

Every mapping class group is generated by 6 involutions

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Abstract

Let $\text{Mod}_{g,b}$ denote the mapping class group of a surface of genus g with b punctures. Luo asked in [Lu] if there is a universal upper bound, independent of genus, for the number of torsion elements needed to generate $\text{Mod}_{g,b}$. We answer Luo's question by proving that 3 torsion elements suffice to generate $\text{Mod}_{g,0}$. We also prove the more delicate result that there is an upper bound, independent of genus, not only for the number of torsion elements needed to generate $\text{Mod}_{g,b}$ but also for the order of those elements. In particular, our main result is that 6 involutions (i.e. orientation-preserving diffeomorphisms of order two) suffice to generate $\text{Mod}_{g,b}$ for every genus $g \geq 3, b = 0$ and $g \geq 4, b = 1$.

1 Introduction

Let $S_{g,b}$ denote a closed, oriented surface of genus g with b punctures, and let $\text{Mod}_{g,b}$ denote its mapping class group, which is the group of homotopy classes of orientation-preserving homeomorphisms preserving the set of punctures. We shall frequently abuse terminology by confusing an individual homeomorphism with its mapping class in $\text{Mod}_{g,b}$.

We begin with a brief survey of some known generating sets for $\text{Mod}_{g,b}$ possessing various properties. Dehn [De] produced a finite set of generators of $\text{Mod}_{g,0}$, proving that $2g(g-1)$ Dehn twists suffice for $g \geq 3$. Lickorish [Li] improved on this result by giving a generating set for $\text{Mod}_{g,0}$ consisting of $3g-1$ twists for any $g \geq 1$. Humphries [Hu] then showed that a certain

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subset of Lickorish's set consisting of $2g+1$ twists suffices to generate $\text{Mod}_{g,0}$ and that this is in fact the minimal number of twist generators for $\text{Mod}_{g,0}$ (here again $g \geq 1$). Johnson [Jo] later proved that Humphries' set also generates $\text{Mod}_{g,1}$.

If one allows generators other than twists, smaller generating sets can be obtained. Lickorish [Li] noted that $\text{Mod}_{g,0}$ can be generated with 4 elements, 3 of which are twists and 1 of which has finite order. N. Lu [Lu] found a generating set with 3 elements, 2 of which have finite order. Wajnryb [Wa1] proved that for $g \geq 1$, $b = 0, 1$, the group $\text{Mod}_{g,b}$ can be generated by 2 elements, one of which has finite order.

The problem of finding small generating sets, torsion generating sets, and generating sets of *involutions* (i.e. elements of order two) is a classical one, and has been studied extensively, especially for finite groups (see, e.g., [DT] for a survey). In 1971, Maclachlan [Mac] proved that $\text{Mod}_{g,0}$ is generated by torsion elements, and deduced from this that moduli space \mathcal{M}_g is simply-connected as a topological space. These results were later extended to $\text{Mod}_{g,b}$, $g \geq 3, b \geq 1$ by Patterson [Pa]. The question of generating mapping class groups by involutions¹ was first investigated by McCarthy and Papadopoulos [MP]. Among other results, they proved that for $g \geq 3$, $\text{Mod}_{g,0}$ is generated by infinitely many conjugates of a certain involution.

Luo [Luo], using work of Harer [Ha], described the first finite set of involutions which generate $\text{Mod}_{g,b}$ for $g \geq 3, b \geq 0$. The order of his generating set depends on both g and b ; in particular, his set consists of $12g + 2$ involutions when $b = 0, 1$. Luo also gives torsion generators, not necessarily involutions, for all other cases except $g = 2, b = 5k + 2$. It should be noted that the existence of a generating set for $\text{Mod}_{g,0}$ consisting of $4g + 4$ torsion elements follows directly from a lemma of Birman [Bi] (see Lemma 3 below). In his paper ([Luo], §1.4), Luo poses the question of whether there is a universal upper bound, independent of g and b , for the number of torsion elements needed to generate $\text{Mod}_{g,b}$. Our first result is a positive answer to Luo's question for $b = 0$.

