Isometries, rigidity and universal covers

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1 Introduction

The goal of this paper is to describe all closed, aspherical Riemannian manifolds M whose universal covers $\widetilde{M}$ have have a nontrivial amount of symmetry. By this we mean that $\mathrm{Isom}(\widetilde{M})$ is not discrete. By the well-known theorem of Myers-Steenrod [MS], this condition is equivalent to $[\mathrm{Isom}(\widetilde{M}) : \pi_1(M)] = \infty$. Also note that if any cover of M has a nondiscrete isometry group, then so does its universal cover $\widetilde{M}$.

Our description of such M is given in Theorem 1.2 below. The proof of this theorem uses methods from Lie theory, harmonic maps, large-scale geometry, and the homological theory of transformation groups.

The condition that $\widetilde{M}$ have nondiscrete isometry group appears in a wide variety of problems in geometry. Since Theorem 1.2 provides a taxonomy of such M, it can be used to reduce many general problems to verifications of specific examples. Actually, it is not always Theorem 1.2 which is applied directly, but the main subresults from its proof. After explaining in §1.1 the statement of Theorem 1.2, we give in §1.2 a number of such applications. These range from new characterizations of locally symmetric manifolds, to the classification of contractible manifolds covering both compact and finite volume manifolds, to a new proof of the Nadel-Frankel Theorem in complex geometry.

1.1 Statement of the general theorem

The basic examples of closed, aspherical, Riemannian manifolds whose universal covers have nondiscrete isometry groups are the *locally homogeneous (Riemannian) manifolds* M, i.e. those M whose universal cover admits a transitive Lie group action whose isotropy subgroups are maximal compact. Of course one might also take a product of such a manifold with an arbitrary manifold. To find nonhomogeneous examples which are not products, one can do the following construction.

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Example 1.1. Let $F \to M \to B$ be any Riemannian fiber bundle with the induced path metric on F locally homogeneous. Let $f : B \to \mathbf{R}^+$ be any smooth function. Now at each point of M lying over b, rescale the metric in the tangent space $TM_b = TF_b \oplus TB_b$ by rescaling TF_b by $f(b)$. Almost any f gives a metric on M with $\dim(\mathrm{Isom}(\widetilde{M})) > 0$ but with $\widetilde{M}$ not homogeneous, indeed with each $\mathrm{Isom}(\widetilde{M})$-orbit a fiber. This construction can be further extended by scaling fibers using any smooth map from B to the moduli space of locally homogeneous metrics on F; this moduli space is large for example when F is an n-dimensional torus.

Hence we see that there are many closed, aspherical, Riemannian manifolds whose universal covers admit a nontransitive action of a positive-dimensional Lie group. The following general result says that the examples described above exhaust all the possibilities for such manifolds.

Before stating the general result, we need some terminology. A *Riemannian orbifold* B is a smooth orbifold where the local charts are modelled on quotients V/G, where G is a finite group and V is a linear G-representation endowed with some G-invariant Riemannian metric. The orbifold B is *good* if it is the quotient of V by a properly discontinuous group action.

A *Riemannian orbibundle* is a smooth map $M \longrightarrow B$ from a Riemannian manifold to a Riemannian orbifold locally modelled on the quotient map $p : V \times_G F \longrightarrow V/G$, where F is a fixed smooth manifold with smooth G-action, and where $V \times F$ has a G-invariant Riemannian metric such that projection to V is an orthogonal projection on each tangent space. Note that in this definition, the induced metric on the fibers of a Riemannian orbibundle may vary, and so a Riemannian orbibundle is not a fiber bundle structure in the Riemannian category.

Theorem 1.2. *Let M be a closed, aspherical Riemannian manifold. Then either* $\mathrm{Isom}(\widetilde{M})$ *is discrete, or M is isometric to an orbibundle*

$$F \longrightarrow M \longrightarrow B \tag{1.1}$$

where:

- *B is a good Riemannian orbifold, and* $\mathrm{Isom}(\widetilde{B})$ *is discrete.*
- *Each fiber F, endowed with the induced metric, is isometric to a closed, aspherical, locally homogeneous Riemannian n-manifold, $n > 0$* [1].

[1] Recall that a manifold F is *locally homogeneous* if its universal cover is isometric to G/K, where G is a Lie group, K is a maximal compact subgroup, and G/K is endowed with a left G-invariant, K bi-invariant metric.

Note that B is allowed to be a single point.

One might hope that the Riemannian orbifold B in the conclusion of Theorem 1.2 could be taken to be a Riemannian manifold, at least after passing to a finite cover of M. This is not the case, however. In §6 we construct a Riemmanian manifold M with the property that M is a Riemannian orbibundle, fibering over a singular orbifold, but such that no finite cover of M fibers over a manifold; further, $\mathrm{Isom}(\widetilde{M})$ is not discrete. This seems to be the first known example of an aspherical manifold with a singular fibration that remains singular in every finite cover. In constructing M we produce a group Γ acting properly discontinuously and cocompactly by diffeomorphisms on $\mathbf{R}^n$, but which is not virtually torsion-free.

1.2 Applications

We now explain how to apply Theorem 1.2 and its proof to a variety of problems in geometry. The proofs of these results will be given in §4 below.

Characterizations of locally symmetric manifolds. We begin with a characterization of locally symmetric manifolds among all closed Riemannian manifolds. The theme is that such manifolds are characterized by some simple properties of their fundamental group, together with the property that their universal covers have nontrivial symmetry (i.e. have nondiscerete isometry group). We say that a smooth manifold M is *smoothly irreducible* if M is not smoothly covered by a nontrivial finite product of smooth manifolds.

Theorem 1.3. *Let M be any closed Riemannian n-manifold, $n > 1$. Then the following are equivalent:*

1. *M is aspherical, smoothly irreducible, $\pi_1(M)$ has no nontrivial, normal abelian subgroup, and $\mathrm{Isom}(\widetilde{M})$ is not discrete.*
2. *M is isometric to an irreducible, locally-symmetric Riemannian manifold of nonpositive sectional curvature.*

The idea here is to apply Theorem 1.2, or more precisely the main results in its proof, and then to show that if the base B were positive dimensional, the manifold M would not be smoothly irreducible; see §4.1 below.

Remark. The proof of Theorem 1.3 gives more: the condition that M is smoothly irreducible can be replaced by the weaker condition that M is not Riemannian covered by a nontrivial Riemannian warped product; see §4.1.

When M has nonpositive curvature, the Cartan-Hadamard Theorem gives that M is aspherical. For nonpositively curved metrics on M, Theorem 1.3 was proved by Eberlein in

[Eb1, Eb2] [2]. While the differential geometry and dynamics related to nonpositive curvature are central to Eberlein's work, for the most part they do not, by necessity, play a role in this paper.

Recall that the Mostow Rigidity Theorem states that a closed, aspherical manifold of dimension at least three admits at most one irreducible, nonpositively curved, locally symmetric metric up to homotheties of its local direct factors. For such locally symmetric manifolds M, Theorem 1.3 has the following immediate consequence:

Up to homotheties of its local direct factors, the locally symmetric metric on M is the unique Riemannian metric with $\mathrm{Isom}(\widetilde{M})$ *not discrete.*

Uniqueness within the set of nonpositively curved Riemannian metrics on M follows from [Eb1, Eb2]. This statement also generalizes the characterization in [FW] of the locally symmetric metric on an arithmetic manifold.

Combined with basic facts about word-hyperbolic groups, Theorem 1.3 provides the following characterization of closed, negatively curved, locally symmetric manifolds.

Corollary 1.4. *Let M be any closed Riemannian n-manifold, $n > 1$. Then the following are equivalent:*

1. *M is aspherical, $\pi_1(M)$ is word-hyperbolic, and* $\mathrm{Isom}(\widetilde{M})$ *is not discrete.*

2. *M is isometric to a negatively curved, locally symmetric Riemannian manifold.*

Theorem 1.3 can also be combined with Margulis's Normal Subgroup Theorem to give a simple characterization in the higher rank case. We say that a group Γ is *almost simple* if every normal subgroup of Γ is finite or has finite index in Γ.

Corollary 1.5. *Let M be any closed Riemannian manifold. Then the following are equivalent:*

1. *M is aspherical, $\pi_1(M)$ is almost simple, and* $\mathrm{Isom}(\widetilde{M})$ *is not discrete.*

2. *M is isometric to a nonpositively curved, irreducible, locally symmetric Riemannian manifold of (real) rank at least* 2.

