclically permute the punctures. Thus, they also obtain an upper bound $L(PB_{2g+1}) \leq (2+1/g)\log(2+\sqrt{3}) \leq 5\log(2+\sqrt{3})/2$.

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2 Torelli groups

The mechanism by which we prove that a pseudo-Anosov element $f \in \mathcal{I}(S)$ is forced to have big dilatation comes from the action of f on simple closed curves. This is best explored via intersection numbers.

2.1 $\mathcal{I}(S)$ and geometric intersection numbers

Let a and b be free homotopy classes of simple closed curves in S. The *geometric intersection number* $i(a,b)$ is defined by

$$i(a,b) = \min\{|\alpha \cap \beta| : \alpha \in a \text{ and } \beta \in b\}$$

We generally do not distinguish between homotopy classes of simple closed curves and particular representatives of the classes, referring to both simply as "curves", with usage dictating what is meant (likewise for mapping classes and representative homeomorphisms). Representative curves a and b of homotopy classes of the same names are in *minimal position* if they are transverse and $i(a,b) = |a \cap b|$. Whenever considering representatives of a pair of homotopy classes we assume that they are in *minimal position* unless stated otherwise.

We will need the following general fact about geometric intersection numbers. We say that a collection of curves *fills* a closed surface if the complement is a disjoint union of disks.

Lemma 2.1. *If a and b are two simple closed curves which together fill the closed surface S_g, then $i(a, b) \geq 2g - 1$. More generally, for any two curves a and b in S, $i(a, b) = -\chi(N)$, where N is a regular neighborhood of $a \cup b$.*

Proof. If a and b fill S_g, then $\chi(N) < \chi(S_g) = 2 - 2g$ since S_g is obtained from N by gluing disks to the boundary components. Because these numbers are integers, the first statement follows from the second.

To prove the second statement, we simply note that N deformation retracts onto $a \cup b$, thought of as a graph in N. Then $\chi(N) = V - E$, where $V = i(a, b)$ is the number of vertices and E is the number of edges. Because each vertex of $a \cup b$ is 4-valent we see that $E = 2V = 2i(a, b)$, and hence $\chi(N) = -i(a, b)$. □

Lemma 2.2. *Suppose $f \in \mathcal{I}(S)$, that c is a separating curve, and that $f(c) \neq c$. Then $i(f(c), c) \geq 4$.*

This lemma is sharp when the genus of S is at least 3, for in this case one can find two separating curves c and d with $i(c, d) = 2$, and in this case $i(T_d(c), c) = 4$ (here and throughout, T_a is the *Dehn twist* about the curve a).

Proof. We first note that since a separating curve is trivial in $H_1(S; \mathbb{Z})$, any two have even geometric intersection number. Two distinct separating curves with intersection number zero clearly induce different splittings of $H_1(S; \mathbb{Z})$, so it is impossible to have $i(c, f(c)) = 0$. Any two separating curves with intersection number 2 are, after composing with a homeomorphism, given as in Figure 1. Again we see that they induce different homology splittings, and so we cannot have $i(c, f(c)) = 2$. □

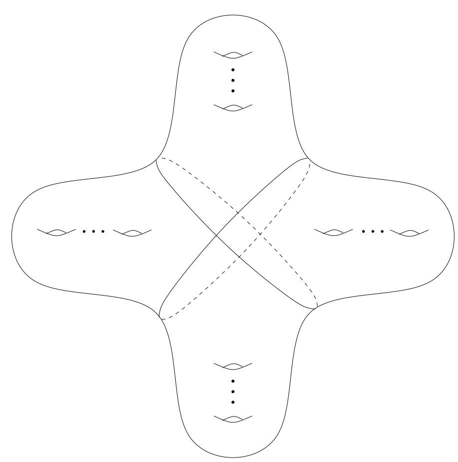

Figure 1: A pair of separating curves which intersect twice.

Lemma 2.3. *Suppose that $f \in \mathcal{I}(S)$, that c is a nonseparating curve, and that $f(c) \neq c$. Then at least one of $i(f(c), c)$ and $i(f^2(c), c)$ is at least 2.*

Note that this lemma is sharp for $g \geq 3$ in the sense that there exists an element $f \in \mathcal{I}(S)$ and a curve c so that $i(c, f(c)) = 0$ and $i(c, f^2(c)) = 2$. Consider, for instance, a *bounding pair* $\{d, e\}$, i.e. a pair of disjoint, nonhomotopic, homologous, nonseparating simple closed curves, and a curve c which intersects both d and e exactly once each; in this case, $i(c, T_d T_e^{-1}(c)) = 0$ and $i(c, (T_d T_e^{-1})^2(c)) = 2$.

Proof. Since $f \in \mathcal{I}(S)$ and $c \neq f(c)$ we know that $c \neq f^2(c)$ (combine Theorem 3 and Corollary 3.7 of [Iv2]). Now suppose $i(f(c), c) = 0$ and $i(f^2(c), c) = 0$, so that $\{c, f(c), f^2(c)\}$ is a collection of 3 distinct, disjoint simple closed curves each representing a fixed nonzero element of $H_1(S; \mathbb{Z})$.

We can choose curves $u_2, v_2, \ldots, u_g, v_g$ which are each disjoint from c, $f(c)$, and $f^2(c)$, and whose corresponding homology classes span a codimension 2 subspace V of $H_1(S; \mathbb{Z})$. Now, f takes the pair $\{c, f(c)\}$ to the pair $\{f(c), f^2(c)\}$. However, it is clear that these two pairs induce different splittings of V, and so we have a contradiction (see Figure 2). Since $f \in \mathcal{I}(S)$ we cannot have $i(f(c), c) = 1$ or $i(f^2(c), c) = 1$, so we are done. □

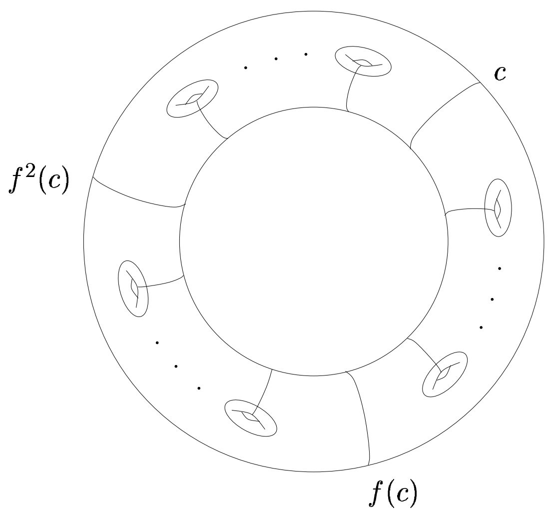

Figure 2: The picture for Lemma 2.3. The unlabeled curves are the u_i and v_i.

Lemma 2.2 and Lemma 2.3 together prove Proposition 1.4, since a pseudo-Anosov mapping class does not fix any curve in the surface.

2.2 Proof of the lower bound

We begin by recalling a few definitions and facts; see [Ab] for a more detailed discussion. If f is pseudo-Anosov, we let $q = q_f$ denote a holomorphic quadratic differential for which the vertical and horizontal foliations are precisely the stable and unstable foliations for f, respectively. The differential q determines a euclidean cone metric, which we also denote q, and f acts as an affine diffeomorphism (off the singularities) whose derivative has eigenvalues $\lambda(f)$ and $\lambda(f)^{-1}$.

For any curve c in S, we let $\ell_q(c)$ denote the infimum of q-lengths of representatives of c, which is equivalently the length of a q-geodesic representative for c. We note that, in general, the q-geodesic representative of a simple closed curve need not be embedded.