Theorem 1 (Three torsion elements generate). *For each $g \geq 1$, the group $\text{Mod}_{g,0}$ is generated by 3 elements of finite order.*

As we will see in Section 2, at least one of the three torsion generators we give has order depending on g . Subsequent to the original posting of this paper, M. Korkmaz [Ko1] has shown that $\text{Mod}_{g,b}$, $b = 0, 1$ is generated by two elements, each of order $4g + 2$.

¹We remind the reader that the only involutions under consideration are orientation-preserving.

Finding a set of generators whose orders are *universally* bounded and whose cardinality is also universally bounded is more delicate, especially if one wants a generating set consisting of involutions. Our main theorem addresses this.

Theorem 2 (Six involutions generate). *For $g \geq 3, b = 0$, and for $g \geq 4, b = 1$, the group $\text{Mod}_{g,b}$ is generated by 6 involutions.*

In [Ka], Kassabov builds on our method to extend and improve this result to the case $b > 1$. Further, in some cases (e.g. $g \geq 8$) he proves that 4 involutions suffice to generate $\text{Mod}_{g,b}$.

In §4, we note that Theorem 2 implies that $\text{Mod}_{g,b}$ is the quotient of a 6-generator Coxeter group.

Remarks.

1. Since $\text{Mod}_{1,0} = \mathbf{Z}/4\mathbf{Z} *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/6\mathbf{Z}$ and $H_1(\text{Mod}_{2,0}, \mathbf{Z}) = \mathbf{Z}/10\mathbf{Z}$, it is easy to see that Theorem 2 does not extend to the genus $g = 1$ or $g = 2$ cases.
2. As $\text{Mod}_{g,0}$ surjects onto the integral symplectic group $\text{Sp}(2g, \mathbf{Z})$, it follows from Theorems 1 and 2 that for all $g \geq 3$, the group $\text{Sp}(2g, \mathbf{Z})$ is generated by 3 torsion elements, and also by 6 involutions.
3. If one allows *orientation-reversing* involutions, it is possible to use Theorem 1, together with other arguments, to prove that the *extended mapping class group*, which includes orientation-reversing mapping classes, is generated by 5 involutions.

It would be interesting to obtain the exact bounds for Theorem 2. It is easy to see the lower bound of 3; we do not know how to improve on this bound.

It is a pleasure to thank Dieter Kotschick, Mustafa Korkmaz, and Dan Margalit for their valuable suggestions. We also thank Ian Agol for suggesting that Remark 3 above might be possible, Nathan Broaddus for pointing out the connection with Coxeter groups, and Martin Kassabov who pointed out a redundancy in our original 7-involution generating set, thus reducing the conclusion in Theorem 2 to 6 involutions.

2 Proof that 3 torsion elements generate

In this section we shall only consider the case of the closed surface, i.e., $b = 0$.

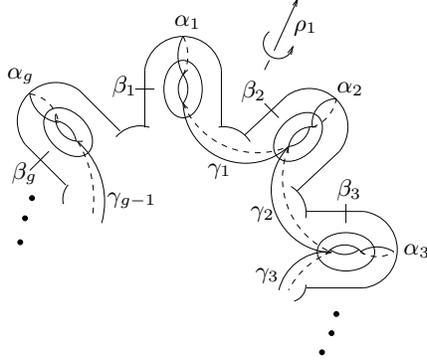


Figure 1: A circular embedding of the surface together with Lickorish's $3g-1$ twist generators and an involution ρ_1 which is rotation by π about the axis indicated.

We begin with a lemma of Birman [Bi], and include an adapted version of her proof for completeness. For a simple closed curve x in $S_{g,b}$, let T_x denote the (right-handed) Dehn twist about x .