The above results distinguish, by a few simple properties, the locally symmetric manifolds among all Riemannian manifolds. We conjecture that a stronger, more quantitative result holds, whereby there is a kind of universal (depending only on π_1) constraint on the amount of symmetry of any Riemannian manifold which is not an orbibundle with locally symmetric fiber.

[2] Eberlein's results are proved not just for lattices but more generally for groups satisfying the so-called *duality condition* (see [Eb1, Eb2]), a condition on the limit set of the group acting on the visual boundary.

Conjecture 1.6. *The hypothesis "*$\mathrm{Isom}(\widetilde{M})$ *is not discrete" in Theorem 1.3, Corollary 1.5, and Corollary 1.4 can be replaced by:* $[\mathrm{Isom}(\widetilde{M}) : \pi_1(M)] > C$*, where* C *depends only on* $\pi_1(M)$*.*

We do not know how to prove Conjecture 1.6. However, we can prove it in the special case of a fixed manifold admitting a locally symmetric metric.

Theorem 1.7. *Let* (M, g_0) *be a closed, irreducible, nonpositively curved locally symmetric* n*-manifold,* $n > 1$*. Then there exists a constant* C*, depending only on* $\pi_1(M)$*, such that for any Riemannian metric* h *on* M*:*

$$[\mathrm{Isom}(\widetilde{M}) : \pi_1(M)] > C \quad \text{if and only if} \quad h \sim g_0$$

where $\sim$ *denotes "up to homothety of direct factors".*

Manifolds with both closed and finite volume quotients. We can also apply our methods to answer the following fundamental question in Riemannian geometry: which contractible Riemannian manifolds X cover both a closed manifold and a (noncompact, complete) finite volume manifold?

This question has been answered for many (but not all) contractible homogeneous spaces X. Recall that a contractible *(Riemannian) homogeneous space* X is the quotient of a connected Lie group H by a maximal compact subgroup, endowed with a left-invariant metric. Mostow proved that solvable H admit only cocompact lattices, while Borel proved that noncompact, semisimple H have both cocompact and noncocompact lattices (see [Ra], Thms. 3.1, 14.1). The case of arbitrary homogeneous spaces is more subtle, and as far as we can tell remains open.

The following theorem extends these results to all contractible manifolds X. It basically states that if X covers both a compact and a noncompact, finite volume manifold, then the reason is that X is "essentially" a product, with one factor a homogeneous space which itself covers both types of manifolds.

To state this precisely, we define a *warped Riemannian product* to be a smooth manifold $X = Y \times Z$ where Z is a (locally) homogeneous space, $f : Y \longrightarrow \mathcal{H}(Z)$ is a smooth function with target the space $\mathcal{H}(Z)$ of all (locally) homogeneous metrics on Z, and the metric on X is given by

$$g_X(y, z) = g_Y \oplus f(y) g_Z$$

We can now state the following.

Theorem 1.8. *Let* X *be a contractible Riemannian manifold. Suppose that* X *Riemannian covers both a closed manifold and a noncompact, finite volume, complete manifold. Then* X *is isometric to a warped product* $Y \times X_0$*, where* Y *is a contractible manifold (possibly a point) and* X_0 *is a homogeneous space which admits both cocompact and noncocompact lattices. In particular, if* X *is not a Riemannian warped product then it is homogeneous.*

Note that the factor Y is necessary, as one can see by taking the product of a homogeneous space with the universal cover of any compact manifold. We begin the deduction of Theorem 1.8 from the other results in this paper by noting that its hypotheses imply that $\mathrm{Isom}(Z)$ is nondiscrete, so that our general result can be applied.

Irreducible lattices in products. Let $X = Y \times Z$ be a Riemannian product. Except in obvious cases, $\mathrm{Isom}(Y) \times \mathrm{Isom}(Z) \hookrightarrow \mathrm{Isom}(X)$ is a finite index inclusion. Recall that a lattice Γ in $\mathrm{Isom}(X)$ is *irreducible* if it is not virtually a product. Understanding which Lie groups admit irreducible lattices is a classical problem; see, e.g., [Ma], §IX.7. Eberlein determined in [Eb1, Eb2] the nonpositively curved X which admit irreducible lattices; they are essentially the symmetric spaces. The following extends this result to all contractible manifolds; it also provides another proof of Eberlein's result.

Theorem 1.9. *Let X be a nontrivial Riemannian product, and suppose that* $\mathrm{Isom}(X)$ *admits an irreducible, cocompact lattice. Then X is isometric to a warped Riemannian product $X = Y \times X_0$, where Y is a contractible manifold (possibly a point), X_0 is a positive dimensional homogeneous space, and X_0 admits an irreducible, cocompact lattice.*

As with Theorem 1.8, Theorem 1.9 is deduced from the other results in this paper by noting that its hypotheses imply that $\mathrm{Isom}(Z)$ is nondiscrete; see §4.6.

Compact complex manifolds. Our results on isometries also have implications for complex manifolds. Kazhdan conjectured that any irreducible bounded domain Ω which admits both a compact quotient M and a one-parameter group of holomorphic automorphisms must be biholomorphic to a bounded *symmetric* domain. Frankel [Fr1] first proved this for convex domains Ω, and subsequent work by Nadel [Na] and Frankel [Fr2], which we now recall, proved it in general.

The Bergman volume form on a bounded domain produces a metric on the canonical bundle so that the first Chern class satisfies $c_1(M) < 0$; equivalently, the canonical line bundle is ample. Hence Kazhdan's conjecture is implied by (and, indeed, inspired) the following.

Theorem 1.10 (Nadel, Frankel)**.** *Let M be a compact, aspherical complex manifold with $c_1(M) < 0$. Then there is a holomorphic splitting $M' = M_1 \times M_2$ of a finite cover M' of M, where M_1 is locally symmetric and M_2 is* locally rigid *(i.e. the biholomorphic automorphism group of the universal cover $\widetilde{M_2}$ is discrete.)*

Theorem 1.10 was first proved in (complex) dimension two by Nadel [Na] and in all dimensions by Frankel [Fr2]. They do not require the asphericity of M, although this is of course the case for quotients of bounded domains. Complex geometry is an essential ingredient in their work.

In §4.8 we give a different proof of Theorem 1.10, using a key proposition from the earlier paper of Nadel [Na] together with (the proof of) our Theorem 1.2 below. In complex dimension two, we give a proof independent of both [Na] and [Fr2]. We do not see, however, how to use our methods without the asphericity assumption.

As with [Na] and [Fr2], our starting point is a theorem of Aubin-Yau, which gives that the biholomorphism group $\mathrm{Aut}(\widetilde{M})$ acts isometrically on a Kahler-Einstein metric lifted from M. Our proof shows that, at least in complex dimension two, this is the only ingredient from complex geometry needed to prove Kazhdan's conjecture.

Remark. Nadel pointed out explicitly in Proposition 0.1 of [Na] that his methods would extend to prove Theorem 1.10 in all dimensions if one could prove that each isotropy subgroup of $\mathrm{Aut}(\widetilde{M})^{\circ}$ were a maximal compact subgroup. The solution to this problem in the aspherical case is given in Claim IV of Section 2 below; it also applies outside of the holomorphic context as well.

Some additional applications. A number of the results from this paper generalize from closed, aspherical Riemannian manifolds to all closed Riemannian manifolds. In §5 we provide an illustrative example, Theorem 5.1, which seems to be the first geometric rigidity theorem for non-aspherical manifolds with infinite fundamental group.

In §4.7 below we give an application of our methods to the Hopf Conjecture about Euler characetristics of aspherical manifolds.

Finally, we would like to mention the work of K. Melnick in [Me], where some of the results here are extended from the Riemannian to the pseudo-Riemannian (especially the Lorentz) case. Melnick combines the ideas here with Gromov's theory of rigid geometric structures, as well as methods from Lorentz dynamics.

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2 Finding the orbibundle (proof of Theorem 1.2)

Our goal in this section is to prove Theorem 1.2. The starting point is the following well-known classical theorem.

Theorem 2.1 (Myers-Steenrod, [MS]). *Let M be a Riemannian manifold. Then* $\mathrm{Isom}(M)$ *is a Lie group, and acts properly on M. If M is compact then* $\mathrm{Isom}(M)$ *is compact.*

Note that the Lie group $\mathrm{Isom}(M)$ in Theorem 2.1 may have infinitely many components; for example, let M be the universal cover of a bumpy metric on the torus.