Lemma 2.4. *Let $f \in \mathrm{Mod}(S)$ be pseudo-Anosov and let $q = q_f$. Then for any closed curve c in S, we have*

$$\frac{\ell_q(f(c))}{\ell_q(c)} < \lambda(f)$$

Proof. Note that because f is affine with respect to q, the image of a geodesic representative for c is a geodesic representative for $f(c)$. Furthermore, since the leading eigenvalue of the derivative of f is $\lambda = \lambda(f)$, the length of the curve $f(c)$ differs from that of c by at most a factor of λ. Moreover, only geodesics which are everywhere tangent to the eigenspace for λ can be maximally stretched. However any such geodesic is a leaf of the stable or unstable foliation, and hence cannot be part of a closed geodesic, so the inequality is strict. □

We are now ready to give the proof of the lower bound on $L(\mathcal{I}(S))$ given in Theorem 1.1.

Proposition 2.5. *If $g \geq 2$, then $L(\mathcal{I}(S_g)) > .197$.*

Proof. Let $q = q_f$, and let c be a shortest curve in S with respect to q. We will assume in what follows that all closed q-geodesics under consideration are embedded and that all pairs of q-geodesics are in minimal position. This is not true in general, but so as not to disrupt the flow of ideas we make this assumption. We will discuss the minor modifications needed for the general case at the end of the proof.

Case 1. $i(c, f(c)) \geq 4$ or $i(c, f^2(c)) \geq 4$.

Let h be either f or f^2, where $i(c, h(c)) \geq 4$. The intersection points $c \cap h(c)$ cut each of c and $h(c)$ into arcs. Since there are at least 4 intersection points, there is an arc a of $h(c)$ which satisfies

$$\ell_q(a) \leq \frac{\ell_q(h(c))}{4} < \frac{\lambda(h)\ell_q(c)}{4}$$

where the second inequality comes from an application of Lemma 2.4. Here we have written $\ell_q(a)$ to denote the q-length of the segment a. The endpoints of a cut c into two arcs. One of which, call it b, has length at most $\ell_q(c)/2$. The union $a \cup b$ is a simple closed curve in S. It is nontrivial for otherwise it would bound a disk, which we could use as a homotopy to show $i(c, h(c)) < |c \cap h(c)|$. Since c is a shortest curve with respect to q, we have

$$\begin{aligned} \ell_q(c) \leq \ell_q(a \cup b) &\leq \ell_q(a) + \ell_q(b) \\ &< \frac{\lambda(h)\ell_q(c)}{4} + \frac{\ell_q(c)}{2} \\ &= \ell_q(c)\left(\frac{\lambda(h)}{4} + \frac{1}{2}\right) \end{aligned}$$

It follows that

$$\frac{\lambda(h)}{4} + \frac{1}{2} > 1$$

and so $\lambda(h) > 2$. Since h is f or f^2 and since $\lambda(f^2) = \lambda(f)^2$, we have $\lambda(f) > \sqrt{2}$.

Case 2. c is nonseparating and $i(c, f(c))$ and $i(c, f^2(c))$ are both less than 4.

By Lemma 2.3, either $i(c, f(c)) = 2$ or $i(c, f^2(c)) = 2$. Let h be either f or f^2, where $i(c, h(c)) = 2$.

The intersection points $c \cap h(c)$ define 2 arcs of $h(c)$, say a and a', and two arcs of c, say b and b'. There are 4 ways to combine these arcs to form curves different from c and $h(c)$. By a basic homology argument, two of these, say d and d', are separating in S, and two are nonseparating. What is more, d and d' are "opposite" each other in the sense that if $d = a \cup b$, then $d' = a' \cup b'$. See Figure 3.

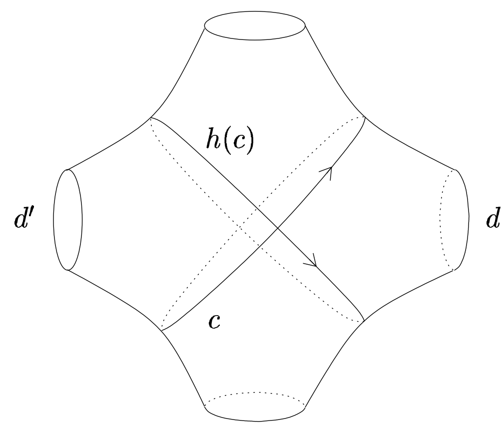

Figure 3: Homologous curves c and $h(c)$ with $i(c, h(c)) = 2$.

Since

$$\begin{aligned} \ell_q(a) + \ell_q(b) + \ell_q(a') + \ell_q(b') &= \ell_q(c) + \ell_q(h(c)) \\ &< \ell_q(c) + \lambda(h)\ell_q(c) \end{aligned}$$

it follows that at least one of d and d', say d, has length bounded above by half of $\ell_q(c) + \ell_q(h(c))$:

$$\ell_q(d) < \frac{\ell_q(c) + \lambda(h)\ell_q(c)}{2} \leq \frac{\ell_q(c) + \lambda(f)^2\ell_q(c)}{2} \tag{1}$$

We now consider d, which is a separating curve, and its image $f(d)$, which intersect in at least four points by Lemma 2.2. As in Case 1, if a is the shortest arc of $f(d)$ and b is the shortest arc of d cut off by a, then

$$\ell_q(a \cup b) < \ell_q(d)\left(\frac{\lambda(f)}{4} + \frac{1}{2}\right) \tag{2}$$

Note that we can always use f (as opposed to f^2) since d is separating.

Also, since c is shortest, we have

$$\ell_q(c) \leq \ell_q(a \cup b) \tag{3}$$

Combining (1), (2), and (3) we see that

$$\ell_q(c) < \frac{\ell_q(c) + \lambda(f)^2 \ell_q(c)}{2} \left(\frac{\lambda(f)}{4} + \frac{1}{2} \right)$$

In other words,

$$\lambda(f)^3 + 2\lambda(f)^2 + \lambda(f) - 6 > 0$$

The cubic polynomial in $\lambda(f)$ on the left has one real root, and so

$$\lambda(f) > -\frac{2}{3} + \frac{1}{3}\sqrt[3]{82 - 9\sqrt{83}} + \frac{1}{3}\sqrt[3]{82 + 9\sqrt{83}} \approx 1.218$$

approximated from below.

By Proposition 1.4, these are all cases and so, after taking the logarithms, we are done if all q-geodesics are embedded and all pairs are in minimal position.

In the general case, we approximate the q-metric on S by a non-positively curved Riemannian metric q_0 which agrees with the q-metric in the complement of a small neighborhood of the singular points. This can be done by an explicit computation; compare, e.g., [BH] or [GT]. Given any positive number $R > 0$, which is not one of the q-lengths of a curve, we can choose this approximation so that the set of curves with q-length at most R is precisely the same as the set of curves with q_0-length at most R. Moreover, given $\epsilon > 1$, we may assume that the ratio of q-length and q_0-length of any curve is between ϵ and $1/\epsilon$. In particular, since we can assume that $\lambda(f) \leq 2$, say, then we may choose q_0 so that for the finite set of curves with q_0-length at most R, we have

$$\frac{\ell_{q_0}(f(c))}{\ell_{q_0}(c)} < \lambda(f)$$

Since a geodesic representative of any simple closed curve in a non-positively curved Riemannian metric on a surface is embedded, and since any two representatives of distinct closed curves are in minimal position, choosing R sufficiently large, the above proof can be carried out verbatim. □

For convenience, we isolate the key idea involved here as it will be used again.