Lemma 3 (Two torsion elements for a twist). *Let x be a simple closed curve which is nonseparating in $S_{g,0}$, $g \geq 1$. Then T_x can be written as the product of two torsion elements.*

Proof. We will find it convenient to fix a “circular embedding” of the surface $S_{g,b}$, as seen in Figure 1. Referring to the curves of the same figure, we define

$$Q = T_{\alpha_g} T_{\beta_g} (T_{\gamma_{g-1}} T_{\beta_{g-1}}) (T_{\gamma_{g-2}} T_{\beta_{g-2}}) \cdots (T_{\gamma_1} T_{\beta_1}) T_{\alpha_1} \quad (1)$$

and

$$S = Q T_{\alpha_1}^{-1} = T_{\alpha_g} T_{\beta_g} (T_{\gamma_{g-1}} T_{\beta_{g-1}}) (T_{\gamma_{g-2}} T_{\beta_{g-2}}) \cdots (T_{\gamma_1} T_{\beta_1}) \quad (2)$$

We say that an ordered set c_1, \dots, c_n of simple closed curves on $S_{g,b}$ forms an n -chain if the geometric intersection $(c_k, c_{k+1}) = 1$ for $1 \leq k \leq n-1$ and $(c_k, c_l) = 0$ if $|k-l| \geq 2$. If n is odd, the boundary of a regular neighborhood of any n -chain has two components d_1 and d_2 ; if n is even the boundary has one component d . The so-called *chain relation* in $\text{Mod}_{g,b}$ tells us that for a given n -chain c_1, \dots, c_n , if n is odd we have

$$(T_{c_1} T_{c_2} \cdots T_{c_n})^{n+1} = T_{d_1} T_{d_2}$$

and if n is even, we have

$$(T_{c_1} T_{c_2} \cdots T_{c_n})^{2n+2} = T_d$$

The curves defining the twists of Q and S form a $(2g+1)$ -chain and a $(2g)$ -chain, respectively. However, any boundary curve of a regular neighborhood of either chain is null-homotopic in the closed surface. Hence the chain relation tells us that Q and S have orders which divide $2g+2$ and $4g+2$, respectively, in $\text{Mod}_{g,0}$. In fact, Birman notes in her original proof that an “ugly but routine” calculation establishes that these values are precisely the orders of the two elements, with explicit calculations recorded in [BH] (see also [HK] for a shorter proof).

Given any nonseparating simple closed curve x , there is a homeomorphism h such that $h(\alpha_1) = x$. Recall that for $h \in \text{Mod}_{g,b}$ and a simple closed curve c contained in $S_{g,b}$, we have

$$hT_ch^{-1} = T_{h(c)} \tag{3}$$

Then we have

$$\begin{aligned} T_x &= hT_{\alpha_1}h^{-1} \\ &= h(S^{-1}Q)h^{-1} \\ &= (hS^{-1}h^{-1})(hQh^{-1}) \end{aligned}$$

which gives the desired result. \diamond

Now to complete the proof of Theorem 1. It is well known that $\text{Mod}_{1,0} \cong \text{SL}(2, \mathbf{Z})$ is generated by an element of order 4 and an element of order 6, so we can assume $g \geq 2$. Let α_i be as above, let Q and S be defined as in (1) and (2), and let

$$U = T_{\alpha_1} T_{\alpha_2}^{-1}$$

Wajnryb [Wa1] showed that $\text{Mod}_{g,0}$, $g \geq 1$ is generated by the two maps U and S . In the proof of Lemma 3, we saw that S has finite order in $\text{Mod}_{g,0}$; hence it remains to deal with U , which is the product of two Dehn twists. As above, we have

$$T_{\alpha_1} = S^{-1}Q$$

From the proof of Lemma 3, we know that

$$T_{\alpha_2} = h(S^{-1}Q)h^{-1}$$

where h is any map taking the curve α_1 to α_2 .

