

Teichmüller geometry of moduli space, II:

$\mathcal{M}(S)$ seen from far away

Benson Farb and Howard Masur *

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1 Introduction

Let $S = S_{g,n}$ be a closed, orientable surface with genus $g \geq 0$ with $n \geq 0$ marked points, and let $\text{Teich}(S)$ be the associated Teichmüller space of marked conformal classes or (fixed area) constant curvature metrics on S . Endow $\text{Teich}(S)$ with the Teichmüller metric $d_{\text{Teich}(S)}(\cdot, \cdot)$. Recall that for marked conformal structures $X_1, X_2 \in \text{Teich}(S)$ we define

$$d_{\text{Teich}(S)}(X_1, X_2) = \frac{1}{2} \log K$$

where $K \geq 1$ is the least number such that there is a K -quasiconformal mapping between the marked structures X_1 and X_2 . The mapping class group $\text{Mod}(S)$ acts properly discontinuously and isometrically on $\text{Teich}(S)$, thus inducing a metric $d_{\mathcal{M}(S)}(\cdot, \cdot)$ on the quotient moduli space $\mathcal{M}(S) := \text{Teich}(S)/\text{Mod}(S)$.

The goal of this paper is to build an “almost isometric” simplicial model for $\mathcal{M}(S)$, from which we will determine the tangent cone at infinity of $\mathcal{M}(S)$. In analogy with the case of locally symmetric spaces, this can be viewed as a step in building a “reduction theory” for the action of $\text{Mod}(S)$ on $\text{Teich}(S)$. Other results in this direction can be found in [Le].

Moduli space seen from far away. Gromov formalized the idea of “looking at a metric space (X, d) from far away” by introducing the notion of the *tangent cone at infinity* of (X, d) . This metric space, denoted $\text{Cone}(X)$, is defined to be a Gromov-Hausdorff limit of based metric spaces (where basepoint $x \in X$ is fixed once and for all):

$$\text{Cone}(X) := \lim_{\epsilon \rightarrow 0} (X, \epsilon d)$$

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So, for example, any compact Riemannian manifold M has $\text{Cone}(X) = *$, a one point space. Let $M = \Gamma \backslash G/K$ be an arithmetic, locally symmetric manifold (or orbifold); so G is a semisimple algebraic \mathbf{Q} -group, K a maximal compact subgroup, and Γ an arithmetic lattice. Hattori, Leuzinger and Ji-MacPherson proved that $\text{Cone}(M)$ is a metric cone over the quotient by Γ of the spherical Tits building $\Delta_{\mathbf{Q}}(G)$ associated to $G_{\mathbf{Q}}$. Here the metric on the cone on a maximal simplex of $\Delta_{\mathbf{Q}}(G)$ makes it isometric to the standard (Euclidean) metric on a Weyl chamber in G/K . In particular they deduce:

$$\mathbf{Q}\text{-rank}(\Gamma) = \dim(\text{Cone}(\Gamma \backslash G/K))$$

Our first result is a determination of the metric space $\text{Cone}(\mathcal{M}(S))$. The role of the rational Tits building will be played by the *complex of curves* $\mathcal{C}(S)$ on S . Recall that, except for some sporadic cases discussed below, the complex $\mathcal{C}(S)$ is defined to be the simplicial complex whose vertices are (isotopy classes of) simple closed curves on S , and whose k -simplices are $(k+1)$ -tuples of distinct isotopy classes which can be realized simultaneously as disjoint curves on S . Note that $\mathcal{C}(S)$ is a d -dimensional simplicial complex, where $d = 3g - 4 + n$. While $\mathcal{C}(S)$ is locally infinite, its quotient by the natural action of $\text{Mod}(S)$ is a *finite orbicomplex*, by which we mean the quotient of a finite simplicial complex by the action of a finite group. The quotient can be made a simplicial complex by looking at the action on the barycentric subdivision of $\mathcal{C}(S)$.

