Parameterized Abel–Jacobi maps and abelian cycles in the Torelli group

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Abstract

Let $\mathcal{I}_{g,*}$ denote the *Torelli group* of the genus $g \geq 2$ surface S_g with one marked point. This is the group of homotopy classes (rel basepoint) of homeomorphisms of S_g fixing the basepoint and acting trivially on $H \coloneqq H_1(S_g, \mathbb{Q})$. In 1983 Johnson constructed a beautiful family of invariants

$$\tau_i \colon H_i(\mathcal{I}_{g,*}, \mathbb{Q}) \to \bigwedge^{i+2} H$$

for $0 \leq i \leq 2g-2$, using a kind of Abel–Jacobi map for families. He used these invariants to detect nontrivial cycles in $\mathcal{I}_{g,*}$. Johnson proved that τ_1 is an isomorphism, and asked if the same is true for τ_i with $i > 1$.

The goal of this paper is to introduce various methods for computing τ_i; in particular we prove that τ_i is not injective for any $2 \leq i < g$, answering Johnson's question in the negative. We also show that τ_2 is surjective. For $g \geq 3$, we find many classes in the image of τ_i and use them to deduce that $H_i(\mathcal{I}_{g,*}, \mathbb{Q}) \neq 0$ for each $1 \leq i < g$. This is in contrast with the case of mapping class groups. Many of our classes are stable, so we can deduce that $H_i(\mathcal{I}_{\infty,1}, \mathbb{Q})$ is infinite-dimensional for each $i \geq 1$. Finally, we conjecture a new kind of "representation-theoretic stability" for the homology of the Torelli group, for which our results provide evidence.

1 Introduction

Let S_g be a connected, closed, oriented surface of genus $g \geq 2$, let $H \coloneqq H_1(S_g, \mathbb{Q})$, and let $H_\mathbb{Z} \coloneqq H_1(S_g, \mathbb{Z})$. The *(pointed) Torelli group* $\mathcal{I}_{g,*}$ is the group of pointed homotopy classes of pointed homeomorphisms of S_g acting trivially on $H_\mathbb{Z}$. Understanding $\mathcal{I}_{g,*}$, and particularly its (co)homology, is an important problem in topology and algebraic geometry (see, e.g., [Jo83], [Ha95] and [Mo99] for discussions).

Parametrized Abel–Jacobi maps. In [Jo83], Johnson produced a beautiful family of invariants, and used them to prove that certain cycles in $H_*(\mathcal{I}_{g,*}, \mathbb{Q})$ are nontrivial. He did this by constructing, for each $0 \leq i \leq 2g-2$, an $\mathrm{Sp}(2g, \mathbb{Z})$–equivariant homomorphism

$$\tau_i \colon H_i(\mathcal{I}_{g,*}, \mathbb{Q}) \to \bigwedge^{i+2} H.$$

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The maps τ_i can be described using a kind of parametrized Abel–Jacobi map, as follows. We first outline an explicit construction of $\tau_1 \colon H_1(\mathcal{I}_{g,*}, \mathbb{Q}) \to \bigwedge^3 H$. Let $S_{g,*}$ denote S_g with a marked point. For any $f \in \mathcal{I}_{g,*}$, the image $\tau_1([f]) \in \bigwedge^3 H$ can be computed as follows. Construct the mapping torus

$$M_f := S_{g,*} \times [0,1]/(f(p),0) \sim (p,1).$$

This 3–manifold is naturally a bundle with section $S_{g,*} \to M_f \to S^1$, and the fact that f acts trivially on $H_1(S_{g,*})$ implies that $H_1(M_f)$ is canonically isomorphic to $H_1(S_{g,*}) \times H_1(S^1)$.