Throughout this paper we will use the following notation:

M= a closed, aspherical Riemannian manifold

$\Gamma = \pi_1(M)$

$X = \widetilde{M}$ = the universal cover of M

$I = \mathrm{Isom}(X)$ = the group of isometries of X

I_0 = the connected component of I containing the identity

$\Gamma_0 = \Gamma \cap I_0$

Here X is endowed with the unique Riemannian metric for which the covering map $X \to M$ is a Riemannian covering. Hence Γ acts on X isometrically by deck transformations, giving a natural inclusion $\Gamma \to I$, where $I = \mathrm{Isom}(X)$ is the isometry group of X.

By Theorem 2.1, I is a Lie group, possibly with infinitely many components. Let I_0 denote the connected component of the identity of I; note that I_0 is normal in I. If I is discrete, then we are done, so suppose that I is not discrete. Theorem 2.1 then gives that the dimension of I is positive, and so I_0 is a connected, positive-dimensional Lie group.

We have the following exact sequences:

$$1 \longrightarrow I_0 \longrightarrow I \longrightarrow I/I_0 \longrightarrow 1 \tag{2.1}$$

and

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow \Gamma/\Gamma_0 \longrightarrow 1 \tag{2.2}$$

We now proceed in a series of steps. Our first step is to construct what will end up as the locally homogeneous fibers of the orbibundle (1.1).

Claim I: The quotient I_0/Γ_0 is compact.

Proof. Let $\mathrm{Fr}(X)$ denote the frame bundle over X. The isometry group I acts freely on $\mathrm{Fr}(X)$. The I_0 orbits in $\mathrm{Fr}(X)$ give a smooth foliation of $\mathrm{Fr}(X)$ whose leaves are diffeomorphic to I_0. This foliation descends via the natural projection $\mathrm{Fr}(X) \longrightarrow \mathrm{Fr}(M)$ to give a smooth foliation $\mathcal{F}$ on $\mathrm{Fr}(M)$, each of whose leaves is diffeomorphic to I_0/Γ_0. Thus we must prove that each of these leaves is compact.

The quotient of $\mathrm{Fr}(X)$ by the smallest subgroup of I containing both Γ and I_0 is homeomorphic to the space of leaves of $\mathcal{F}$. We claim that this quotient is a finite cover of $\mathrm{Fr}(X)/I$.

To prove this, it is clearly enough to show that the natural injection $\Gamma/\Gamma_0 \longrightarrow I/I_0$ has finite index image.

To this end, we first recall the following basic principle of Milnor-Svarc (see, e.g., [H]). Let G be a compactly generated topological group, generated by a compact subspace $S \subset G$. Endow G with the *word metric*, i.e. let $d_G(g,h)$ be defined to be the minimal number of elements of S needed to represent gh^{-1}; this is a left-invariant metric on G. Now suppose that G acts properly and cocompactly by isometries on a proper, geodesic metric space X. Then G is *quasi-isometric* to X, i.e. for any fixed basepoint $x_0 \in X$, the orbit map $G \longrightarrow X$ sending g to $g \cdot x_0$ satisfies the following two conditions:

- (Coarse Lipschitz): For some $K, C > 0$,

$$\frac{1}{K} d_G(g,h) - C \leq d_X(g \cdot x_0, h \cdot x_0) \leq K d_G(g,h) + C$$

- (C-density) $\mathrm{Nbhd}_C(G \cdot x) = X$

While the standard proofs of this fact (see, e.g., [H]) usually assume that S is finite, they apply verbatim to the more general case of S compact.

Applying this principle, the cocompactness of the actions of both Γ and of I on X give that the inclusion $\Gamma \longrightarrow I$ is a quasi-isometry. The quotient map $I \longrightarrow I/I_0$ is clearly distance nonincreasing, and so the image Γ/Γ_0 of Γ under this quotient map is C-dense in I/I_0. As both groups are discrete, this clearly implies that the inclusion $\Gamma/\Gamma_0 \longrightarrow I/I_0$ is of finite index. Thus the claim is proved.

Now note that $\mathrm{Fr}(X)/I$ is clearly compact, and is a manifold since I is acting freely and properly. Hence the leaf-space of $\mathcal{F}$ is also a compact manifold. Since each leaf of $\mathcal{F}$ is the inverse image of a point under the map from $\mathrm{Fr}(M)$ to the leaf space, we have that each leaf of $\mathcal{F}$ is compact. $\diamond$

It will be useful to know that I_0 cannot have compact factors.

Claim II: I_0 has no nontrivial compact factor.

Proof. In proving this claim, we will use degree theory for noncompact manifolds, phrased in terms of locally finite homology H_*^{lf} (see, e.g., [Iv] for a discussion). *Locally finite homology* is the theory of cycles which pair with cohomology with compact support. Perhaps the quickest description of $H_*^{lf}(X)$ is as the usual reduced homology $\widetilde{H}_*(\widehat{X})$ of the one-point compactification $\widehat{X}$ of X. Alternatively, it can be described (for locally finite simplicial complexes) as the homology of the chain complex of infinite formal combinations of simplices for which only finitely many simplices with nonzero coefficients intersect any given compact region.

With this definition, it is easliy verified (see [Iv]) that the usual degree theory holds for continuous quasi-isometries between (possibly noncompact) manifolds X, with the fundamental class of the n-manifold X now being an element of $H_n^{lf}(X, \mathbf{Z})$. As one example, the universal cover X of a closed, aspherical n-manifold M has a nonzero fundamental class lying in $H_n^{lf}(X, \mathbf{Z})$. With this degree theory in place, we can now begin the proof of Claim II.

Now suppose that I_0 has a nontrivial compact factor K. Since I_0 is connected and $\dim(I_0) > 0$ by assumption, we have that K is connected and $\dim(K) > 0$.

Since M is closed, and so there is a compact fundamental domain for the Γ-action on X, we easily see that there exists a constant C so that each K-orbit has diameter at most C. But then X/K is quasi-isometric to X. Now the standard "connect the dots" trick (see, e.g., p.527 of [BW], or Appendix A of [BF] for exact details) states that such quasi-isometries are a bounded distance (in the sup norm) from a *continuous* quasi-isometry (i.e. Lipschitz map). Hence there are continuous maps $X \longrightarrow X/K$ and $X/K \longrightarrow X$ inducing the given quasi-isometry. Since $\dim(K) > 0$ we have that $\dim(X/K) < \dim(X) = n$. This implies that the fundamental class of X in $H_n^{lf}(X, \mathbf{R})$, where $n = \dim(X)$, must vanish, contradicting the fact that X is the universal cover of a closed, aspherical n-manifold. $\diamond$

The next step in our proof of Theorem 1.2 is to determine information which will help us construct the orbifold base space B of the orbibundle (1.1).

Claim III: X/I_0 is contractible.

Proof. The Conner Conjecture, proved by Oliver [Ol], gives that the quotient of a contractible manifold by a connected, compact, smooth transformation group is contractible. Our claim that X/I_0 is contractible follows directly from the following simple extension of Oliver's theorem.

Proposition 2.2. *Let G be a connected Lie group acting properly by diffeomorphisms on a contractible manifold X. Then the underlying topological space of the orbit space X/G is contractible.*

Proposition 2.2 is a consequence of Oliver's Theorem and the following.

Proposition 2.3. *Let G be a connected Lie group acting properly by diffeomorphisms on an aspherical manifold X. Denote by K the maximal compact subgroup of G. Then there exists an aspherical manifold Y such that X is diffeomorphic to $Y \times G/K$, the manifold Y has a K-action, and the original action is given by the product action. In particular, X/G is diffeomorphic to Y/K.*

Proof. Let $\underline{EG}$ be the classifying space for proper CW G-complexes, so that $\underline{EG}/G$ is the classifying space for proper G-bundles (see, e.g., the appendix of [BCH]). Now G/K is an

$\underline{EG}$ space. Hence there is a proper G-map $\psi : X \longrightarrow \underline{EG}$. But $\underline{EG}$ has only one G-orbit, so ψ is surjective. Now let $Y = \psi^{-1}([K])$, where $[K]$ denotes the identity coset of K. Hence X is diffeomorphic to $G \times_K Y$, and we are done.

$\diamond$

We are now ready to construct, on the level of universal covers, the orbibundle (1.1), and in particular to prove that the base space B is a Riemannian orbifold. The crucial point is to understand stabilizers of the I_0 action on X. For $x \in X$, denote the stabilizer of x under the I_0 action by $I_x := \{g \in I_0 : gx = x\}$. Let K_0 denote the maximal compact subgroup of I_0; it is unique up to conjugacy.