Proposition 2.6. *If f is a pseudo-Anosov element of* $\mathrm{Mod}(S_g)$ *with the property that $i(c, f(c)) \geq n \geq 3$ for every simple closed curve c, then*

$$\log(\lambda(f)) > \log\left(\frac{n}{2}\right)$$

Proof. As in the proof above, fix the metric $q = q_f$ on S, and let c be a shortest curve in S with respect to q. We again assume geodesics are embedded and pairs are in minimal position, with the general case handled as above. Take the shortest segment a of $f(c)$ cut by c (which is one of $i(c, f(c))$ segments of $f(c)$), and the shortest segment b of c cut by a, and we obtain

$$\ell_q(c) \leq \ell_q(a \cup b) \leq \frac{\ell_q(f(c))}{i(c, f(c))} + \frac{\ell_q(c)}{2} < \frac{\lambda(f)\ell_q(c)}{i(c, f(c))} + \frac{\ell_q(c)}{2}$$

Dividing the left and right by $\ell_q(c)$, and simplifying and taking logarithms, we obtain

$$\log(\lambda(f)) > \log\left(\frac{i(c, f(c))}{2}\right) \geq \log\left(\frac{n}{2}\right)$$

□

Remark. Wolpert [Wo] has shown that a K-quasiconformal map f of S with respect to a hyperbolic metric X distorts lengths in X by a factor of at most K. That is, $\ell_X(f(c))/\ell_X(c) < K$, where $\ell_X(c)$ is the length of c with respect to X. The dilatation λ is related to the minimal quasiconformal distortion K of a pseudo-Anosov homeomorphism (over all hyperbolic metrics) by the equation $\lambda = \sqrt{K}$. In a previous version we used this result and the same argument above to produce a lower bound of .197 for $\log(\lambda^2)$. J. Franks suggested using the quadratic differential metric, thus improving the lower bound by a factor of 2.

2.3 Examples with small dilatation

In this section we give an upper bound for $L(\mathcal{I}(S))$ by constructing, for every $S = S_g$ ($g \geq 2$), an element $f \in \mathcal{I}(S)$ with $\log(\lambda(f)) < 4.127$. We do this by appealing to a general construction for pseudo-Anosov mapping classes given by Thurston [Th]; we refer the reader to that paper for the notation and details of the construction.

A *multicurve* is the isotopy class of a collection of pairwise disjoint simple closed curves, and a *multitwist* is the product of Dehn twists about the curves in a multicurve.

We begin by fixing a pair of multicurves $A = a_1 \cup \cdots \cup a_{\lceil g/2 \rceil}$ and $B = b_1 \cup \cdots \cup b_{\lceil g/2 \rceil}$ in S with the following three properties:

1. $A \cup B$ fills S.
2. $i(a_i, b_i) = i(a_i, b_{i-1}) = 4$ and $i(a_i, b_j) = 0$ otherwise (indices taken modulo $\lceil g/2 \rceil$).
3. Each a_i and b_j is a separating curve.

We can construct such an A and B explicitly as follows. Start with a sphere with $2g+2$ marked points arranged symmetrically as in Figure 4; the arrangement depends on whether g is odd (on the left) or even (on the right—there is one more marked point "in back"). Let $\bar{A} = \cup \bar{a}_i$ and $\bar{B} = \cup \bar{b}_i$ be multicurves in the marked sphere as shown, and let S be the two-fold cover, branched over the marked points, with A and B the preimages of $\bar{A}$ and $\bar{B}$, respectively. Since each component of $\bar{A}$ and $\bar{B}$ surrounds exactly three marked points, each component of A and B is separating; in fact it bounds a genus 1 subsurface.

Next, we consider the matrix $N_{ij} = i(a_i, b_j)$, and compute the matrix NN^t. This has entries given by

$$(NN^t)_{ij} = \sum_{k=1}^{\lceil g/2 \rceil} i(a_i, b_k) i(a_j, b_k)$$

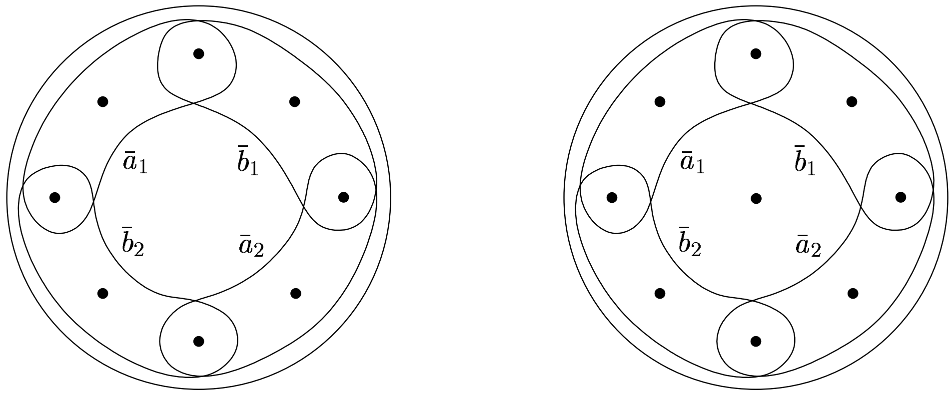

Figure 4: $\bar{A}$ and $\bar{B}$ for $g = 3$ (left) and $g = 4$ (right; one marked point is "in back").

and the above description of intersection numbers easily implies that for i and j modulo $\lceil g/2 \rceil$ we have

$$(NN^t)_{ij} = \begin{cases} 32 & \text{for} \quad i = j \\ 16 & \text{for} \quad |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

In particular, note that the row sum of any row is 64. It follows that the Perron–Frobenius eigenvalue is 64: take as an eigenvector the vector with all entries equal to 1.

Now let T_A denote the multitwist which is the composition of the Dehn twists about each of the a_i and T_B the composition of Dehn twists about each of the b_j. Thurston constructed a homomorphism $\langle T_A, T_B \rangle \to \mathrm{PSL}_2(\mathbb{R})$ given (in this case) by

$$T_A \mapsto \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T_B \mapsto \begin{pmatrix} 1 & 0 \\ -8 & 1 \end{pmatrix}$$

Remark. In the general case of Thurston's construction, the off-diagonal entries are given by the square root of the Perron–Frobenius eigenvalue of NN^t.

Thurston proved that any element of $\langle T_A, T_B \rangle$ which maps to a hyperbolic element of $\mathrm{PSL}_2(\mathbb{R})$ is pseudo-Anosov. Moreover, he proved that the dilatation of such an element is given by the absolute value of the leading eigenvalue of its image.

In our case, a direct computation gives that the mapping class $f = T_A T_B$ maps to a matrix with trace $= -62$. It follows that f is pseudo-Anosov, and that $\lambda = \lambda(f)$ satisfies

$$\lambda^2 - 62\lambda + 1 = 0$$

Solving for the largest root, we find $\log(\lambda) < 4.127$, as required. We have thus proven the upper bound of Theorem 1.1.

Proposition 2.7. *If $g \geq 2$, then $L(\mathcal{I}(S_g)) < 4.127$.*

We mention that $T_A T_B$ has the smallest dilatation among all pseudo-Anosov elements of $\langle T_A, T_B \rangle$; see [Le].

This construction also provides a universal upper bound for $L(PB_n)$.