Let ρ_1 denote the involution which is rotation by π about the axis indicated in Figure 1. Clearly, $\rho_1(\alpha_1) = \alpha_2$, so we can write

$$\begin{aligned} U &= T_{\alpha_1} T_{\alpha_2}^{-1} \\ &= [S^{-1}Q][\rho_1(Q^{-1}S)\rho_1^{-1}] \end{aligned}$$

Thus we can generate Wajnryb's two generators, and hence all of $\text{Mod}_{g,0}$, with three torsion elements: Q, S and ρ_1 . \diamond

Remarks.

1. We can replace the involution ρ_1 in the above proof with a rotation of order g .
2. Wajnryb [Wa1] also shows that the two maps U and S generate $\text{Mod}_{g,1}$. However the proof of Theorem 1 does not go through for $b = 1$ since Lemma 3 fails in this case.

We also note that, using Wajnryb's two generators S, U , we can generate $\text{Mod}_{g,b}$ when $b = 0, 1$ with a set consisting of two involutions and one element of order $4g + 2$. Since $U = T_{\alpha_1} T_{\alpha_2}^{-1}$, and $\rho_1(\alpha_2) = \alpha_1$, we have that $U = T_{\alpha_1}(\rho_1 T_{\alpha_1}^{-1} \rho_1)$. Shifting the parentheses, we have that U is the product of the involutions ρ_1 and its conjugate $T_{\alpha_1} \rho_1 T_{\alpha_1}^{-1}$. We will see the usefulness of this "parentheses shifting" technique in the next section.

3 A universal bound on involutions

For the remainder of the paper, we assume that $g \geq 3$ and $b = 0, 1$. For simplicity of exposition, we provide explicit arguments only for $b = 0$. In the case $b = 1$ the arguments are the same, although some involutions must be replaced with certain conjugates which move the puncture to a fixed point of the involution.

3.1 Writing Dehn twists as a product of involutions

We begin by recalling the so-called *lantern relation* in $\text{Mod}_{g,b}$. This relation was discovered by Dehn [De] in the 1930s and was rediscovered by Johnson [Jo] over forty years later. For convenience, we will use the notation X in addition to T_x to denote the (right-handed) Dehn twist about the simple closed curve x in $S_{g,b}$.

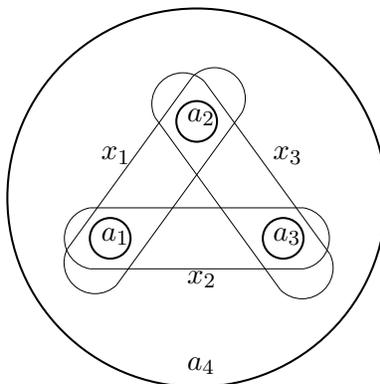


Figure 2: The Lantern Relation: $X_1X_2X_3 = A_1A_2A_3A_4$.

Referring to the four boundary curves a_1, a_2, a_3 and a_4 of the surface $S_{0,4}$ together with the interior curves x_1, x_2 and x_3 , as shown in Figure 2, the lantern relation is the following relation amongst the corresponding Dehn twists:

$$X_1X_2X_3 = A_1A_2A_3A_4 \quad (4)$$

The symmetry of Figure 2 clearly shows that the twists on the left-hand side of (4) can be cyclically permuted. Moreover, each twist on the right-hand side commutes with all other twists in the relation since the corresponding curves are disjoint from all others in the lantern. Thus we can rewrite the relation in the following way, which will be convenient for our purposes:

$$A_4 = (X_1A_1^{-1})(X_2A_2^{-1})(X_3A_3^{-1}) \quad (5)$$

The following lemma improves on a result of Luo [Luo] and Harer [Ha]; they showed that a Dehn twist about a nonseparating curve can be written as a product of six involutions.