Claim IV: $I_x = K_0$ for each $x \in X$. Hence the following hold:

1. X/I_0 is a manifold.
2. Each I_0-orbit in X is isometric to the contractible, homogeneous manifold I_0/K_0, endowed with some left-invariant Riemannian metric.
3. The natural quotient map gives a Riemannian fibration

$$I_0/K_0 \longrightarrow X \longrightarrow X/I_0 \tag{2.3}$$

Proof. Clearly $I_x \subseteq K_0$. Iwasawa proved ([Iw], Theorem 6) that any maximal compact subgroup of a connected Lie group is connected. Hence it is enough to prove that $\dim(I_x) = \dim(K_0)$.

To this end we consider rational cohomological dimension $\mathrm{cd}_{\mathbf{Q}}$. By Claim III we have X/I_0 is contractible. Since Γ/Γ_0 acts properly on X/I_0, we then have

$$\mathrm{cd}_{\mathbf{Q}}(\Gamma/\Gamma_0) \leq \dim(X/I_0) \tag{2.4}$$

Since K_0 is maximal, we know I_0/K_0 is contractible. By Claim I, we have that Γ_0 is a uniform lattice in I_0, and so

$$\mathrm{cd}_{\mathbf{Q}}(\Gamma_0) = \dim(I_0/K_0) \tag{2.5}$$

Since X is contractible and $M = X/\Gamma$ is a closed manifold, by general facts about cohomological dimension (see [Bro], Chapter VIII (2.4)), we have

$$\dim(X) = \mathrm{cd}_{\mathbf{Q}}(\Gamma) \leq \mathrm{cd}_{\mathbf{Q}}(\Gamma_0) + \mathrm{cd}_{\mathbf{Q}}(\Gamma/\Gamma_0)$$

which combined with (2.4) and (2.5) gives

$$\dim(X) \leq \dim(X/I_0) + \dim(I_0/K_0) \tag{2.6}$$

But for each $x \in X$, we have

$$\dim(X) \geq \dim(X/I_0) + \dim(I_0/I_x) \tag{2.7}$$

which combined with (2.6) gives

$$\dim(I_0/I_x) \leq \dim(I_0/K_0)$$

and so $\dim(I_x) \geq \dim(K_0)$, as desired. Thus $I_x = K_0$.

It follows that each orbit $I_0 \cdot x$ is diffeomorphic to a common Euclidean space I_0/K_0, so by the Slice Theorem (see, e.g., [Br], Chap. IV, §3,4,5) it follows that X/I_0 is a manifold. We note that while the Slice Theorem is usually stated for actions of compact groups, the proof extends immediately to the case of proper actions of noncompact groups; one simply produces an invariant Riemannian metric by translating a compactly supported pseudometric, and this gives the required structure via exponentiation. $\diamond$

Finishing the proof. The action of Γ on X induces actions of Γ_0 on I_0/K_0, and of Γ/Γ_0 on X/I_0, compatible with the Riemannian fibration (2.3). By Myers-Steenrod, Γ/Γ_0 acts properly discontinuously on X/I_0; we denote the quotient space of this action by B. We thus have a Riemannian orbibundle (as defined in the introduction):

$$F \longrightarrow M \longrightarrow B \tag{2.8}$$

where F denotes the closed, locally homogeneous Riemannian manifold $\Gamma_0\backslash I_0/K_0$, endowed with the quotient metric of a left I_0-invariant metric on I_0/K_0. This completes the proof of Theorem 1.2

3 The case when I_0 is semisimple

The main goal of this section is to prove Proposition 3.1 below, which shows that when I_0 is semisimple with finite center, a much stronger conclusion holds in Theorem 1.2.

Proposition 3.1. *Suppose that I_0 is semisimple with finite center. Then M has a finite cover which is a Riemannian warped product $N \times B$, where N is nonempty, locally symmetric with nonpositive curvature, and has no local torus factors. In particular, $\pi_1(B) \lhd \pi_1(M)$, and any nontrivial, normal abelian subgroup of $\pi_1(M)$ lies in $\pi_1(B)$.*

Remark on semisimplicity. We would like to emphasize that by calling a connected Lie group G "semisimple" we mean only that the Lie algebra of G is semisimple. Thus the center $Z(G)$ may be infinite. Such examples do exist (for example the universal cover of $\mathrm{U}(n,1)$) , and must be taken into account. We also point out that G may in general have compact factors. For the connected component I_0 of the isometry group of the universal cover of a closed, aspherical Riemannian manifold, however, we have already proven in Claim II of §2 above that I_0 has no nontrivial compact factor. Even so, the semisimple part $(I_0)^{\mathrm{ss}}$ may have nontrivial compact factors coming from $Z(I_0)$.

After proving Proposition 3.1, we show in §3.2 that the hypothesis that I_0 is semisimple with finite center is more common than one might guess. Indeed, in Proposition 3.3 we prove that I_0 is always semisimple with finite center unless $\Gamma = \pi_1(M)$ contains an infinite, normal abelian subgroup.

3.1 The proof of Proposition 3.1

The structure of the proof of Proposition 3.1 is to first prove it at the level of fundamental groups, mostly using Lie theory. The theory of harmonic maps, as well as the existence of arithmetic lattices, is then used to build so many isometries of the universal cover of M that it is forced to fiber in the claimed way.

Triviality of the extension. Our first goal will be to prove that, after replacing Γ by a finite index subgroup if necessary, the exact sequence

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow \Gamma/\Gamma_0 \longrightarrow 1 \tag{3.1}$$

splits as a direct product. As with every extension, (3.1) is determined by two pieces of data:

1. A representation $\rho : \Gamma/\Gamma_0 \longrightarrow \mathrm{Out}(\Gamma_0)$, and
2. A cohomology class in $H^2(\Gamma/\Gamma_0, Z(\Gamma_0)_\rho)$, where $Z(\Gamma_0)_\rho$ is a Γ/Γ_0-module via ρ.

We analyze these pieces in turns. Let $< I_0, \Gamma >$ be the smallest subgroup of I containing I_0 and Γ. Consider the exact sequence

$$1 \longrightarrow I_0 \longrightarrow < I_0, \Gamma > \longrightarrow \Gamma/\Gamma_0 \longrightarrow 1 \tag{3.2}$$

and let

$$\rho_1 : \Gamma/\Gamma_0 \longrightarrow \mathrm{Out}(I_0)$$

denote the induced action; this is just the action induced by the conjugation action of Γ on I. Since I_0 is semisimple, we know (see, e.g. [He], Theorem IX.5.4) that $\mathrm{Out}(I_0)$ is finite. Hence, after passing to a finite index subgroup of Γ if necessary, we may assume that ρ_1 is trivial. In other words, the Γ-action on I_0 is by inner automorphisms, giving a representation

$$\rho_2 : \Gamma/\Gamma_0 \longrightarrow I_0/Z(I_0)$$

Now, the conjugation action of Γ on I_0 preserves Γ_0, and so the image of ρ_2 lies in the normalizer $N_H(\Gamma_0)$ of Γ_0 in $H := I_0/Z(I_0)$. Note that $\Gamma_0 \cap Z(I_0)$ is finite and hence trivial, as is $Z(\Gamma_0)$, since Γ_0 is torsion free, and so Γ_0 can be viewed as a subgroup of H. Since H is semisimple and Γ_0 is a cocompact lattice in H (by Claim I in the proof of Theorem 1.2), it follows that $N_H(\Gamma_0)/\Gamma_0$ is finite [3].

Hence, by replacing Γ with a finite index subgroup if necessary, we may assume ρ_2 has trivial image. We thus have that the conjugation action of Γ on Γ_0 is by inner automorphisms of Γ_0. Since $Z(\Gamma_0)$ is trivial, the representation $\rho : \Gamma/\Gamma_0 \longrightarrow \mathrm{Out}(\Gamma_0)$ is trivial [4]. We also know that

$$H^2(\Gamma/\Gamma_0, Z(\Gamma_0)_\rho) = 0$$

since $Z(\Gamma_0) = 0$. It follows that, up to finite index, the exact sequence (3.1) splits, and in fact that

$$\Gamma \approx \Gamma_0 \times \Gamma/\Gamma_0 \tag{3.3}$$

Recall (2.8), where we found a Riemannian orbibundle

$$F \longrightarrow M \longrightarrow B$$

Our goal now is to use (3.3) to find a section of this fibration, and to use this to prove that M is a Riemannian warped product. In order to do this we will use the following tool.