Theorem 2.8. *For all $n \geq 3$, we have*

$$1.443 < L(PB_n) < 2.634$$

Proof. As mentioned in the introduction, Song [So] proved the lower bound $1.443 \approx \log(2+\sqrt{5})$. To prove the upper bound, consider the sphere with marked points which we described above. We can puncture every marked point and turn one puncture into a boundary component, making the surface into a $(2g+1)$-times punctured disk. Then $T_{\bar{A}} T_{\bar{B}}$ represents a pseudo-Anosov braid in PB_{2g+1}. Indeed, we can use the same method of Thurston described above to find the homomorphism $\langle T_{\bar{A}}, T_{\bar{B}} \rangle \to \mathrm{PSL}_2(\mathbb{R})$. The eigenvalue for the analogous NN^t matrix is 16, and so by a calculuation, we obtain the upper bound

$$\log(\lambda(T_{\bar{A}} T_{\bar{B}})) < 2.634$$

Whenever a marked point is not contained in a bigon, we can erase the marking, and $\bar{A}$ and $\bar{B}$, as drawn, are still in minimal position. Puncturing the remaining marked points and turning one puncture into a boundary component, we can obtain examples proving the upper bound $L(PB_n) < 2.634$ for all for $n \geq 3$. □

2.4 Principal congruence subgroups

We now give the proof of Theorem 1.7, which states that the bounds given in Propositions 2.5 and 2.7 for $L(\mathcal{I}(S))$ can be extended to $\mathrm{Mod}(S)[r]$ when $r \geq 3$.

Question 2.9. *Is it true that $L(\mathrm{Mod}(S_g)[2]) \asymp 1$?*

Proof of Theorem 1.7. Since $\mathcal{I}(S) < \mathrm{Mod}(S)[r]$, the upper bound is immediate. For $r \geq 4$, the proof of the lower bound is essentially the same as that of Proposition 2.5; all that needs to be verified is that Lemmas 2.2 and 2.3 hold under the weaker hypothesis that $f \in \mathrm{Mod}(S)[r]$, $r \geq 4$. Indeed, the same arguments work, using splittings of $H_1(S; \mathbb{Z}/r\mathbb{Z})$ (and its quotients) in place of $H_1(S; \mathbb{Z})$.

For the case $r = 3$, the proof is the same as in the $r \geq 4$ case, except we need to include the possibilities $i(c, f(c)) = 3$ and $i(c, f^2(c)) = 3$ in Case 1 of Proposition 2.5. By Proposition 2.6, the lower bound for this case becomes $.202 > .197$, and the argument for Case 2 still gives a lower bound of .197, so we are done. □

It follows from the discreteness of $\mathrm{spec}(\mathrm{Mod}(S))$, and the fact that $\mathcal{I}(S) < \mathrm{Mod}(S)[r]$, that there is a (minimal) $r = r(g)$ such that $L(\mathrm{Mod}(S_g)[n]) = L(\mathcal{I}(S_g))$ whenever $n \geq r$ (see the proof of Proposition 4.1).

Question 2.10. *What are the values of $r(g)$? What are the asymptotics of $r(g)$?*

3 The Johnson kernel

Johnson [Jo1] proved that $\mathcal{K}(S_g)$ is an infinite index subgroup of $\mathcal{I}(S_g)$ for $g \geq 3$ (when $g = 2$, the two groups agree). We have $L(\mathcal{K}(S)) \geq L(\mathcal{I}(S))$ since $\mathcal{K}(S) < \mathcal{I}(S)$, and it is natural to ask the following.

Question 3.1. *Is $L(\mathcal{K}(S_g)) > L(\mathcal{I}(S_g))$ for $g \geq 3$?*

While we do not know the answer to this question, we are able to give a better lower bound for $L(\mathcal{K}(S))$ than we did for $L(\mathcal{I}(S))$ in Theorem 1.1. As with $\mathcal{I}(S)$, the key is to understand how elements of $\mathcal{K}(S)$ act on curves.

3.1 $\mathcal{K}(S)$ and geometric intersection numbers

The conclusions of Lemmas 2.2 and 2.3 can be improved by assuming $f \in \mathcal{K}(S)$.

Proposition 3.2. *For $f \in \mathcal{K}(S)$, and any curve c, if $c \neq f(c)$, then $i(c, f(c)) \geq 4$.*

The proposition is sharp, since for $g \geq 2$ one can find a curve c and a separating curve d with $i(c, d) = 2$, and in this case $i(T_d(c), c) = 4$.

Proof. When c is separating the proposition was already proven in Lemma 2.2 for any $f \in \mathcal{I}(S)$. So assume that c is nonseparating. Since $f(c)$ is homologous to c, it suffices to rule out $i(c, f(c)) = 0$ and $i(c, f(c)) = 2$. As $\mathcal{K}(S)$ is normal in $\mathrm{Mod}(S)$, the mapping class

$$T_c f T_c^{-1} f^{-1} = T_c T_{f(c)}^{-1}$$

must lie in $\mathcal{K}(S)$. The proposition now follows from Lemma 3.3 below. □

Lemma 3.3. *$\mathcal{K}(S)$ contains no elements of the form $T_c T_d^{-1}$ where c and d are distinct homologous curves with $i(c, d)$ either 0 or 2.*

Proof. Johnson [Jo1] constructed a homomorphism $\tau : \mathcal{I}(S) \to (\wedge^3 H)/(\langle \omega \rangle \wedge H)$, where $H = H_1(S; \mathbb{Z})$ and ω is the symplectic intersection pairing, and he proved that the kernel is exactly $\mathcal{K}(S)$. Moreover, Johnson [Jo1, Corollary to Lemma 4B] gave an explicit formula for the τ-image of a bounding pair map:

$$\tau(T_a T_b^{-1}) = \left(\sum_{i=1}^{k} u_i \wedge v_i \right) \wedge [a] \tag{4}$$

Here, $[a]$ denotes the homology class of a and b and $u_1, v_1 \ldots, u_k, v_k$ is any symplectic basis for $H_1(R; \mathbb{Z})/\langle [a] \rangle$, where R is the component of $S - (a \cup b)$ not containing the base point for $\pi_1(S)$ (the orientation of $[a]$ is chosen so that R lies on the left of the curve a).

It immediately follows that $\mathcal{K}(S)$ contains no bounding pair maps, and hence it remains to show that $\mathcal{K}(S)$ contains no elements of the form $T_c T_d^{-1}$ where c and d are homologous curves with $i(c, d) = 2$. In this case, c and d are necessarily configured as in Figure 5. Using the notation of the picture, the lantern relation (see [De, §7g]) gives

$$T_e T_c T_d = T_x T_y T_z T_w$$

which implies

$$T_c T_d^{-1} = T_e^{-1} T_x T_y T_z T_w T_d^{-2}$$

As T_e, T_x, and T_y are elements of $\mathcal{K}(S)$, we see that $T_c T_d^{-1}$ is an element of $\mathcal{K}(S)$ if and only if $T_z T_w T_d^{-2}$ is an element of $\mathcal{K}(S)$. The latter is a product of two bounding pair maps: $(T_z T_d^{-1})(T_w T_d^{-1})$. To prove the lemma then, it suffices to check that $\tau(T_z T_d^{-1}) \neq \tau(T_d T_w^{-1})$. But this is apparent from equation (4) (consult Figure 2). □

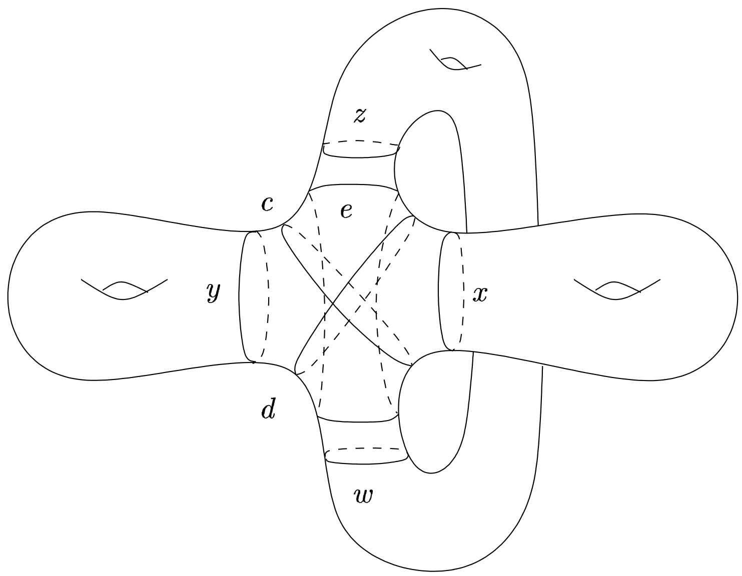

Figure 5: Homologous curves c and d with $i(c, d) = 2$. The genera of the subsurfaces bounded by x, y, and the pair $\{w, z\}$ may vary.