We now build a metric space which will serve as a coarse metric model for $\mathcal{M}(S)$. Let $\tilde{\mathcal{V}}(S)$ denote the topological cone

$$\tilde{\mathcal{V}}(S) := \frac{[0, \infty) \times \mathcal{C}(S)}{\{0\} \times \mathcal{C}(S)}$$

For each maximal simplex σ of $\mathcal{C}(S)$, we will think of the cone over σ as an orthant with coordinates (x_1, \dots, x_d) . We endow this orthant with the standard sup metric:

$$d((x_1, \dots, x_d), (y_1, \dots, y_d)) := \frac{1}{2} \max_{1 \leq i \leq d} |x_i - y_i|$$

The factor of $\frac{1}{2}$ is designed to be consistent with the definition of the Teichmüller metric.

The metrics on the cones on any two such maximal simplices clearly agrees on (the cone on) any common face. We can thus endow $\tilde{\mathcal{V}}(S)$ with the corresponding path metric. Note that the natural action of $\text{Mod}(S)$ on $\tilde{\mathcal{V}}(S)$ induces an isometric action of $\text{Mod}(S)$ on $\tilde{\mathcal{V}}(S)$. The quotient

$$\mathcal{V}(S) := \tilde{\mathcal{V}}(S)/\text{Mod}(S)$$

thus inherits a well-defined metric. The example $\mathcal{V}(S_{1,2})$ is described in Figure 1. To endow $\mathcal{V}(S)$ with the structure of a simplicial complex instead of an orbicomplex, we can simply replace $\mathcal{C}(S)$ with its barycentric subdivision in the construction above.

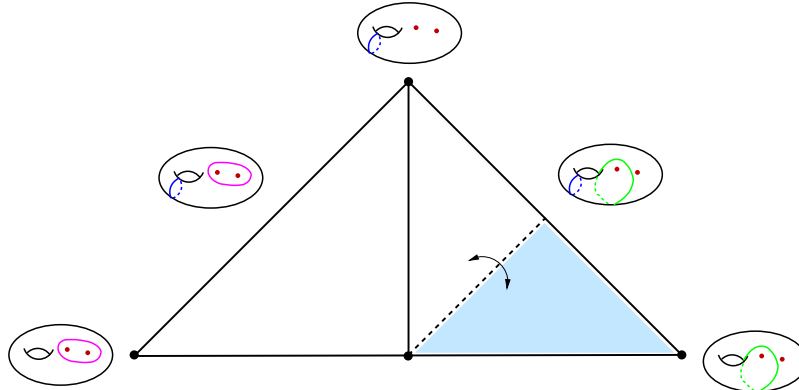


Figure 1: The metric space $\mathcal{V}(S_{1,2})$. The fundamental domain for the action of $\text{Mod}(S)$ on $\mathcal{C}(S)$ is the union of two edges, one corresponding to a separating/nonseparating pair of curves, the other to a nonseparating/nonseparating pair. Note that the latter edge has an order two symmetry, corresponding to the mapping class which switches the curves. Thus $\mathcal{V}(S)$ is the union of a Euclidean quadrant and a quotient of a Euclidean quadrant by a reflection along the $y = x$ ray.

Our first result is that $\mathcal{V}(S)$ provides a simple and reasonably accurate geometric model for $\mathcal{M}(S)$.

Theorem 1. *There is a continuous map $\Psi : \mathcal{V}(S) \rightarrow \mathcal{M}(S)$ which is a $(1, D)$ -quasi-isometry; that is, there is a constant $D = D(S) \geq 0$ such that :*

- $|d_{\mathcal{V}(S)}(x, y) - d_{\mathcal{M}(S)}(\Psi(x), \Psi(y))| \leq D$ for each $x, y \in \mathcal{V}(S)$, and
- The D -neighborhood of $\Psi(\mathcal{V}(S))$ in $\mathcal{M}(S)$ is all of $\mathcal{M}(S)$.

The main ingredient in our proof of Theorem 1 is a theorem of Minsky [Mi], which determines up to an additive factor the Teichmüller metric near infinity in $\text{Teich}(S)$.