Lemma 4 (Four involutions for a twist). *Let c be a simple closed curve which is nonseparating in $S_{g,b}$, $g \geq 3$. Then T_c can be written as the product of four involutions.*

Proof. We begin with the argument of Luo and Harer, which we include here for completeness. If $g \geq 3$, we can find a lantern as in Figure 2 embedded in $S_{g,b}$ such that the given curve c plays the role of a_4 and such that the complement of the lantern is connected and its four boundary curves are distinct and nonseparating in $S_{g,b}$. Let us call such an embedding a

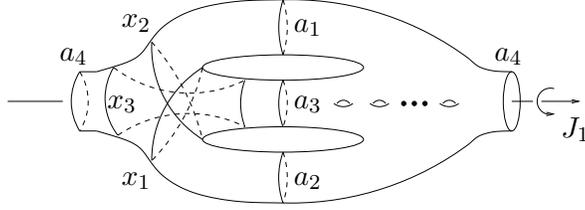


Figure 3: The *pair swap involution* J_1 , taking (a_1, x_1) to (a_2, x_2) . Here the boundary curve a_4 on the left is identified with that on the right.

good lantern. We now observe that for each j , $1 \leq j \leq 3$, the pair (x_j, a_j) consists of two disjoint, nonseparating simple closed curves such that the complement of the union $x_j \cup a_j$ is connected in $S_{g,b}$ for each $1 \leq j \leq 3$. Then for each j we can find an involution I_j such that $I_j(x_j) = a_j$.

By (3) we can write

$$X_j A_j^{-1} = X_j (I_j X_j^{-1} I_j) = (X_j I_j X_j^{-1}) I_j$$

for each $1 \leq j \leq 3$. Note that each $X_j I_j X_j^{-1}$ is an involution, so that each $X_j A_j^{-1}$, $1 \leq j \leq 3$, can be realized as a product of two conjugate involutions. Hence $T_c = A_4$ is the product of six involutions by (5).

To improve upon this result, consider the surface shown in Figure 3. If the curves labelled a_4 are identified in the obvious way, then the four curves a_i together bound a good lantern with interior curves x_j as shown. Given the non-separating curve $c = a_4$, we can find a good lantern with a_4 as a boundary component. We can choose its remaining labels a_i, x_j , $1 \leq i, j \leq 3$, in such a way that there is a homeomorphism of $S_{g,b}$ to the surface of Figure 3 taking this good lantern to the lantern of Figure 3. It is clear now that there exists an involution J_1 of $S_{g,b}$ which takes the pair (a_1, x_1) to the pair (a_2, x_2) . We note that the surface of Figure 3 could be embedded in \mathbf{R}^3 in such a way that the involution J_1 of the surface is a restriction of an isometry of \mathbf{R}^3 ; one can simply add a tube connecting the two ends which encloses the surface of Figure 3. (We shall abuse notation throughout by using J_1 to refer to both the involution of $S_{g,b}$ as well as its conjugate which is the involution of the surface of Figure 3.)

In fact, given a particular choice of labels for a good lantern, a similar process yields 6 distinct “pair swaps” of order 2, i.e., involutions taking one pair (a_i, x_j) to another pair (a_k, x_l) , assuming $i \neq k, j \neq l$. In particular, there is a conjugate involution J_2 taking the pair (a_1, x_1) to the pair (a_3, x_3) . This is clear if we act on the surface of Figure 3.1 by the homeomorphism

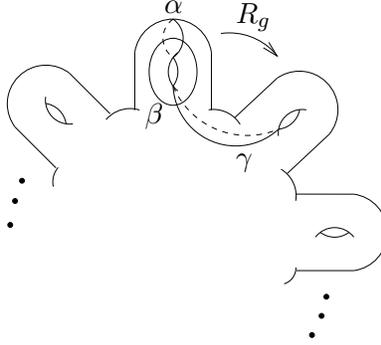


Figure 4: A generating set for $\text{Mod}_{g,b}$.

which interchanges the “tube” containing the curve a_2 with the tube containing a_3 , fixing the curve x_1 while simultaneously moving the curve x_3 to the current position of x_2 .