3.2 Bounds for $L(\mathcal{K}(S))$

We are now ready to give the following improvement for the bounds on $L(\mathcal{K}(S))$ given by Theorem 1.1.

Proposition 3.4. *For $g \geq 2$, we have*

$$.693 < L(\mathcal{K}(S_g)) < 4.127$$

Proof. The mapping class $T_A T_B$ constructed in §2.3 is a composition of Dehn twists about separating curves. Thus this mapping class already lies in $\mathcal{K}(S)$, giving the upper bound. The lower bound follows immediately from Propositions 3.2 and 2.6, as $\log(2) \approx .693$ approximated from below. □

4 The Johnson filtration

We will now prove Theorem 1.2. Both the upper and lower bounds will follow from generalized versions of our arguments for $\mathcal{I}(S)$ and $\mathcal{K}(S)$.

4.1 Asymptotic lower bounds

Before proving the lower bound in Theorem 1.2, we give the following weaker statement which holds for any *normal filtration* of $\text{Mod}(S)$, by which we mean a filtration of $\text{Mod}(S)$ by normal subgroups.

Proposition 4.1. *For any normal filtration* $N_1 > N_2 > \cdots$ *of* $\text{Mod}(S)$*, we have* $L(N_k) \to \infty$ *as* $k \to \infty$*.*

Proof. Given $M > 0$, there are only finitely many conjugacy classes of pseudo-Anosov mapping classes f with $\lambda(f) \leq M$ (see [Iv1]). The proposition then follows from the definition of a normal filtration. □

The first step towards the proof of Theorem 1.2 is to generalize Lemma 3.3. In the proof of this lemma, it was essential that there were unique pictures for homologous curves with geometric intersection number 0 or 2. We are forced to replace this precise description of how our two curves sit in S with a rough finiteness statement.

A *configuration* is a triple (S, c, d), where S is a closed surface, and c and d are distinct curves in S which are in minimal position (i.e. their union does not bound any bigon). Let $N = N(c, d)$ denote a closed regular neighborhood of $c \cup d$. There is a natural partial ordering on configurations where $(\hat{S}, \hat{c}, \hat{d}) < (S, c, d)$ if $\hat{S} \ncong S$ and there is a continuous map $\eta : (S, c, d) \to (\hat{S}, \hat{c}, \hat{d})$ which restricts to a homeomorphism of triples

$$\eta|_{N(c,d)} : (N(c, d), c, d) \to (N(\hat{c}, \hat{d}), \hat{c}, \hat{d})$$

We call such a map η a *crushing map.* Because the composition of crushing maps is a crushing map, $<$ is a partial order. Any minimal configuration with respect to this partial ordering is called a *terminal configuration.* We declare two configurations (S, c, d) and $(\hat{S}, \hat{c}, \hat{d})$ to be equal provided they are homeomorphic as triples.

Lemma 4.2. *For every* $n > 0$ *there are only finitely many terminal configurations* (S, c, d) *with* $i(c, d) \leq n$*.*

Proof. To begin, we show that the topology of the complement of $N(c, d)$ for any terminal configuration (S, c, d) is limited.

Claim: If (S, c, d) is terminal then every complementary component U of $S - N(c, d)$ has genus at most one.

Proof of claim: Suppose (S, c, d) is a configuration, and some component U has genus at least two. We produce a crushing map $\eta : (S, c, d) \to (\hat{S}, \hat{c}, \hat{d})$ as follows. Let $\eta : S \to \hat{S}$ be the quotient of S obtained by first fixing a compact genus 1 subsurface $R \subset U \subset S$ with exactly one boundary component and identifying ("crushing") R to a point. We let $\hat{c} = \eta(c)$ and $\hat{d} = \eta(d)$ and note that the restriction of η to $N(c, d)$ is a homeomorphism onto $N(\hat{c}, \hat{d})$. To prove that η is a crushing map, all that remains is to verify that $(\hat{S}, \hat{c}, \hat{d})$ is indeed a configuration. That is, we must check that $\hat{c}$ and $\hat{d}$ are essential, not homotopic to one another, and in minimal position. For this, it suffices to verify that no component of the complement of $N(\hat{c},\hat{d})$ is a disk or annulus if $\hat{c}\cap\hat{d}=\emptyset$ or a bigon if $\hat{c}\cap\hat{d}\neq\emptyset$. However, the components of the complement of $N(\hat{c},\hat{d})$ are all homeomorphic to those of $N(c,d)$ with the exception of $\eta(U)$, and since c and d are essential, homotopically distinct, and in minimal position, it suffices to verify this statement for the single component $\eta(U)$. By construction, $\eta(U)$ has genus at least one, so it is not a disk, annulus, or bigon, and therefore $(\hat{S},\hat{c},\hat{d})$ is a configuration. It follows that $(\hat{S},\hat{c},\hat{d}) < (S,c,d)$, and (S,c,d) is not terminal, proving the claim.

Now let (S,c,d) be a terminal configuration with $N = N(c,d)$. By Lemma 2.1, we have $\chi(N) = -i(c,d) \geq -n$, and so there are finitely many possibilities for N, up to homeomorphism. Since c is a curve in N, there are only finitely many possibilities for the homeomorphism type of (N,c). Further, since c cuts d into $i(c,d)\leq n$ arcs, it follows that there are only finitely many possibilities for the homeomorphism type of (N,c,d).

Now note that there are only finitely many possibilities for the number of boundary components of N, and hence finitely many possibilities for the number of boundary components of $\overline{S-N}$. Because each component of $\overline{S-N}$ has genus at most 1, there are only finitely many possibilities for the homeomorphism type of $\overline{S-N}$. Finally, the homeomorphism type of (S,c,d) can be specified by $\overline{S-N}$ and (N,c,d) and the (finite) combinatorial gluing data matching boundary components of the former with those of the latter. □

The next lemma allows us to say that, given a particular terminal configuration (S,c,d), it is not possible to "push" $T_cT_d^{-1}$ further down the Johnson filtration by adding genus to S outside $N(c,d)$. The proof applies in a much more general context, so we state it in this generality.

Suppose we are given a map of pairs $\eta:(S,N)\to(\hat{S},\hat{N})$ where $N\subset S$ and $\hat{N}\subset\hat{S}$ are subsurfaces and the restriction $\eta|_N : N\to\hat{N}$ is a homeomorphism; for example η might be a crushing map. Then any $f\in\mathrm{Mod}(S)$ which is supported in N pushes forward via η to an element $\hat{f}\in\mathrm{Mod}(\hat{S})$ supported in $\hat{N}$ given by $f|_{\hat{N}} = \eta|_N\circ f|_N\circ\eta|_N^{-1}$.

Lemma 4.3. *Suppose $\eta:(S,N)\to(\hat{S},\hat{N})$, $f\in\mathrm{Mod}(S)$, and $\hat{f}\in\mathrm{Mod}(\hat{S})$ are as above. Then $\hat{f}\in\mathcal{N}_k(\hat{S})$ whenever $f\in\mathcal{N}_k(S)$.*

Proof. Suppose that $f\in\mathcal{N}_k(S)$, i.e. after picking a representative of f and fixing a base point, the induced action $f_\star$ of f on Γ/Γ_k is inner (where $\Gamma=\pi_1(S)$ and Γ_k is the k^{th} term of its lower central series, as above).