It is clear that Theorem 1 implies that $\text{Cone}(\mathcal{M}(S)) = \text{Cone}(\mathcal{V}(S))$. Further, it is clear that multiplying the given metric on $\mathcal{V}(S)$ by any fixed constant gives a metric space which is isometric (via the dilatation) to the original metric. In particular, $\text{Cone}(\mathcal{V}(S))$ is isometric to $\mathcal{V}(S)$ itself. We thus deduce the following.

Corollary 2. $\text{Cone}(\mathcal{M}(S))$ is isometric to $\mathcal{V}(S)$.

Using different methods, Leuzinger [Le] has independently proven that $\mathcal{V}(S)$ is bilipschitz homeomorphic to $\text{Cone}(\mathcal{M}(S))$. His methods do not seem to yield the isometry type of $\text{Cone}(\mathcal{M}(S))$.

Remarks.

1. Corollary 2 has applications to metrics of positive scalar curvature. Namely, it is a key ingredient in the proof by Weinberger and the first author that, while $\mathcal{M}(S)$ admits a metric of positive scalar curvature for most S (e.g. when $\text{genus}(S) > 2$), it admits no metric with the same quasi-isometry type as the Teichmüller metric on $\mathcal{M}(S)$. See [FW].
2. For locally symmetric M , we know that $\text{Cone}(M)$ is nonpositively curved in the CAT(0) sense. In contrast, $\mathcal{V}(S)$ strongly exhibits aspects of *positive* curvature, since even within the cone on a single simplex, any two points $x, y \in \mathcal{V}(S)$ have whole families of distinct geodesics between them, and these geodesics get arbitrarily far apart as $d(x, y) \rightarrow \infty$. This is a basic property of the sup metric on a quadrant.
3. Corollary 2 implies that any metric on $\mathcal{M}(S)$ quasi-isometric to the Teichmüller metric must have a cone which is bilipschitz homeomorphic to $\mathcal{V}(S)$. This in particular applies to McMullen’s $1/\ell$ metric.

2 The proof of Theorem 1

2.1 Defining the map Ψ

We will define a map $\tilde{\Psi} : \tilde{\mathcal{V}}(S) \rightarrow \text{Teich}(S)$ and will prove that it is $\text{Mod}(S)$ -equivariant, and so descends to a map $\Psi : \mathcal{V}(S) \rightarrow \mathcal{M}(S)$. Let $d = 3g - 3 + n$.

The Collar Lemma in hyperbolic geometry gives that, for a fixed topological type of surface S , there exists a constant $\epsilon_0 = \epsilon_0(S)$ with the following property: For any hyperbolic surface X homeomorphic to S , and any two simple closed curves α, β on X , if $\ell_X(\alpha) < \epsilon_0$ and $\ell_X(\beta) < \epsilon_0$ then α and β can be realized disjointly.

Fix a maximal simplex σ of $\mathcal{C}(S)$. Say σ represents the maximal collection of disjoint simple closed curves $\{\alpha_1, \dots, \alpha_d\}$. This choice of curves determines a set of *Fenchel-Nielsen coordinates* on $\text{Teich}(S)$, where a point $X \in \text{Teich}(S)$ is given by coordinates as follows:

$$X \mapsto (\ell_X(\alpha_1), \dots, \ell_X(\alpha_d), \theta_1(X), \dots, \theta_d(X))$$

where $\ell_X(\alpha_i)$ is the length of α_i with respect to the hyperbolic metric on X , and where the θ_j are the so-called “twist coordinates”. Following Minsky [Mi], we note that changing coordinates to

$$X \mapsto (\theta_1(X), 1/\ell_X(\alpha_1), \dots, \theta_d(X), 1/\ell_X(\alpha_d))$$

gives a coordinate system which we can think of as coordinates in a product of hyperbolic planes.

We will think of the cone on σ , as a subspace of $\tilde{\mathcal{V}}(S)$, as an octant in \mathbf{R}^{3g-3} with coordinates x_1, \dots, x_{3g-3} , endowed with the sup metric.