Harmonic maps. We recall that a map $f : N \longrightarrow M$ between Riemannian manifolds is *harmonic* if it minimizes the energy functional

$$E(f) = \int_N ||Df_x||^2 d\mathrm{vol}_N$$

The key properties of harmonic maps between closed Riemannian manifolds which we will need are the following (see, e.g. [SY]):

- (Eels-Sampson) When the target manifold has nonpositive sectional curvatures, a harmonic map exists in each homotopy class.
- (Hartman, Schoen-Yau) If a harmonic map $f : M \longrightarrow N$ induces a surjection on π_1, and if $\pi_1(N)$ is centerless, then f is unique in its homotopy class. This follows directly from Theorem 2 of [SY].

[3] Since H has no compact factors, this follows for example from Bochner's classical result that the closed manifold $M = \Gamma\backslash H/K$ has finite isometry group since it has negative Ricci curvature, and $\mathrm{Isom}(M) = N_H(\Gamma_0)$. For another proof, see [Ma], II.6.3.

[4] Note that there are cases when $\mathrm{Out}(\Gamma_0)$ is nontrivial; for example when Γ_0 is a surface group then $\mathrm{Out}(\Gamma_0)$ is the mapping class group of that surface.

- (easy) The precomposition and postcomposition of a harmonic map with an isometry gives a harmonic map.

Showing that X is a warped product. The isomorphism in (3.3) gives via projection to a direct factor a natural surjective homomorphism $\pi : \Gamma \longrightarrow \Gamma_0$. Recall that $0 = Z(\Gamma_0) \supseteq Z(I_0) \cap \Gamma_0$, and so the injection $\Gamma_0 \longrightarrow I_0$ gives an injection $\Gamma_0 \longrightarrow I_0/Z(I_0)$. Our first goal is to extend the projection π to a projection $\widehat{\pi} :< I_0, \Gamma > \longrightarrow I_0/Z(I_0)$.

To this end, note that $Z(I_0)$ is characteristic in I_0, and so $Z(I_0) \lhd I$; in particular, $Z(I_0) \lhd < I_0, \Gamma >$. Taking the quotient of the exact sequence (3.2) by the finite normal subgroup $Z(I_0)$ gives an exact sequence

$$1 \longrightarrow I_0/Z(I_0) \longrightarrow < I_0, \Gamma > /Z(I_0) \longrightarrow \Gamma/\Gamma_0 \longrightarrow 1 \tag{3.4}$$

We claim that the kernel of (3.4) is centerless. Indeed, if G is any connected semisimple Lie group, then its center $Z(G)$ is clearly closed, hence discrete since G is semsimple. But for any connected Lie group G with $Z(G)$ discrete, the center of $G/Z(G)$ is trivial (see, e.g., Exercise 7.11(b) of [FH]). The reason this fact is true can be seen from the fact that the discreteness of $Z(G)$ implies that the quotient map $G \longrightarrow G/Z(G)$ is a covering map of Lie groups, and so both G and $G/Z(G)$ have isomorphic Lie algebras and isomorphic universal covers..

Since the kernel of (3.4) is centerless, the exact argument as above gives that (3.4) splits, so that

$$< I_0, \Gamma > /Z(I_0) \approx I_0/Z(I_0) \times \Gamma/\Gamma_0 \tag{3.5}$$

This isomorphism, composed with the natural projections, then gives us a surjective homomorphism

$$\widehat{\pi} :< I_0, \Gamma > \longrightarrow I_0/Z(I_0)$$

Let K_0 denote a maximal compact subgroup of the semisimple Lie group I_0. We then have that I_0 acts isometrically on the contractible, nonpositively curved symmetric space of noncompact type $X_0 := I_0/K_0$. Since $Z(I_0)$ is finite, it lies in K_0, and so the I_0 action on X_0 factors through a faithful action of $I_0/Z(I_0)$. As X_0 is contractible, the homomorphism π is induced by some continuous map $h : X/\Gamma \longrightarrow X_0/\Gamma_0$. Thus f is homotopic to a harmonic map h. By the theorem of Hartman and Schoen-Yau stated above, f is the unique harmonic map in its homotopy class.

Claim 3.2. *The lifted map $\widetilde{f} : X \longrightarrow X_0$ is equivariant with respect to the representation $\widehat{\pi} :< I_0, \Gamma > \longrightarrow I_0/Z(I_0)$.*

To prove this claim, first note that $\widetilde{f}$ is equivariant with respect to the representation π, by construction; we want to promote this to $\widehat{\pi}$-equivariance. One strange aspect of this is that we use an auxilliary arithmetic group, which seems to have nothing to do with the situation.

To begin, consider any cocompact lattice Δ in $I_0/Z(I_0)$. By (3.5), $\Delta \times \Gamma/\Gamma_0$ is a cocompact lattice in $< I_0, \Gamma > /Z(I_0)$, so it pulls back under the natural quotient to a cocompact lattice, which we will also call Δ, in $< I_0, \Gamma >$ (recall that $Z(I_0)$ is finite).

Then, up to translation by elements of Δ, there is a unique harmonic map $\phi_\Delta : X \longrightarrow X_0$ equivariant with respect to the restriction $\widehat{\pi}|_{\Delta\times(\Gamma/\Gamma_0)}$. Suppose Δ' is any other lattice in I_0 which is commensurable with Δ. Since both ϕ_Δ and $\phi_{\Delta'}$ are harmonic and equivariant with respect to the representation $\widehat{\pi}$ restricted to $(\Delta \cap \Delta') \times (\Gamma/\Gamma_0)$, and since $\Delta \cap \Delta'$ has finite index in both Δ and in Δ', we have by uniqueness of harmonic maps that $\phi_\Delta = \phi_{\Delta'}$. We remark that this "uniqueness implies equivariance" principle is also a key trick in [FW].

Since I_0 is semisimple with finite center, the quotient $I_0/Z(I_0)$ is semisimple and centerless (as proven just after equation (eq:ss4b) on page 15 above). By a theorem of Borel ([Bo], Theorem C), there exists a cocompact arithmetic lattice Δ_1 in $I_0/Z(I_0)$. Since $I_0/Z(I_0)$ is centerless, it follows that the commensurator $\mathrm{Comm}_{I_0/Z(I_0)}(\Delta_1)$ is dense in $I_0/Z(I_0)$; see, for example, Proposition 6.2.4 of [Zi], where this is clearly explained.

Let Δ_0 denote the pullback of Δ_1 under the natural quotient map $I_0 \longrightarrow I_0/Z(I_0)$. Since Δ_0 contains $Z(I_0)$, and so $\mathrm{Comm}_{I_0}(\Delta_0)$ is the central extension of $\mathrm{Comm}_{I_0/Z(I_0)}(\Delta_1)$ associated to $Z(I_0)$, it follows that $\mathrm{Comm}_{I_0}(\Delta_0)$ is dense in I_0. At this point, a verbatim application of the proof of the "arithmetic case" of Theorem 1.4 in [FW] completes the proof of Claim 3.2; for completeness, we briefly recall this proof.

Let U denote the set of $g \in < I_0, \Gamma >$ for which the equation

$$\phi_{\Delta_0} g = g \phi_{\Delta_0} \tag{3.6}$$

holds. Now U is closed, and the uniqueness of harmonic maps gives that U is a subgroup of I_0. Hence U is a Lie subgroup of I_0. Applying the above paragraphs with $\Delta = \Delta_0$ and with Δ' running through the collection $\mathcal{L}$ of lattices commensurable with Δ_0 in I_0, gives that U contains every lattice in $\mathcal{L}$. Since $\mathrm{Comm}_{I_o}(\Delta_0)$ is dense in I_0, there are infinitely many distinct members of $\mathcal{L}$ conjugate to Δ_0, namely the conjugates of Δ_0 by elements of $\mathrm{Comm}_{I_o}(\Delta_0)$. Hence U is nondiscrete, hence positive dimensional. Under the adjoint representation, Δ_0 preserves the Lie algebra of U. But Δ_0 is a lattice in I_0, hence is Zariski dense by the Borel Density Theorem (see, e.g. [Ma], Theorem II.2.5). Thus $U = I_0$, finishing the proof of Claim 3.2.