Let $\hat{\Gamma}=\pi_1(\hat{S})$, and denote by $\{\hat{\Gamma}_i\}$ its lower central series. The map η induces a surjective homomorphism $\Gamma\to\hat{\Gamma}$ which restricts to a surjection $\Gamma_k\to\hat{\Gamma}_k$, so we have an induced map $\eta_\star:\Gamma/\Gamma_k\to\hat{\Gamma}/\hat{\Gamma}_k$. Finally, let $\hat{f}_\star$ be the induced action of $\hat{f}$ on $\hat{\Gamma}/\hat{\Gamma}_k$. We encode this information in the following diagram

$$\begin{array}{ccc} \Gamma/\Gamma_k & \xrightarrow{f_\star} & \Gamma/\Gamma_k \\ \downarrow{\scriptstyle \eta_\star} & & \downarrow{\scriptstyle \eta_\star} \\ \hat{\Gamma}/\hat{\Gamma}_k & \xrightarrow{\hat{f}_\star} & \hat{\Gamma}/\hat{\Gamma}_k \end{array}$$

The diagram is commutative by the definition of $\hat{f}_\star$, which implies that $\hat{f}_\star$ is also inner; indeed, if $f_\star$ is conjugation by γ, then $\hat{f}_\star$ is conjugation by $\eta_\star(\gamma)$. Therefore, $\hat{f} \in \mathcal{N}_k(\hat{S})$. $\square$

We now arrive at the desired generalization of Lemma 3.3. Define

$$C(n) = 1 + \sup\{k \mid (S, c, d) \text{ is terminal, } i(c, d) \le n, \text{ and } T_c T_d^{-1} \in \mathcal{N}_k(S)\}$$

By Lemma 4.2 and the definition of a normal filtration, $C(n)$ is finite for each n.

Lemma 4.4. *Let $n > 0$. If $c \neq d$ and $i(c, d) \le n$, then $T_c T_d^{-1} \notin \mathcal{N}_{C(n)}(S)$. Furthermore, $\lim_{n\to\infty} C(n) = \infty$.*

Proof. Suppose c and d are curves in S with $i(c, d) \le n$ and $T_c T_d^{-1} \in \mathcal{N}_k(S)$. If (S, c, d) is a terminal configuration, then by the definition of $C(n)$, we see that $k < C(n)$. If (S, c, d) is not terminal, then there is a terminal configuration $(\hat{S}, \hat{c}, \hat{d})$ and crushing map $\eta : (S, c, d) \to (\hat{S}, \hat{c}, \hat{d})$. The induced map of pairs $\eta : (S, N(c, d)) \to (\hat{S}, N(\hat{c}, \hat{d}))$ and the map $T_c T_d^{-1}$ satisfy the hypothesis of Lemma 4.3 with

$$\widehat{T_c T_d^{-1}} = T_{\hat{c}} T_{\hat{d}}^{-1}$$

so $T_{\hat{c}} T_{\hat{d}}^{-1} \in \mathcal{N}_k(\hat{S})$, which again implies $k < C(n)$ as required.

To complete the proof and show that $C(n) \to \infty$, we first notice that $C(n)$ is nondecreasing, since the set of terminal configurations used to define $C(n)$ contains the set of configurations used to define $C(n-1)$. Thus, it suffices to show that for any k there exists a c and d such that $T_c T_d^{-1} \in \mathcal{N}_k(S)$ for some S. Let $f \in \mathcal{N}_k(S)$ be any nontrivial element and c any curve with $f(c) \neq c$. Since $\mathcal{N}_k(S) \lhd \mathrm{Mod}(S)$, it follows that $[T_c, f] = T_c T_{f(c)}^{-1}$ is an element of $\mathcal{N}_k(S)$. $\square$

We are finally ready to give the "asymptotic version" of Proposition 3.2. For the statement, define

$$B(k) = \sup\{n : C(n) \le k\} + 1$$

In the case where B is not defined by this equation, we artificially set $B = 0$ (B is not defined for any integer which is smaller than the smallest value of C). Note that $B(k)$ is well-defined and finite for each $k \ge 0$ since C is unbounded and nondecreasing. We require the following alternate characterization of $B(k)$, which follows immediately from the definition.

Lemma 4.5. *$B(k)$ is the smallest integer valued function for which $C(B(k)) > k$ for all k.*

The next proposition gives the desired generalization of Proposition 3.2.

Proposition 4.6. *Let $k > 0$ and S any surface. If $f \in \mathcal{N}_k(S)$, then $i(c, f(c)) \ge B(k)$ for every simple closed curve c in S with $f(c) \neq c$. Moreover, $\lim_{k\to\infty} B(k) = \infty$.*

Proof. First, $B(k) \to \infty$ as $k \to \infty$ since C is unbounded and nondecreasing. Now, given k, choose any S, any $f \in \mathcal{N}_k(S)$, and any simple closed curve c in S with $f(c) \neq c$. Since $\mathcal{N}_k(S)$ is normal in $\mathrm{Mod}(S)$, we have $[T_c, f] = T_c T_{f(c)}^{-1} \in \mathcal{N}_k(S)$. By Lemma 4.5, if $i(c, f(c)) < B(k)$, then $C(i(c, f(c))) \le k$. By Lemma 4.4, we have $T_c T_{f(c)}^{-1} \notin \mathcal{N}_k(S)$, which is a contradiction. $\square$

Using Propositions 4.6 and 2.6, it is now straightforward to prove the lower bound of Theorem 1.2.

Proof of Theorem 1.2. If $f \in \mathcal{N}_k(S)$ is pseudo-Anosov, then $i(c, f(c)) \geq B(k)$ for every curve c, by Proposition 4.6 and the fact that a pseudo-Anosov mapping class does not fix any curve. We set

$$m(k) = \log\left(\frac{B(k)}{2}\right)$$

By Proposition 2.6, $\log(\lambda(f)) > m(k)$, and so this completes the proof. □

Among several questions which now arise, we pose the following.

Question 4.7. *What are the asymptotics of $B(k)$? What are the asymptotics of $L(\mathcal{N}_k(S))$?*

4.2 Asymptotic upper bounds

We now prove the upper bound in Theorem 1.2.

Proposition 4.8. *For any $k \geq 1$, there is an $M(k)$ so that $L(\mathcal{N}_k(S_g)) < M(k)$, for all $g \geq 2$.*

Since $m(k) \to \infty$, it follows that $M(k) \to \infty$ as $k \to \infty$.

Proof. Let $k \geq 1$ be fixed. To prove the proposition, we need to find a pseudo-Anosov mapping class $f \in \mathcal{N}_k(S)$ whose dilatation depends on k, but not on S. We begin by recalling that since $\{\mathcal{N}_i(S)\}_{i\geq 1}$ is a central series for $\mathcal{I}(S)$, then the $(k-1)^{\text{st}}$ term of the lower central series of $\mathcal{I}(S)$ is contained in $\mathcal{N}_k(S)$; note that $\mathcal{N}_1(S) = \mathcal{I}(S)$ is the zero term of the lower central series.

Now, without specifying a particular surface S, we consider the group $\langle T_A, T_B\rangle$ generated by the multitwists T_A and T_B of Section 2.3. The group $\langle T_A, T_B\rangle$ is a free group on the given generators (see [Le, §6.1] for a discussion). Therefore, there is a nontrivial element f in the $(k-1)^{\text{st}}$ term of the lower central series of $\langle T_A, T_B\rangle$. Since T_A and T_B are both elements of $\mathcal{I}(S)$, it follows that f is an element of the $(k-1)^{\text{st}}$ term of the lower central series of $\mathcal{I}(S)$, and hence $f \in \mathcal{N}_k(S)$.