We now define the map $\tilde{\Psi} : \tilde{\mathcal{V}}(S) \rightarrow \text{Teich}(S)$ restricted to the simplex σ to be:

$$\tilde{\Psi}(x_1, \dots, x_d) = (\epsilon_0 e^{-x_1}, \dots, \epsilon_0 e^{-x_d}, 0, \dots, 0)$$

where $\text{Teich}(S)$ is endowed with Fenchel-Nielsen coordinates. Under the change of coordinates as above, this becomes

$$\tilde{\Psi}(x_1, \dots, x_d) = (0, e^{x_1}/\epsilon_0, \dots, 0, e^{x_d}/\epsilon_0) \in (\mathbf{H}^2)^d$$

We give \mathbf{H}^2 the metric $ds^2 = \frac{1}{4}(dx^2 + dy^2)/y^2$, and endow $(\mathbf{H}^2)^d$ with the sup metric. It is then immediate that the map $\tilde{\Psi}$ is an isometry from the sup metric on each cone as above and this metric. Note that the factor of $\frac{1}{4}$ leads to a factor of $\frac{1}{2}$ in the distance, and is consistent with the factor of $\frac{1}{2}$ in the metric on the Euclidean octant.

$\tilde{\Psi}$ is well-defined: We need to prove that $\tilde{\Psi}$ restricted to (the cone over) any two simplices σ_1, σ_2 agrees on (the cone over) $\sigma_1 \cap \sigma_2$. To see this, note that on each of σ_1 and σ_2 , the coordinates are Fenchel-Nielsen coordinates, consisting of curves in $\sigma_1 \cap \sigma_2$ together with complimentary curves. On the complimentary curves, the corresponding x_i are 0, and $\tilde{\Psi}$ takes the value $1/\epsilon_0$. On $\sigma_1 \cap \sigma_2$, the value of $\tilde{\Psi}$ restricted to σ_1 agrees with the value restricted to σ_2 because it is a function of the Fenchel-Nielsen coordinates, which agree.

$\tilde{\Psi}$ is $\text{Mod}(S)$ -equivariant: An orbit type in $\mathcal{V}(S)$ is precisely the cone on an orbit type in $\mathcal{C}(S)$. A maximal simplex in an orbit type is a topological type of a pair of pants in S . On the other hand the quotient of Fenchel-Nielsen coordinates from $\text{Teich}(S)$ to $\mathcal{M}(S)$ in a single coordinate patch corresponding to a pants decomposition is given by the topological type of a pants decomposition on S . The claim follows.

From $\text{Mod}(S)$ -equivariance it follows that $\tilde{\Psi}$ descends to a well-defined map

$$\Psi : \mathcal{V}(S) \rightarrow \mathcal{M}(S)$$

.

Ψ is continuous: Since the restriction of $\tilde{\Psi}$ to a single closed simplex is given explicitly in coordinates by continuous functions, this restriction is continuous. As $\tilde{\Psi}$ agrees on the common boundaries of these closed simplices, $\tilde{\Psi}$ is continuous. As $\text{Mod}(S)$ acts by homeomorphisms on both $\widehat{\mathcal{V}(S)}$ and on $\text{Teich}(S)$, the continuous map $\tilde{\Psi}$ descends to a continuous map $\Psi : \mathcal{V}(S) \rightarrow \mathcal{M}(S)$.

A technical remark. The action of $\text{Mod}(S)$ on $\mathcal{C}(S)$ induces an action of $\text{Mod}(S)$ on the barycentric subdivision $\mathcal{C}'(S)$. This action has the property that for any $\phi \in \text{Mod}(S)$, any ϕ -invariant simplex must be pointwise fixed. As a consequence, the quotient space $\mathcal{C}'(S)/\text{Mod}(S)$ has a simplicial structure so that the natural quotient map is simplicial.