We now have a map

$$X/(\Gamma \times (\Gamma/\Gamma_0)) \longrightarrow (X_0/\Gamma_0) \times (X/I_0)/(\Gamma/\Gamma_0)$$

given by the product of $\widetilde{f}$ and the natural orbit map. This map harmonic when composed with projection to the first factor, and is clearly a diffeomorphism, since we have just shown that the first coordinate is equivariant with respect to $\widehat{\pi}$.

3.2 Consequences of no normal abelian subgroups

The assumption that $\Gamma = \pi_1(M)$ contains no nontrivial normal abelian subgroup has strong consequences for our setup. The main one is the following.

Proposition 3.3. *Suppose Γ contains no infinite, normal abelian subgroup. Then I_0 is semisimple with finite center.*

Proof. Note that since Γ is torsion free, it follows that Γ has no normal abelian subgroups; in particular $Z(\Gamma) = 1$.

For any connected Lie group G there is an exact sequence

$$1 \longrightarrow G^{\mathrm{sol}} \longrightarrow G \longrightarrow G^{\mathrm{ss}} \longrightarrow 1 \tag{3.7}$$

where G^{sol} denotes the *solvable radical* of G (i.e. the maximal connected, normal, solvable Lie subgroup of G), and where G^{ss} is the connected semisimple Lie group G/G^{sol}.

Let Γ^{sol} denote the unique maximal normal solvable subgroup of Γ_0; the existence of such a subgroup is exactly the statement of Corollary 8.6 of [Ra]. Since Γ^{sol} is unique it is characteristic. It is also torsion free since Γ is torsion free. We claim that Γ^{sol} is trivial. Suppose not. Being a nontrivial torsion-free solvable group, Γ^{sol} would then have an infinite, characteristic, torsion-free abelian subgroup H, namely the last nontrivial term in its derived series. Since H is characteristic in the normal subgroup Γ^{sol} of Γ, it would follow that H is normal in Γ. Since H is infinite abelian, this contradicts the hypothesis on Γ.

We now quote a result of Prasad, namely Lemma 6 in [Pr]. For a lattice Γ in a connected Lie group I_0, Conclusion (2) of Prasad's Lemma gives, in the terminology of [Pr]:

$$\mathrm{rank}(\Gamma^{\mathrm{sol}}) = \chi(I_0^{\mathrm{sol}}) + \mathrm{rank}(Z(I_0^{\mathrm{ss}}))$$

Here $\chi(I_0^{\mathrm{sol}})$ denotes the dimension of I_0^{sol} minus that of its maximal compact subgroup, $\mathrm{rank}(Z(I_0^{\mathrm{ss}}))$ denotes the the rank of the center of I_0^{ss}, and *rank* denotes the sum of the ranks of the abelian quotients in the derived series. Since in our case we have proven that $\Gamma^{\mathrm{sol}} = 0$, it follows both that $\chi(I_0^{\mathrm{sol}}) = 0$, i.e. that I_0^{sol} is compact, and that the rank of $Z(I_0)$ is 0, so that $Z(I_0)$ is finite.

Since I_0^{sol} is both solvable and compact, it is a torus T. Since the automorphism group of T is discrete (namely it is $\mathrm{GL}(\dim(T), \mathbf{Z})$), the natural conjugation action of the connected group I_0^{ss} on T given by (3.7) must be trivial, so that T is a direct factor of I_0. But we have already proven (Claim II of §2) that I_0 has no nontrivial compact factors, a contradiction unless T is trivial. Thus $I_0^{\mathrm{sol}} = T$ is trivial; that is, I_0 is semisimple. $\diamond$

Remark. It is possible to weaken the hypothesis of Proposition 3.3, and hence of all of the results which rely on it, to assuming only that Γ contains no *finitely generated*, infinite normal abelian subgroups. To do this, we begin by recalling that Prasad's result used above also gives that the group Γ^{sol} is a lattice in some connected solvable subgroup S of I_0. It follows from Proposition 3.4 below that Γ^{sol} is polycyclic. But it is well-known and easy to see that any polycyclic group has the property that each of its subgroups is finitely generated (see, e.g. [Ra], Prop. 3.8). Hence the subgroup H constructed in the proof of Proposition 3.3 would in fact be finitely generated.

In the argument just given we needed the following proposition, proved by Mostow in the simply connected case.

Proposition 3.4. *Every lattice Λ in a connected solvable Lie group S is polycyclic.*

Proof. First note that $\pi_1(S)$ is finitely-generated and abelian, and so the universal cover $\widetilde{S}$ is a central $\mathbf{Z}^d$ extension of S for some $d \geq 0$. The lattice Λ pulls back to a lattice $\widetilde{\Lambda}$ in $\widetilde{S}$, which is a central $\mathbf{Z}^d$ extension of Λ. Mostow proved (see, e.g. [Ra], Prop. 3.7) that any lattice $\widetilde{\Lambda}$ in a connected, simply-connected solvable Lie group $\widetilde{S}$ must be polycyclic. It follows easily that Λ is polycyclic. $\diamond$

The use of Prasad's result simplifies the approach to Proposition 3.3 given in an earlier version of this paper. As part of that earlier approach, we proved the following proposition. We include this result here since we believe it might prove useful in the future, since the proof is direct, and since we were not able to find this result in the literature. The argument was kindly supplied to us by the referee.

Proposition 3.5. *Let G be a connected semsimple Lie group, and let Λ be a lattice in G. If $Z(G)$ is infinite then $Z(\Lambda)$ is infinite.*

Proof. Let T be the identity component of the closure of $Z(G)\Lambda$ in G. First note that T is abelian; indeed, the commutator subgroup $[T,T]$ of T is contained in the closure of the subgroup

$$[Z(G)\Lambda, Z(G)\Lambda] = [\Lambda, \Lambda] \subset \Lambda$$

and hence $[T,T]$, being connected, is trivial.

Now let C be the unique maximal compact, connected normal subgroup of G. Then the Borel Density Theorem applied to the image of $Z(G)\Lambda$ in G/C gives that the image of T in G/C is a connected, normal abelian subgroup. Hence it must be trivial. Thus $T \subseteq C$, and so it is a torus normalized by $Z(G)\Lambda$, and $TZ(G)\Lambda$ is a closed subgroup of G containing the lattice Λ. Thus $TZ(G)\Lambda/\Lambda$ has finite volume, which in turn implies that $TZ(G) \cap \Lambda$ is a lattice in $TZ(G)$. Since $Z(G)$ is infinite by hypothesis, and since T is a torus, we conclude that $\Lambda' := TZ(G) \cap \Lambda$ is an infinite normal abelian subgroup of Λ. Since $[\Lambda, \Lambda'] \subset [\Lambda, T] \subset T$, and since T is compact, we have that $[\Lambda, \Lambda']$ is finite. Now since Λ

is finitely generated (every lattice in a connected Lie group is finitely generated), we can conclude easily that a subgroup of Λ' of finite index is contained in $Z(\Lambda)$. This proves that $Z(\Lambda)$ is infinite. $\diamond$

4 Some applications

In this section we finish the proof of Theorem 1.3. We then use Theorem 1.2 and its proof, and also Theorem 1.3, to prove the other theorems and corollaries stated in the introduction.

4.1 No normal abelian subgroups (proof of Theorem 1.3)

The fact that (2) implies (1) follows immediately from well-known properties of closed, locally symmetric Riemannian manifolds. Such M are aspherical by the Cartan-Hadamard theorem. Any normal abelian subgroup is trivial since the symmetric space $\widetilde{M}$ has no Euclidean factors. The other two properties follow from the definitions.

To prove that (1) implies (2), we first quote Proposition 3.3 followed by Proposition 3.1. This gives that M has a finite-sheeted Riemannian cover M' of M which is a smooth (indeed Riemannian warped) product $M' = N \times B$, where N is is isometric to a nonempty, irreducible, locally symmetric, nonpositively curved manifold. But M' is smoothly irreducible by hypothesis, so that B must be a single point. It follows that $M' = N$ is locally symmetric. Since the metric on M' was lifted from M, we have that M is locally symmetric.

4.2 Word-hyperbolic groups (proof of Corollary 1.4)

Again, (2) implies (1) follows immediately from the basic properties of closed, rank one locally symmetric manifolds.

To prove that (1) implies (2), first note that no torsion-free word-hyperbolic group can virtually be a nontrivial product, since then it would contain a copy of $\mathbf{Z} \times \mathbf{Z}$. It then follows from Theorem 1.3 that M is locally symmetric. But every closed, locally symmetric manifold M either contains $\mathbf{Z} \times \mathbf{Z}$ in its fundamental group, or M must be negatively curved; hence the latter must hold for M.