The key feature here is this: the image of f in $\mathrm{PSL}_2(\mathbb{R})$ does not depend of the choice of S. This is because f was chosen independently of S as a word in T_A and T_B, and the images of T_A and T_B in $\mathrm{PSL}_2(\mathbb{R})$ do not depend on the choice of S (see Section 2.3).

Since $\langle T_A, T_B\rangle$ is a free group and the only elements of this group which are not pseudo-Anosov are conjugates of T_A and T_B (see, e.g., [Le]), it follows that f is pseudo-Anosov. Since its dilatation only depends on its image in $\mathrm{PSL}_2(\mathbb{R})$, and the latter is independent of the choice of S, we are done. □

Remark. Note that the word in T_A and T_B given as a simple nested commutator has word length on the order of 2^k, where k is the number of nested commutators involved (i.e. the depth in the lower central series). Thus the order of the translation length of the word acting on the Teichmüller space is at most exponential in k.

5 Translation lengths on the complex of curves

Our goal in this section is to prove Theorems 1.5 and 1.6. These will follow rather quickly from the following theorem.

Theorem 5.1. *For any $g \geq 2$ and any pseudo-Anosov $f \in \text{Mod}(S_g)$ with $\lambda(f) \leq g - 1/2$, we have*

$$\tau_{\mathcal{C}}(f) < \frac{4\log(\lambda(f))}{\log\left(g - \frac{1}{2}\right)}$$

We remark that it seems likely that the hypothesis $\lambda(f) \leq g - 1/2$ is not necessary, but it is required for our argument. The proof of Theorem 5.1 is an application of Proposition 2.6.

Proof. Let n be the smallest integer so that $2 < n\tau_{\mathcal{C}}(f)$. Note that $n\tau_{\mathcal{C}}(f) \leq 4$ whenever $n > 1$.

Let c be any curve on S. We now claim that $d_{\mathcal{C}}(f^n(c), c) \geq 3$, which (by the definition of $d_{\mathcal{C}}$) would imply that c and $f^n(c)$ fill S. Indeed, if not, we have $d_{\mathcal{C}}(f^n(c), c) \leq 2$, so by the triangle inequality $d_{\mathcal{C}}(f^{nj}(c), c) \leq 2j$. Dividing both sides by nj and taking the lim inf, we get

$$\liminf_{j\to\infty} \frac{d_{\mathcal{C}}(f^{nj}(c), c)}{nj} \leq \frac{2}{n}$$

The lim inf used to define $\tau_{\mathcal{C}}(f)$ is no larger than the left hand side, and so we arrive at $n\tau_{\mathcal{C}}(f) \leq 2$, a contradiction.

Lemma 2.1 implies $i(c, f^n(c)) \geq 2g - 1$, and hence Proposition 2.6 applied to f^n says

$$n\log(\lambda(f)) = \log(\lambda(f^n)) > \log\left(g - \frac{1}{2}\right)$$

which we write as

$$\frac{1}{n} < \frac{\log(\lambda(f))}{\log\left(g - \frac{1}{2}\right)}$$

By hypothesis, the right hand side is at most 1, and so $n > 1$. As mentioned above, this means that $n\tau_{\mathcal{C}}(f) \leq 4$. Thus, we have

$$\tau_{\mathcal{C}}(f) \leq \frac{4}{n} < \frac{4\log(\lambda(f))}{\log\left(g - \frac{1}{2}\right)}$$

□

We can now deduce Theorems 1.5 and 1.6 as corollaries of Theorem 5.1.

Proof of Theorem 1.5. Let $f_g \in \text{Mod}(S_g)$ be a minimal dilatation pseudo-Anosov mapping class. Hironaka–Kin [HK] showed that $\log(\lambda(f_g)) \leq \log(2 + \sqrt{3})/g$, and so if $g \geq 3$, then $\lambda(f_g) < g - 1/2$. The theorem thus follows for $g \geq 3$ from Theorem 5.1. The case of genus 2 can be handled by explicit examples. □

Proof of Theorem 1.6. For any fixed k, with $M(k)$ as in Theorem 1.2, we have $M(k) \leq \log(g - 1/2)$ for g sufficiently large. That is, for large enough g, we have some $f_g \in \mathcal{N}_k(S_g)$ with $\lambda(f_g) \leq g - 1/2$. Letting g tend to infinity, Theorem 5.1 implies

$$L_{\mathcal{C}}(\mathcal{N}_k(S_g)) \leq \tau_{\mathcal{C}}(f_g) < \frac{4\log(\lambda(f_g))}{\log\left(g - \frac{1}{2}\right)} \leq \frac{4M(k)}{\log\left(g - \frac{1}{2}\right)} \to 0$$

□

6 Brunnian subgroups

Let $S_{g,p}$ be the orientable surface of genus g with $p > 0$ punctures, and let $\mathrm{PMod}(S_{g,p})$ be the subgroup of $\mathrm{Mod}(S_{g,p})$ consisting of elements which fix each puncture. There are p natural surjective homomorphisms

$$F_i : \mathrm{PMod}(S_{g,p}) \to \mathrm{PMod}(S_{g,p-1})$$

obtained by filling in the i^{th} puncture, for $1 \leq i \leq p$. The *Brunnian subgroup* of $\mathrm{Mod}(S_{g,p})$ is the (nonempty!) intersection of the kernels:

$$\mathrm{Brun}(S_{g,p}) = \bigcap_{i=1}^{p} \ker(F_i)$$

A topological description of each F_i is given by the Birman exact sequence [Bi, Theorem 1.4].

Proof of Theorem 1.8. This is similar to the proof of the lower bound in Proposition 2.5 and comes in two parts. We begin by uniformly bounding $i(c, f(c))$ from below for any $f \in \mathrm{Brun}(S_{g,p})$ and any curve c with $f(c) \neq c$. To do this, we first note that by definition $F_i(f)(c) = c$ for every curve c and every $i = 1, ..., p$ (since $F_i(f) = 1$). In other words, if we fill in any puncture, $f(c)$ becomes isotopic to c. Therefore, the complement of $c \cup f(c)$ contains p punctured bigons, one for each puncture of $S_{g,p}$. In the present case, an endpoint of a punctured bigon can lie in at most two punctured bigons and so we have $i(c, f(c)) \geq p$.

For the second part of the proof we would like to apply Proposition 2.6. However, this is unavailable: the hypothesis of that proposition requires that the surface involved be closed. Indeed, that proof breaks down when the surface has punctures since the curve which is produced by the cut-and-paste may be *peripheral* (homotopic to a puncture), and hence has no geodesic representative.

Proposition 6.1 below is a version of Proposition 2.6 for punctured surfaces, and it completes the proof. □

Proposition 6.1. *If $f \in \mathrm{Mod}(S_{g,p})$ is pseudo-Anosov and has the property that $i(c, f(c)) \geq n \geq 5$ for every simple closed curve c, then*

$$\log(\lambda(f)) > \log\left(\frac{n}{4}\right)$$

Proof. As in the proof of Proposition 2.6, we let $q = q_f$. One must be more careful in the setting of punctured surfaces: curves need not have geodesic representatives in S since the q-metric is incomplete. It is true however that a sequence of representatives of a curve c for which the q-lengths converge to the infimum $\ell_q(c)$, converges in the completion (obtained by filling in the punctures) to a path which is geodesic, except at the completion points; see [Ab] for more. It follows that Lemma 2.4 still holds in this situation.