The Jacobian of a Riemann surface S_g is the complex torus $\mathrm{Jac}(S_g) \approx T^{2g}$ given by $(H^1(S_g, \mathbb{C}))^*/H_1(S_g, \mathbb{Z})$. The Abel–Jacobi map is a holomorphic map $j \colon (S_g, *) \to (\mathrm{Jac}(S_g), 0)$ unique in its homotopy class, with the property that the induced map on fundamental group $j_* \colon \pi_1(S_g, *) \to \pi_1(\mathrm{Jac}(S_{g,*}), 0) \approx H_{\mathbb{Z}}$ is the abelianization. The composition

$$\pi_1(M_f) \to H_1(M_f) \approx H_1(S_{g,*}) \times H_1(S^1) \to H_1(S_{g,*})$$

induces a map

$$J \colon M_f \to \mathrm{Jac}(S_g) \qquad (1)$$

unique up to homotopy. Now $\tau_1([f])$ is defined as the image of the fundamental class $[M_f] \in H_3(M_f, \mathbb{Q})$ under the induced map $J_* \colon H_3(M_f, \mathbb{Q}) \to H_3(\mathrm{Jac}(S_g), \mathbb{Q}) \approx \bigwedge^3 H$.

For the definition of τ_i for all $i \geq 1$, let $\mathcal{T}_{g,*}$ denote the *Torelli space* of Riemann surfaces diffeomorphic to S_g endowed with a homology marking and a marked point; that is, $\mathcal{T}_{g,*}$ is the quotient of the Teichmüller space $\mathrm{Teich}_{g,*}$ by the (free) action of $\mathcal{I}_{g,*}$. As $\mathrm{Teich}_{g,*}$ is contractible, we have that $H_i(\mathcal{I}_{g,*}) \approx H_i(\mathcal{T}_{g,*})$. Let

$$S_{g,*} \to \mathcal{T}^*_{g,*} \xrightarrow{\pi} \mathcal{T}_{g,*} \qquad (2)$$

denote the universal $S_{g,*}$–bundle over $\mathcal{T}_{g,*}$. There is also a universal bundle

$$\mathrm{Jac}(S_g) \to \mathcal{T}^{\mathrm{Jac}}_{g,*} \to \mathcal{T}_{g,*} \qquad (3)$$

of Jacobians, and the Abel–Jacobi map j globalizes to give a (holomorphic) map

$$J \colon \mathcal{T}^*_{g,*} \to \mathcal{T}^{\mathrm{Jac}}_{g,*}. \qquad (4)$$

Note that the zero element in each fiber gives a section of the bundle (3), and that $\mathcal{I}_{g,*}$ acts trivially on the fiber $\mathrm{Jac}(S_g)$. These are the only obstructions to triviality for a torus bundle, so the bundle $\mathcal{T}^{\mathrm{Jac}}_{g,*} \to \mathcal{T}_{g,*}$ is topologically trivial. Let $p \colon \mathcal{T}^{\mathrm{Jac}}_{g,*} \to \mathrm{Jac}(S_g)$ denote projection to the torus factor.

We can now define $\tau_i(\sigma)$ for an i–cycle σ in $\mathcal{T}_{g,*}$, where $i \leq 2g-2$. The inverse image $\pi^{-1}(\sigma)$ in $\mathcal{T}^*_{g,*}$ gives an $(i+2)$–cycle (this operation is sometimes called the *Gysin homomorphism*), and we can take the image of this $(i+2)$–cycle under the composition $p \circ J$, giving us an element $\tau_i(\sigma) \in H_{i+2}(\mathrm{Jac}(S_g), \mathbb{Q}) \approx \bigwedge^{i+2} H$.

The images of the maps τ_0 and τ_1 are worked out explicitly in Section 2 below. In particular, as proved by Hain in [Ha97] (see also Proposition 2.2 below), the map

τ_1 agrees with the original, purely algebraic definition of the *Johnson homomorphism*, which plays a central role in the study of $\mathcal{I}_{g,*}$. In a series of papers, Johnson proved that τ_1 is, modulo torsion, an isomorphism (see [Jo83] for a summary of this work). In his 1983 paper [Jo83], Johnson constructed τ_i as above, and as Question C he asked if τ_i is a rational isomorphism for all $i \geq 1$. In [Ha97] Hain used continuous cohomology and representation theory to prove that τ_2 is not injective; it seems that Hain's method cannot be extended to the case when $i \geq 3$. In this paper we develop a method for concretely computing the values of the τ_i. Our first main result answers Johnson's question negatively in degrees $2 \leq i < g$.