We can now rewrite (5) as

$$T_c = A_4 = [(X_1 I_1 X_1^{-1}) I_1] [J_1 (X_1 I_1 X_1^{-1}) I_1 J_1] [J_2 (X_1 I_1 X_1^{-1}) I_1 J_2] \quad (6)$$

and hence T_c is a product of four involutions. \diamond

3.2 Proof of the main result: 6 involutions generate

Lickorish [Li] proved that Dehn twists about the $3g - 1$ curves given in Figure 1 suffice to generate $\text{Mod}_{g,b}$. In the same paper, he also described a generating set for $\text{Mod}_{g,b}$ consisting of four elements, one of which is not a Dehn twist. We shall now give a rigorous description of such a generating set. Let $R_g \in \text{Mod}_{g,b}$ denote clockwise rotation of $S_{g,b}$ by $\frac{2\pi}{g}$ about the axis which is perpendicular to the plane of the page and intersects the surface twice in the center of our “circle of handles”. Applying Equation (3) to R_g conjugating the twists T_α, T_β and T_γ (referring to the curves of Figure 4), we see that the group generated by R_g together with these three twists contains

$$\{R_g^m T_\alpha R_g^{-m}, R_g^m T_\beta R_g^{-m}, R_g^m T_\gamma R_g^{-m} : 0 \leq m \leq g - 1\}$$

Hence the four elements $T_\alpha, T_\beta, T_\gamma, R_g$ generate all of Lickorish’s twist generators and thus they generate all of $\text{Mod}_{g,b}$. Clearly, one could replace any of the three twists generators T_c , $c \in \{\alpha, \beta, \gamma\}$, with $R_g^k T_c R_g^{-k} = T_{R_g^k(c)}$ for any $k \in \mathbf{Z}$ and the resulting set still generates $\text{Mod}_{g,b}$.

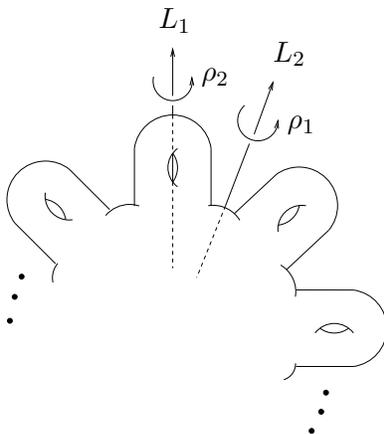


Figure 5: Two involutions generating R_g .

Consider the rotation R_g . For $i = 1, 2$, let ρ_i denote rotation in \mathbf{R}^3 by π about the axis labelled L_i , as shown in Figure 5. Note that if the genus g is even, then L_1 intersects the surface in six points and L_2 in two points. If the genus g is odd, then L_1 and L_2 each intersect the surface in four points. In both cases, we observe that

$$R_g = \rho_1 \rho_2 \tag{7}$$

Combining (7) with Lemma 4, together with the fact proved above that $\text{Mod}_{g,b}$ is generated by R_g and the three twists $T_\alpha, T_\beta, T_\gamma$, we have that $\text{Mod}_{g,b}$ can be generated by $2 + 3 \cdot 4 = 14$ involutions.

We now show how to reduce this number. Since, as previously noted, we can replace T_β with $T_{R_g^2(\beta)}$ in our generating set, we shall abuse notation by using β to refer to the curve $R_g^2(\beta)$ for the remainder of the proof. As shown in Figure 6, we can embed a good lantern in our surface which contains α, β , and γ as three of its four boundary components. In order to motivate our choices of notation, we first observe that, with this particular choice of a good lantern, we can choose x_1 to be the curve labelled as such in Figure 6. If we then let α play the role of a_1 , we see that $\rho_1(a_1) = x_1$, and thus ρ_1 plays the role of I_1 in the proof of Lemma 4.