Approximating the q-metric by a Riemannian metric q_0 of non-positive curvature as in the proof of Proposition 2.5, we may assume that all geodesics are embedded and pairs are in minimal position. Moreover, for all sufficiently short curves (in particular, all those curves that we will encounter), we may assume that

$$\frac{\ell_{q_0}(f(c))}{\ell_{q_0}(c)} < \lambda(f)$$

We let c be a shortest (nonperipheral) curve in the q_0-metric and consider two arcs a_1 and a_2 of $f(c)$ cut along c which share an endpoint, and for which

$$\ell_{q_0}(a_1) + \ell_{q_0}(a_2) \leq 2\,\frac{\ell_{q_0}(f(c))}{i(c, f(c))}$$

Let b_1 and b_2 be the shortest arcs of c cut by a_1 and a_2, respectively. We also consider the concatenated arc $a = a_1 \cup a_2$, and let b denote the shortest arc of c cut by a.

Suppose now that $a_1 \cup b_1$, say, is not peripheral. Then as in the proof of Proposition 2.6 we obtain

$$\ell_{q_0}(c) \leq \ell_{q_0}(a_1 \cup b_1) < \frac{2\lambda(f)\ell_{q_0}(c)}{n} + \frac{\ell_{q_0}(c)}{2}$$

and hence

$$\log(\lambda(f)) > \log\left(\frac{n}{4}\right)$$

Since each of a_1, a_2, and a has length at most $2\ell_{q_0}(f(c))/i(c, f(c))$, and each of b_1, b_2, and b has length at most $\ell_{q_0}(c)/2$, we obtain the same bound if any of $a \cup b$, $a_1 \cup b_1$, or $a_2 \cup b_2$ is nonperipheral. Thus the proof will be complete if we can show that this is the case.

We label the endpoints of a_1 and a_2 as x, y and y, z, respectively (so the endpoints of a are x and z). We also orient c and $f(c)$, thus assigning signs to the intersection points of $c \cap f(c)$, and so in particular, to the points x, y,and z. Two of the signs on x, y, and z must agree. If x and y, say, have the same sign, then the curve $a_1 \cup b_1$ is nonseparating since it has geometric intersection number 1 with the curve $a_1 \cup (c - b_1)$. Therefore, $a_1 \cup b_1$ would be nonperipheral, and we would be done. Similarly, if y and z have the same sign, then $a_2 \cup b_2$ is nonseparating and hence nonperipheral. Therefore, we may assume that the signs of intersection alternate.

It follows that a regular neighborhood of $a_1 \cup a_2 \cup c = a \cup c$ is as shown in Figure 6, where we have decomposed c into three arcs $c_1 \cup c_2 \cup c_3$ by the intersection points x, y, and z.

Each of the arcs b_1, b_2, and b is made from unions of the three arcs c_1, c_2, and c_3, depending on the relative lengths of c_1, c_2, and c_3. There are three cases to consider.

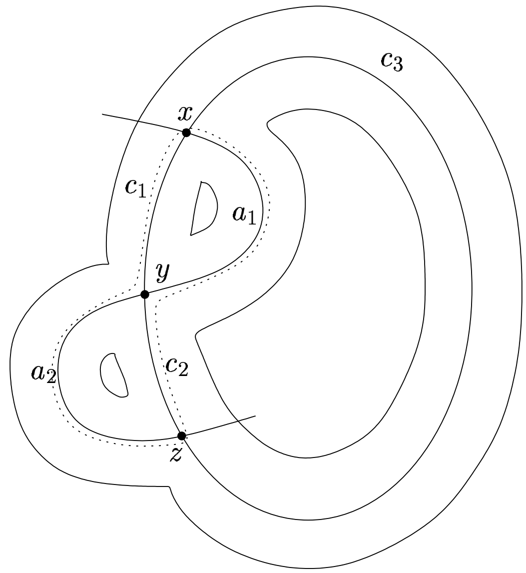

Figure 6: Neighborhood of $a_1 \cup a_2 \cup c$ with $c = c_1 \cup c_2 \cup c_3$. The dotted curve is d (Case 3).

Case 1. $\ell_{q_0}(c_i) \leq \ell_{q_0}(c)/2$ for all $i = 1, 2, 3$.

In this case, we have $b_1 = c_1$, $b_2 = c_2$, and $b = c_3$. We consider the regular neighborhood N of $a \cup c$ shown in Figure 6. We claim that the inclusion of N into S injects on the level of fundamental groups, i.e. N is *incompressible*. We recall the elementary fact that a subsurface is incompressible if and only if each of the boundary curves is homotopically nontrivial (i.e. none of the boundary curves is homotopic to a point). It follows that N is incompressible since each of the boundary components is (homotopic to) a union of two segments in c and $f(c)$ (which were in minimal position), hence homotopically nontrivial. Now note that $a \cup b$ is not peripheral in N, hence cannot be peripheral in S.

Case 2. $\ell_{q_0}(c_i) > \ell_{q_0}(c)/2$ for $i = 1$ or $i = 2$.

We consider only the situation $\ell_{q_0}(c_1) > \ell_{q_0}(c)/2$, with the proof for $\ell_{q_0}(c_2) > \ell_{q_0}(c)/2$ obtained by simply changing the labels. In this case, we have $b_1 = c_2 \cup c_3$, $b_2 = c_2$, and $b = c_3$. We now consider the regular neighborhood of N of $a \cup b_1 = a \cup c_2 \cup c_3$, which is a pair of pants. Note that $a_1 \cup b_1$, $a_2 \cup b_2$, and $a \cup b$ are all contained in N. In fact, these curves are precisely the three boundary components. Since each of these curves is a union of two segments in c and $f(c)$, these are homotopically nontrivial, and so as in Case 1, N is incompressible. Finally, if all three curves were peripheral, the complement of N in S would have to be three once-punctured disks, and hence S would be a thrice-punctured sphere. This is a contradiction since there are no pseudo-Anosov homeomorphisms of a thrice-punctured sphere. Thus, one of the curves must be nonperipheral.

Case 3. $\ell_{q_0}(c_3) > \ell_{q_0}(c)/2$.

In this final case, we must have $b_1 = c_1$, $b_2 = c_2$, and $b = c_1 \cup c_2$. Here we let N be the regular neighborhood of $a \cup b = a_1 \cup a_2 \cup c_1 \cup c_2$. Again N is a pair of pants, and it contains our three curves $a_1 \cup b_1$, $a_2 \cup b_2$, and $a \cup b$. As above $a_1 \cup b_1$ and $a_2 \cup b_2$ are homotopic to two of the three boundary components, and are both homotopically nontrivial. If we show that the third boundary component, d (the dotted curve in Figure 6), is homotopically nontrivial, then $a \cup b$, which is an immersed essential curve in N, will be nonperipheral, and this will complete the proof.

If d is homotopically trivial, then it bounds a disk D in S. Since D cannot contain the other two boundary components of N, as these are nontrivial, it follows that D must be "outside" of d in Figure 6. We orient $f(c)$ so that it passes through x, y, and z in that order. After passing through z, $f(c)$ enters D. Since $f(c)$ has no further intersection with a_1, a_2, c_1, and c_2 other than the ones shown, it must cross c_3 upon leaving D. But this creates a bigon between c and $f(c)$, contradicting our standing assumption on minimal position. It follows that d cannot be homotopically trivial, and hence N is incompressible and $a \cup b$ is nonperipheral, as required. □

We believe that a much stronger result is true; namely, that dilatations increase exponentially in the number of punctures for Brunnian pseudo-Anosov mapping classes.

Conjecture 6.2. *There exist constants $A, B > 0$ so that*

$$L(\text{Brun}(S_{g,p})) \geq Ap + B$$

for all $p \geq 1$ and any g.

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