4.3 Almost simple groups (proof of Corollary 1.5)

This follows just as the proof of Corollary 1.4, but using the following fact: an irreducible, cocompact lattice in a noncompact semisimple Lie group G is almost simple if and only if $\mathrm{rank}_{\mathbf{R}} \geq 2$. The "if" direction is the statement of the Margulis Normal Subgroup Theorem (see [Ma], Thm. IX.5.4). For the "only if" direction, first recall that cocompact lattices in rank one semisimple Lie groups are non-elementary word-hyperbolic. Such groups are never almost simple; for example, a theorem of Gromov-Olshanskii (see [Ol]) gives that all such groups have infinite torsion quotients.

4.4 Universal bound (proof of Theorem 1.7)

A theorem of Kazhdan-Margulis (see, e.g., [Ra], Corollary XI.11.9) shows that, for every connected semisimple Lie group G, there exists $\epsilon = \epsilon(G)$ such that the covolume of every lattice in G is greater than ϵ. Let ϵ be this constant for $G = \mathrm{Isom}(\widetilde{M}, g_{loc})$, where g_{loc} denotes the lift of the locally symmetric metric on M.

Now let X denote $\widetilde{M}$ endowed with any fixed Riemannian metric h lifted from M. If $\mathrm{Isom}(X)$ is not discrete, then by Theorem 1.3 we have that h is homothetic to g_{loc}, and we are done. If $\mathrm{Isom}(X)$ is discrete, then $X/\mathrm{Isom}(X)$ is a compact orbifold, Riemannian covered by the compact manifold M, with degree of the cover $d := [\mathrm{Isom}(X) : \pi_1(M)]$. Let M' be a degree $e \geq 1$ cover of M which is a regular cover of $X/\mathrm{Isom}(X)$, and let F denote the covering group of this regular cover.

By Proposition 1.1 of [FW], F is isomorphic to a subgroup of $\mathrm{Isom}(M', g_{loc})$. Since the volume of a cover is multiplicative in degree, we have that

$$\epsilon < \mathrm{Vol}((M', g_{loc})/F) = \mathrm{Vol}(M', g_{loc})/de = e\,\mathrm{Vol}(M, g_{loc})/de$$

from which it follows that $d < \mathrm{Vol}(M, g_{loc})/\epsilon$, and we are done.

4.5 Models for compact and finite volume manifolds (proof of Theorem 1.8)

Let Γ_1 (resp. Γ_2) be a cocompact (resp. noncocompact) lattice in $\mathrm{Isom}(X)$. Since $\mathrm{Isom}(X)$ contains Γ_1, it acts cocompactly on X. Suppose $\mathrm{Isom}(X)$ were discrete, so that $\mathrm{Isom}(X)$ acts properly and cocompactly on X. Then the (orbifold) quotient $\mathrm{Isom}(X)\backslash X$ would have finite volume. Since covolume is multiplicative in index, it would follow that $\Gamma_2 < \mathrm{Isom}(X)$ has finite index. But then Γ_2 would act cocompactly since $\mathrm{Isom}(X)$ does, a contradiction. Hence $\mathrm{Isom}(X)$ is not discrete and we may apply Theorem 1.2.

We thus obtain a Riemannian orbibundle, which at the level of universal covers gives a Riemannian warped product structure $Y \tilde{\times} X_0$, with Y the universal cover of B, where the metric has the property that for each $x \in X$, the metric on $x \times X_0$ is an I_0- homogeneous metric, depending on x.

Let $\Lambda_i := \Gamma_i \cap I_0, i = 1, 2$. Claim I of the proof of Theorem 1.2 gives that Λ_1 is cocompact. We must now prove that Λ_2 is a noncocompact lattice.

To this end, first note that $\pi := \Gamma_2/\Lambda_2$ is a cocompact lattice in $\mathrm{Isom}(Y)$. First suppose B is not 1-dimensional. We can then perturb the metric on B to get a new universal (in the category of orbifolds) cover Y' with $\mathrm{Isom}(X) = \mathrm{Isom}(Y' \times X_0)$ but with $\mathrm{Isom}(Y') = \pi_1^{orb}(B)$. Now π is a lattice in $\mathrm{Isom}(Y')$, so it has finite index in $\pi_1(B)$.

We thus have that each of $\Gamma_i, i = 1, 2$ can be written as a group extension with kernel Λ_i and quotient a group with the same rational cohomological dimension $\mathrm{cd}_{\mathbf{Q}}$ as $\pi_1(B)$. We consider rational cohomological dimension $\mathrm{cd}_{\mathbf{Q}}$ in order to deal with the fact that $\pi_1(B)$ might not be virtually torsion-free.

Since each Γ_i acts properly on the contractible manifold X, and since Γ_1 acts cocompactly and Γ_2 does not, we have that $\mathrm{cd}_{\mathbf{Q}}(\Gamma_2) < \mathrm{cd}_{\mathbf{Q}}(\Gamma_1)$ (see, e.g. [Bro], VIII.8).

Now, if Λ_2 were cocompact, then Γ_2 would be an extension of fundamental groups of closed, aspherical manifolds, and so $\mathrm{cd}_{\mathbf{Q}}(\Gamma_2)$ would be the sum of the $\mathrm{cd}_{\mathbf{Q}}$ of the kernel and quotient; but this sum equals $\mathrm{cd}_{\mathbf{Q}}(\Gamma_1)$ (see, e.g., Theorem 5.5 of [Bi]), a contradiction. Hence Λ_2 is not cocompact.

If B is one-dimensional, then no perturbation as above exists. To remedy this, we simply take the product of B with a closed, genus 2 surface, endowed with a Riemannian metric with trivial isometry group. We then run the rest of the argument verbatim.

4.6 Irreducible lattices (proof of Theorem 1.9)

We first note that the hypothesis implies $\mathrm{Isom}(X)$ is not discrete. Since in addition X has a cocompact discrete subgroup, we may apply Theorem 1.2. Hence X is isometric to a warped Riemannian product $X = Y \tilde{\times} X_0$. By the proof of Theorem 1.2, the group $\mathrm{Isom}(X_0)$ corresponds with connected component of the identity of $\mathrm{Isom}(X)$. In particular $\mathrm{Isom}(Y)$ must be discrete. The theorem follows easily.

4.7 The Hopf Conjecture

A well-known conjecture of Hopf-Chern-Thurston states that the Euler characteristic of any closed, aspherical manifold M^{2k} satisfies $(-1)^k\chi(M^{2k}) \geq 0$. A stronger conjecture of Singer posits that the L^2-cohomology of M^{2k} vanishes except in dimension k (see, e.g. [CG, Lu]). These conjectures are completely open except when $k = 1$.

We will now prove that the Hopf Conjecture holds for those smooth, aspherical manifolds M^{2k} which admit *some* Riemannian metric with symmetry.

Theorem 4.1. *Let M^{2k} be any closed, aspherical, smooth manifold which is smoothly irreducible. If M admits some Riemannian metric so that the induced metric on the universal cover $\widetilde{M}$ satisfies* $[\mathrm{Isom}(\widetilde{M}) : \pi_1(M)] = \infty$, *then the Singer Conjecture (and hence the Hopf Conjecture) is true for M^{2k}.*

The Singer Conjecture clearly holds for products of surfaces and also for products of any 3-manifolds with S^1. Thus Theorem 4.1 holds in dimension four without the assumption that M^4 is smoothly irreducible.

Proof of Theorem 4.1. If $\pi_1(M^{2k})$ has no nontrivial normal abelian subgroups and is smoothly irreducible, then by Theorem 1.3 it admits a locally symmetric Riemannian metric. The Singer Conjecture is known for such manifolds (see, e.g. [Lu], Cor. 5.16). If $\pi_1(M^{2k})$ does contain a nontrivial, normal abelian subgroup A, then one may apply a theorem of Cheeger-Gromov ([CG], Cor. 0.6) which gives, even more generally for amenable A, that Singer's Conjecture holds. $\diamond$

4.8 Complex manifolds (proof of Theorem 1.10)

As pointed out above, Theorem 1.10 follows immediately from Proposition 0.1 in [Na] together with Claim IV in the proof of Theorem 1.2.