Now, in order to make further use of our work in proving Lemma 4, we assign to the curve γ the role of a_4 , so that a_1 and a_4 are separated by x_1 from a_2 and a_3 . Of the two remaining curves, we now assign to β the role of a_3 , and the remaining curve we label a_2 . We also choose interior curves

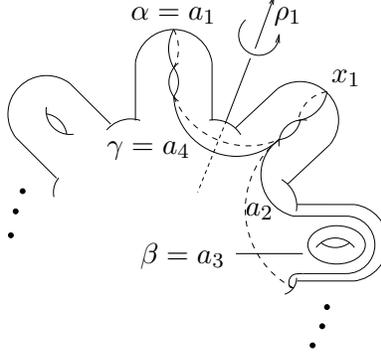


Figure 6: A good lantern embedded with α, β and γ as three boundary curves and x_1 an interior curve.

x_2 and x_3 so the labels of the good lantern of Figure 6 match those of the pair swaps defined in the previous section. Thus we have

$$T_\gamma = A_4 = (X_1 T_\alpha^{-1})(X_2 A_2^{-1})(X_3 T_\beta^{-1}) \quad (8)$$

$$= (X_1 A_1^{-1})(X_2 A_2^{-1})(X_3 A_3^{-1}) \quad (9)$$

$$= [(X_1 I_1 X_1^{-1}) I_1] [J_1 (X_1 I_1 X_1^{-1}) I_1 J_1] [J_2 (X_1 I_1 X_1^{-1}) I_1 J_2]$$

$$= [(X_1 \rho_1 X_1^{-1}) \rho_1] [J_1 (X_1 \rho_1 X_1^{-1}) \rho_1 J_1] [J_2 (X_1 \rho_1 X_1^{-1}) \rho_1 J_2]$$

Given the two involutions ρ_1, ρ_2 used to generate R_g , we have used only three new involutions to write T_γ , namely $X_1 \rho_1 X_1^{-1}, J_1$, and J_2 . Now (8) and (9) can be rewritten, respectively, as

$$\begin{aligned} T_\beta = A_3 &= (X_1 T_\alpha^{-1})(X_2 A_2^{-1})(X_3 T_\gamma^{-1}) \\ &= (X_1 A_1^{-1})(X_2 A_2^{-1})(X_3 A_4^{-1}) \end{aligned} \quad (10)$$

We now observe that we can swap (a_i, x_j) with (a_k, x_j) for any i, j, k by using a slight variation on the embedding of Figure 3. The involution J_3 shown in Figure 7 interchanges (a_3, x_3) with (a_4, x_3) , as well as (a_1, x_3) with (a_2, x_3) , (a_1, x_1) with (a_2, x_1) , and finally (a_3, x_1) with (a_4, x_1) . Note that in the first two cases, the a_i -curves that are interchanged are not separated in the lantern by the fixed curve x_3 , but in the last two cases, the a_i -curves being swapped are separated by the fixed curve x_1 . Thus any possible configuration of such pair swaps can be achieved, although a different embedding may be required in order to align the x_j -curve to be fixed in the swap either

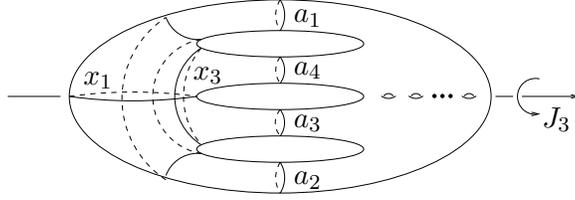


Figure 7: A pair swap J_3 which takes (a_3, x_3) to (a_4, x_3) ; both x_1 and x_3 are interior lantern curves fixed by J_3 .