2.2 Properties of Ψ

Ψ is almost onto: By a theorem of Bers, there is a constant $C = C(g)$ such that every $X \in \mathcal{M}(S)$ has a pants decomposition corresponding to a maximal simplex σ such that every curve of σ has length at most C on X . With respect to these pants curves, each of the twist coordinates is bounded, modulo the action of Dehn twists about the curves in σ , by $2\pi C$. Now given X , we find a point of $\Psi(\mathcal{V}(S))$ whose corresponding simplex has the topological type of σ . For each curve α in σ whose length is at most ϵ_0 , we choose the corresponding Fenchel-Nielsen coordinate of a point in $\Psi(\mathcal{V}(S))$ to be $\ell_X(\alpha)$. For each curve β in σ whose length is between ϵ_0 and C , we choose the corresponding Fenchel-Nielsen coordinate of a point of $\Psi(\mathcal{V}(S))$ to be ϵ_0 . In this way we have chosen all the coordinates which determine a point Z in $\Psi(\mathcal{V}(S))$. Since X and Z have bounded ratios of hyperbolic lengths and bounded differences in twist coordinates, the Teichmüller distance from X to that point is therefore bounded by some constant $D = D(g)$.

Ψ is an almost isometry: Suppose that $X_i \in \Psi(\sigma_i)$, $i = 1, 2$ and that $\sigma_1 \cap \sigma_2$ is a collection of curves reindexed to be $\{\alpha_1, \dots, \alpha_p\}$. It follows from the main theorem of Minsky in [Mi] that there is a constant D' such that

$$d_{\text{Teich}(S)}(X_1, X_2) \geq \sup_{j=1, \dots, p} d_{\mathbf{H}^2}((\theta_j(X_1), 1/\ell_{X_1}(\alpha_j)), (\theta_j(X_2), 1/\ell_{X_2}(\alpha_j))) - D' \quad (1)$$

and if both X_1, X_2 belong to the same simplex $\Psi(\sigma)$ then

$$d_{\text{Teich}(S)}(X_1, X_2) \leq \sup_{j=1, \dots, d} d_{\mathbf{H}^2}(\tilde{\Psi}(X_1), \tilde{\Psi}(X_2)) + D'. \quad (2)$$

We need the following lemma.

Lemma 3 (Path Lemma). *1. Any two points of $\mathcal{V}(S)$ can be joined by a geodesic that enters each simplex of $\mathcal{V}(S)$ at most once.*

2. There is a constant C' such that any two points of $\Psi(\mathcal{V}(S))$ can be joined by a $(1, C')$ quasi-geodesic in the metric $d_{\mathcal{M}(S)}$ that enters each simplex of $\Psi(\mathcal{V}(S))$ at most once.

A first step in proving Lemma 3 is the following.

Lemma 4. 1. Suppose x, y lie in the same simplex Δ of $\mathcal{V}(S)$. Then there is a geodesic joining x and y that stays in Δ .

2. There is a constant C'' such that if $\Psi(x), \Psi(y)$ lie in the same simplex $\Psi(\Delta)$ of $\mathcal{M}(S)$. then there is a $(1, C'')$ quasi-geodesic $\rho(x, y)$ in the metric $d_{\mathcal{M}(S)}$ joining $\Psi(x)$ and $\Psi(y)$ that stays in $\Psi(\Delta)$.

Proof. [of Lemma 4] We prove the first statement. Lift to $\tilde{\mathcal{V}}(S)$ and consider again x, y with the same names and whose coordinates are $(x_1, \dots, x_d), (y_1, \dots, y_d)$. Suppose Δ is defined by the curves $\gamma_1, \dots, \gamma_d$ of a pants decomposition. Without loss of generality assume that $d_{\Delta}(x, y) = \frac{1}{2}(y_1 - x_1)$. We must show that, for every $\phi \in \text{Mod}(S)$, there is no shorter path ρ in $\tilde{\mathcal{V}}(S)$ from $\phi(x)$ to y .

Suppose first that γ_1 is not a vertex in the simplex $\phi(\Delta)$. Then the path from x to y for a last time must enter a simplex for which γ_1 is a vertex at a point z . At z the coordinate corresponding to γ_1 is 0, and thus $d_{\mathcal{V}(S)}(y, z) \geq y_1/2$. Thus we may assume that the path ρ joining $\phi(x)$ to y lies completely in simplices for which γ_1 is a vertex. Break up this path into segments $\rho = \rho_1 * \rho_2 * \dots * \rho_N$, where each ρ_i lies in a single simplex. Let z_1^i (resp. z_1^{i+1}) be the coordinate of γ_1 at the beginning (resp. end) of ρ_i , where $z_1^1 = x_1$ and $z_1^{N+1} = y_1$. Then $|\rho_i| \geq \frac{1}{2}|z_1^{i+1} - z_1^i|$. Thus

$$|\rho| = \sum |\rho_i| \geq \sum \frac{1}{2}|z_1^{i+1} - z_1^i| \geq \frac{1}{2}(y_1 - x_1) = d_{\tilde{\mathcal{V}}(S)}(x, y).$$