Actually, we can do without Nadel's result under certain mild assumptions. Since $\mathrm{Aut}(\widetilde{M})$ acts isometrically in some Riemannian metric, we can apply Theorem 1.2 directly to obtain the claimed splitting, but with two problems: first, the splitting is an isometric (not holomorphic) one; and second, the factor M_1 is only locally homogeneous, not necessarily locally symmetric.

The second problem is corrected once we know that $\mathrm{Aut}(\widetilde{M})^0$ is semisimple with finite center; equivalently, M_1 doesn't fiber with nontrivial solvmanifold fiber. One way to rule this out is to assume that $\pi_1(M)$ contains no infinite normal abelian subgroups, giving us a new proof of Theorem 1.10 in this case. In complex dimension two, M_1 must be locally symmetric, for otherwise it would have a fibering with torus fiber, giving that $\chi(M) = 0$. However, in complex dimension two, we have

$$\chi(M) = c_2(M) \geq \frac{1}{3}c_1(M)^2 > 0$$

by the Bogomolov-Miyaoka-Yau inequality.

To correct the first problem, once we have that the factor M_1 is locally symmetric, we apply Siu's Rigidity Theorem (also used in [Fr2]) to obtain that M_1 is biholomorphic to a closed, Hermitian locally symmetric space.

5 An extension to the non-aspherical case

A number of the results from this paper generalize from closed, aspherical Riemannian manifolds to all closed Riemannian manifolds. As an illustrative example we give the following theorem. As far as we know, this is the first geometric rigidity theorem for non-aspherical manifolds with infinite fundamental group.

We say that the universal cover $\widetilde{M}$ of a Riemannian manifold M has *essential extra symmetry* if, for a compact subset $K \subset \widetilde{M}$, and for all $\epsilon > 0$, there exists $g \in \mathrm{Isom}(\widetilde{M})$ such that

- $d(g(m), m) < \epsilon$ for all $m \in K$

- $\sup d(g^n(m), m) = \infty$ for any fixed m.

The condition that $\widetilde{M}$ have an essential symmetry is equivalent to the identity component of $\mathrm{Isom}(\widetilde{M})$ being noncompact. If $\pi_1(M)$ is torsion free, this is equivalent to $\mathrm{Isom}(\widetilde{M})$ not being an extension of $\pi_1(M)$ by any compact group. (If the identity component of $\mathrm{Isom}(\widetilde{M})$ were compact, then $\pi_1(M)$ would intersect this group in a normal lattice by Claim I, and this must be trivial if $\pi_1(M)$ is torsion free.)

Theorem 5.1. *Let M be any closed Riemannian manifold which, for simplicity, is smoothly irreducible. Suppose that $\pi_1(M)$ contains no infinite, finitely-generated, normal abelian subgroup, and that M has essential extra symmetries. Then there exists a finite Riemannian cover M' of M which is a fiber bundle over a closed, irreducible, locally symmetric manifold, with all fibers isometric. In particular the structure group of the bundle is compact.*

The proof of Theorem 5.1 follows quite closely the proof of Theorem 1.3, with only a few adaptations. Hence instead of a detailed proof, we now just indicate the adaptations that are necessary.

Proof. We will describe how to modify the proof of Theorem 1.3 in order to prove Theorem 5.1. As in the proof of Theorem 1.3, the isometry group $\mathrm{Isom}(\widetilde{M})$ is a semidirect product of I_0 and Δ, where $\Gamma = \pi_1(M)$ intersects I_0 in a lattice Γ_0. As before, the hypothesis that Γ has no infinite, finitely-generated, normal abelian subgroup implies that I_0 is semisimple with finite center.

This implies as in §3.1 that, replacing M by a finite cover if necessary, we have a product structure on Γ. We then have a harmonic map $f : M \longrightarrow K_0\backslash I_0/\Gamma_0$, where K_0 is a maximal compact subgroup of I_0. This will provide the orbibundle structure on M, and the Lie group I_0 will be responsible for the isometries among the fibers.

The proof of this is again based on the uniqueness of harmonic maps to nonpositively curved manifolds, when one has a surjection of fundamental groups, and no center in the image. Here, since Γ splits as a product, having such a center would give us a normal abelian subgroup, contradicting the hypothesis. On $\widetilde{M}$ the Γ-equivariant map given by the lift of f extends, by the arithmetic group trick of Claim 3.2, to an I_0-equivariant map $F : \widetilde{M} \longrightarrow I_0/K_0$. Consequently the fibers are isometric to each other as the I_0-action on the target is transitive. $\diamond$

6 A truly singular orbibundle

In this section we construct a (7-dimensional) Riemmanian manifold M with the property that M is a Riemannian orbibundle, but no finite cover of M is a Riemannian fiber bundle. M will also have the property that $\mathrm{Isom}(\widetilde{M})$ is not discrete. This will prove that the "orbibundle" conclusion of Theorem 1.2 cannot be improved to a fiber bundle, even if one is willing to pass to a finite cover.

We begin with the following construction, which we believe is of independent interest.

Theorem 6.1. *There is a group Γ which acts cocompactly, properly discontinuously by diffeomorphisms on $\mathbf{R}^n$ for some n, but which is not virtually torsion free. Moreover, there exists $\xi \in H^2(\Gamma, \mathbf{Z})$ which restricts to a nonzero class on some $\mathbf{Z}/p\mathbf{Z} \in \Gamma$, where p is a prime, and which also restricts to a nonzero class on every finite index subgroup.*

Proof. Let $G = \mathbf{Z}/p\mathbf{Z} * \mathbf{Z}$. Then G acts properly discontinuously and cocompactly on its Bass-Serre tree T. In [Bri], Bridson shows that there is an amalgamated free product $H = G *_F G$, with F a nonabelian free group, which has no finite quotients. Let Y be the universal cover of the standard Cayley 2-complex for H; hence H acts properly discontinuously and cocompactly on Y. It is easy to see that Y is contractible.

Equivariantly thicken Y (see [As] for the details of equivariant thickening). The action on this thickening gives a properly discontinuous, cocompact Γ-action on a contractbile 6-dimensional manifold. Since the action is simplicial, we can perform the Davis reflection group construction (see [Da]) equivariantly to build a cocompact action of a group Λ on a contractible manifold, and with H being a retract of Λ. This action can in fact be made smooth, as is explained in §17 of [Da]. By a theorem of Stallings [St], the group $\Gamma := \Lambda \times \mathbf{Z}$ then acts on $\mathbf{R}^n$ properly discontinuously and cocompactly by diffeomorphisms.

Note that any finite index subgroup of Γ intersects H in a finite index subgroup of H, which must therefore be all of H since H has no finite index subgroups. Hence the fixed $\mathbf{Z}/p\mathbf{Z}$ subgroup of Γ must lie in each finite index subgroup of Γ.

Now by construction there is a surjection $\Gamma \longrightarrow G$. Let $\xi \in H^2(\Gamma, \mathbf{Z})$ denote the pullback of the class generating $H^2(G, \mathbf{Z})$. As the amalgamating subgroup F is free, the amalgamation dimension of $H^2(H, \mathbf{Z})$ is at most one greater than $H^2(G, \mathbf{Z})$. Since H is a retract of Λ, we know that $\xi \in H^2(\Lambda, \mathbf{Z})$ is also nonzero. It follows easily that ξ pulls back to a nonzero class in Γ. $\diamond$

We now build the manifold M. Let Γ, ξ be given as in Theorem 6.1. Let $\widetilde{\Gamma}$ denote the central extension of Γ given by the cocycle $\xi \in H^2(\Gamma, \mathbf{Z})$. Since this cocycle vanishes in $H^2(\Gamma, \mathbf{R})$, we have that $\widetilde{\Gamma}$ lies in $\Gamma \times \mathbf{R}$. Now fix any Γ-invariant metric on $\mathbf{R}^n$, and extend this to any $(\Gamma \times \mathbf{R})$-invariant metric on $\mathbf{R}^n \times \mathbf{R} = \mathbf{R}^{n+1}$; call the resulting Riemannian manifold Y.

Now $\Gamma \times \mathbf{R}$ acts properly discontinuously and cocompactly by isometries on $Y \approx \mathbf{R}^{n+1}$. The quotient M clearly satisfies the claimed properties. Note too that $\mathrm{Isom}(\widetilde{M}) = \mathrm{Isom}(Y)$ contains $\mathbf{R}$, and so is not discrete.

Remark. in the examples above, the dimension of M is at least 7. We do not know whether this dimension can be lowered. Indeed, it seems difficult to obtain information about the geometry of such examples, although they do seem compatible with at least large-scale nonpositive curvature.

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