“horizontally” or “vertically”, depending, respectively, on whether the two a_i -curves one wishes to swap are separated by the x_j -curve, or lie on the same side.

We now have that the product J_3J_2 takes the pair (x_1, a_1) to the pair (x_3, a_4) and we can rewrite (10) as

$$T_\beta = A_3 = (X_1A_1^{-1})[J_1(X_1A_1)^{-1}J_1][(J_3J_2)(X_1A_1^{-1})(J_2J_3)]$$

Since $(X_1A_1^{-1}) = (X_1\rho_1X_1^{-1})\rho_1$ as above, we see that we need just one new involution, J_3 , in order to generate T_β . Similarly, we can find a pair-swap involution J_4 taking the pair (x_1, a_1) to (x_1, a_4) , so that we can write

$$\begin{aligned} T_\alpha = A_1 &= (X_1T_\beta^{-1})(X_2A_2^{-1})(X_3T_\gamma^{-1}) \\ &= (X_1A_4^{-1})(X_2A_2^{-1})(X_3A_3^{-1}) \\ &= [J_4(X_1A_1^{-1})J_4][J_1(X_1A_1^{-1})J_1][J_2(X_1A_1^{-1})J_2] \end{aligned}$$

with J_4 being the only new involution required to write T_γ .

Thus our count for the number of involutions used to generate each of the four elements $R_g, T_\gamma, T_\beta, T_\alpha$ stands, respectively, at $2 + 3 + 1 + 1 = 7$ involutions. Finally, as pointed out to us by M. Kassabov, the involution J_4 is in fact redundant, and can be taken to be the product $J_2J_3J_2$. Thus we can generate with 6 involutions, as claimed. \diamond

4 Final remarks

Recall that an *Artin group* A is generated by elements x_1, \dots, x_n , subject to the following relations. Let $m_{s,t} = \{2, 3, 4, \dots, \infty\}$. Then for each $s \neq t$, the elements x_s and x_t satisfy the relations:

$$(x_s x_t)^{m_{s,t}/2} = (x_t x_s)^{m_{s,t}/2} \quad \text{if } m_{s,t} \text{ is even}$$

$$(x_s x_t)^{(m_{s,t}-1)/2} = (x_t x_s)^{(m_{s,t}-1)/2} \quad \text{if } m_{s,t} \text{ is odd}$$

Here $m_{s,t} = \infty$ means that there is no relation between the generators x_s and x_t . The Coxeter group \bar{A} associated to an Artin group A is the quotient of A by the extra relation $x_i^2 = 1$ for all $i = 1, \dots, n$. We thank Nathan Broaddus for pointing out the following corollary of our main theorem.

Corollary 5 (Mod $_{g,b}$ is a Coxeter quotient). *For $g \geq 3, b = 0$, and for $g \geq 4, b = 1$, the group Mod $_{g,b}$ can be realized as a quotient of a Coxeter group on 6 generators.*

There has been some effort to understand various mappings of Artin groups both into and onto mapping class groups. For example, Wajnryb studied classes of Artin groups which do not inject into mapping class groups in [Wa2], while Matsumo [Mat] as well as Labruere and Paris [LP] have given explicit presentations of mapping class groups as quotients of certain Artin groups. However, we are not aware of similar investigations of the relationship between Coxeter groups and mapping class groups.

The results of this paper beg several further questions. For example, besides the previously raised question of whether we can do better than 6 involutions, we can also ask whether there exists a constant C , perhaps such that $C = C(g)$, so that every element of Mod $_{g,b}$ can be written as a product of at most C torsion elements². Furthermore, what kinds of relations exist amongst the torsion (or involution) generators? In particular, what kind of Coxeter groups arise in the context of Corollary 5?

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²After initial circulation of this paper, M. Korkmaz [Ko2] and D. Kotschick [Kot] answered the latter question in the negative by proving that no such C exists. Both arguments build on a theorem of Endo-Kotschick [EK].

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