We conclude that a shortest path can be found by a geodesic that lies entirely in Δ

We prove the second statement. Assume, without loss of generality, that the distance in the sup of hyperbolic metrics is given by $\log(\ell_y(\gamma_1)/\ell_x(\gamma_1))$. Using (2) choose a $(1, D')$ quasi-geodesic $\rho(x, y)$ joining lifts $\tilde{\Psi}(x)$ and $\tilde{\Psi}(y)$ that lies entirely in $\tilde{\Psi}(\Delta)$. Now let ϕ a mapping class group element. The first conclusion is that if γ_1 is not a vertex of $\phi(\Delta)$ then again, any path ρ from $\Psi(\phi(x))$ to $\Psi(y)$ must enter a simplex for which γ_1 is a vertex at a last time. By (1) we then have

$$|\rho| \geq \log(\ell_y(\gamma_1)/\ell_x(\gamma_1)) - D' \geq |\rho(x, y)| - 2D'.$$

Thus again we can assume ρ lies completely in simplices for which γ_1 is a vertex. But now the conclusion again follows from (1).

◇

Proof. [of Lemma 3] Suppose $x \in \Delta_1$ and $y \in \Delta_2$. If $y \in \Delta_1$ then we are done by Lemma 4. Thus we can assume that $y \notin \Delta_1$. Suppose ρ is a geodesic from x to y . Suppose ρ leaves Δ_1 and returns to it for a last time at some $z \in \Delta_1 \cap \Delta_3$ for some simplex Δ_3 . Then by the first part of Lemma 4 we can replace ρ by a geodesic that stays in Δ_1 from x to z and then follows ρ from z to y never returning to Δ_1 . We now find the last point w that lies in Δ_3 and replace a segment of ρ with one that stays in Δ_3 and never returns again to Δ_3 . Since there are only a finite number of simplices in $\tilde{\mathcal{V}}(S)/\text{Mod}(S)$ we are done. This proves the first statement.

The proof of the second statement is similar where we now use the second part of Lemma 4. \diamond

We now continue with the final step in the proof of Theorem 1, that the map Ψ is an almost isometry. We first prove that

$$d_{\mathcal{M}(S)}(\Psi(x), \Psi(y)) \leq d_{\mathcal{V}(S)}(x, y) + D$$

for some constant D . To prove this, consider a geodesic path $\gamma \subset \mathcal{V}(S)$ connecting x to y . By the first statement of Lemma 3, there exists $c = c(S)$ so that γ can be written as a concatenation $\gamma = \gamma_1 * \cdots * \gamma_c$ with each γ_i in the cone on some simplex Δ_i of $\mathcal{V}(S)$. By (2) there exists $D' \geq 0$ so that each $\Psi(\gamma_i)$ is a $(1, D')$ -quasigeodesic in the metric $d_{\mathcal{M}(S)}$. It follows that $\Psi(\gamma)$ is a $(1, cD')$ -quasigeodesic.

The proof of the opposite inequality

$$d_{\mathcal{V}(S)}(x, y) \leq d_{\mathcal{M}(S)}(\Psi(x), \Psi(y)) + D$$

uses the second conclusion of Lemma 3. Any two points can be joined by $(1, C')$ quasigeodesic in the metric $d_{\mathcal{M}(S)}$ and which intersects a fixed number of simplices. We now apply (2) to conclude that $d_{\mathcal{V}(S)}(x, y)$ is only larger by an additive constant.

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Dept. of Mathematics, University of Chicago
5734 University Ave.
Chicago, IL 60637
E-mail: farb@math.uchicago.edu, masur@math.uic.edu