Theorem 1.1. *The map τ_i is not injective for any $2 \leq i < g$.*

Remark. The map τ_i can be defined on integral homology, with target $\bigwedge^{i+2} H_{\mathbb{Z}}$. Since the target is free abelian, and since the elements we construct in the kernel of τ_i are integral classes, Theorem 1.1 implies that our classes also lie in the kernel of this integral version of τ_i.

We will find a number of sources for nontrivial cycles in $\ker \tau_i$. One source will be certain "abelian cycles" coming from bounding pair maps (see below). These cycles are determined by certain collections of simple closed curves. The (non)vanishing of τ_i on such cycles will depend on the topological configuration of the collection of curves, namely whether or not they are "truly nested" (see Definition 3.3). The nontriviality of cycles in the kernel of τ_i is detected by combining certain operations in the homology of Torelli groups with other τ_j for $j \neq i$. We remark that Bestvina–Bux–Margalit [BBM] found nontrivial elements of $H_{3g-3}(\mathcal{I}_{g,*}, \mathbb{Q})$; there is no τ_i defined in this dimension since $3g - 3 > 2g - 2$.

In the positive direction of Johnson's question, we show that the τ_i detect nontrivial classes in each dimension; in particular we prove that τ_2 is surjective. Our general theorem in this direction is most simply stated in the language of symplectic representation theory. From the standard exact sequence

$$1 \to \mathcal{I}_{g,*} \to \mathrm{Mod}_{g,*} \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1,$$

the conjugation action of $\mathrm{Mod}_{g,*}$ on $\mathcal{I}_{g,*}$ descends to an action of $\mathrm{Sp}(2g, \mathbb{Z})$ by outer automorphisms, which gives $H_i(\mathcal{I}_{g,1}, \mathbb{Q})$ the structure of an $\mathrm{Sp}(2g, \mathbb{Z})$–module. The construction of the homomorphism τ_i shows that it is $\mathrm{Sp}(2g, \mathbb{Z})$–equivariant.

Irreducibility remark. Let V be an irreducible $\mathrm{Sp}(2g, \mathbb{Q})$–representation. It follows from Proposition 3.2 of [Bo] that V is an irreducible $\mathrm{Sp}(2g, \mathbb{Z})$–module (this is close to the statement of the Borel Density Theorem in this case). Henceforth we will not make the distinction of irreducibility over $\mathbb{Q}$ versus irreducibility over $\mathbb{Z}$.

The algebraic irreducible representations of $\mathrm{Sp}(2g, \mathbb{Q})$ are classified by their highest weight vectors (a good reference is [FH]). Choose a set $\lambda_1, \ldots, \lambda_g$ of fundamental weights for $\mathrm{Sp}(2g, \mathbb{Q})$. A highest weight vector is a linear combination $\lambda = \sum c_i \lambda_i$, where the coefficients are nonnegative integers. We denote the irreducible representation of

$\mathrm{Sp}(2g,\mathbb{Q})$ with highest weight vector λ by $V(\lambda)$. For example, $V(\lambda_i)$ is the kernel of the contraction $C_i : \bigwedge^i H \to \bigwedge^{i-2} H$ defined by:

$$C_i(x_1 \wedge \cdots \wedge x_i) = \sum_{j<k} (-1)^{j+k+1} \omega(x_j, x_k) x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge \widehat{x}_k \wedge \cdots \wedge x_i \qquad (5)$$

The $\mathrm{Sp}(2g,\mathbb{Q})$–module $\bigwedge^k H$ for $k \leq g$ decomposes into irreducible representations as

$$\bigwedge^k H \approx V(\lambda_k) \oplus V(\lambda_{k-2}) \oplus \cdots \oplus V(\lambda_\varepsilon)$$

where $\varepsilon = 0$ or 1 depending on whether k is even or odd. Our second main result is the following.

Theorem 1.2. *Suppose $g \geq 2$. Then for $1 \leq i \leq g-2$, we have*

$$\tau_i(H_i(\mathcal{I}_{g,*},\mathbb{Q})) \supseteq V(\lambda_{i+2}) \oplus V(\lambda_i). \qquad (6)$$

In addition, for $1 \leq i \leq g$ and i even, we have

$$\tau_i(H_i(\mathcal{I}_{g,*},\mathbb{Q})) \supseteq V(\lambda_{i-2}). \qquad (7)$$

For $i = g-1$, the term $V(\lambda_{i+2})$ in (6) is not meaningful, but the proof of Theorem 1.2 will show that $\tau_{g-1}(H_{g-1}(\mathcal{I}_{g,*},\mathbb{Q}))$ contains $V(\lambda_{g-1})$.

Since $\bigwedge^4 H = V(\lambda_4) \oplus V(\lambda_2) \oplus V(\lambda_0)$, we have the following.

Corollary 1.3. Let $g \geq 2$. Then τ_2 is surjective.

We wish to point out that Morita announced in [Mo89] that a map closely related to τ_2 is surjective. As another corollary of Theorem 1.2 we deduce the following.

Corollary 1.4. Let $g \geq 2$. Then $H_i(\mathcal{I}_{g,*},\mathbb{Q})$ is nonzero for each $1 \leq i < g$. When g is even, $H_g(\mathcal{I}_{g,*},\mathbb{Q})$ is also nonzero.

Theorem 1.2 also provides evidence for a "homological stability" conjecture for the Torelli group, which we now outline.

Stable classes. The nontrivial classes we construct above are stable. In order to explain this we need to extend our picture to surfaces with boundary. Let $\mathcal{I}_{g,1}$ denote the group of homotopy classes of homeomorphisms of the compact genus $g \geq 2$ surface $S_{g,1}$ with one boundary component, acting trivially on $H_1(S_{g,1},\mathbb{Z})$. Here both the homeomorphisms and homotopies are taken to be the identity on $\partial S_{g,1}$.

The map $S_{g,1} \to S_{g,*}$ that identifies $\partial S_{g,1}$ to a single (marked) point gives a homomorphism $\nu\colon \mathcal{I}_{g,1} \to \mathcal{I}_{g,*}$ whose kernel is the cyclic group generated by the Dehn twist about $\partial S_{g,1}$. We define a homomorphism

$$\widehat{\tau}_i\colon H_i(\mathcal{I}_{g,1},\mathbb{Q}) \to \bigwedge^{i+2} H$$

by composing τ_i with the map on homology induced by ν. From the proof of Theorem 1.2, we immediately obtain, for $1 \leq i \leq g-2$, that

$$\widehat{\tau}_i(H_i(\mathcal{I}_{g,1},\mathbb{Q})) \supseteq V(\lambda_{i+2}) \oplus V(\lambda_i).$$

Now, the natural inclusion $S_{g,1} \hookrightarrow S_{g+1,1}$ induces a natural inclusion $\mathcal{I}_{g,1} \hookrightarrow \mathcal{I}_{g+1,1}$. We can thus form the direct limit

$$\mathcal{I}_{\infty,1} := \lim_{g\to\infty} \mathcal{I}_{g,1},$$

called the *stable Torelli group*. It is easy to see from the definitions that the following diagram is commutative, where $H_g = H_1(S_{g,1}, \mathbb{Q})$ and $H_{g+1} = H_1(S_{g+1,1}, \mathbb{Q})$:

$$\begin{array}{ccc} H_i(\mathcal{I}_{g,1}, \mathbb{Q}) & \longrightarrow & H_i(\mathcal{I}_{g+1,1}, \mathbb{Q}) \\ \downarrow{\scriptstyle \widehat{\tau}_i} & & \downarrow{\scriptstyle \widehat{\tau}_i} \\ \bigwedge^i H_g & \longrightarrow & \bigwedge^i H_{g+1} \end{array}$$

It follows that each nontrivial class in $H_i(\mathcal{I}_{g,1}, \mathbb{Q})$ constructed above is *stable*, in that its image in $H_i(\mathcal{I}_{g+k,1}, \mathbb{Q})$ is nontrivial for each $k \geq 0$. As homology preserves direct limits, and since $\dim V(\lambda_i) \to \infty$ as $g \to \infty$, we have the following corollary.

Corollary 1.5. For each $i \geq 1$, the vector space $H_i(\mathcal{I}_{\infty,1}, \mathbb{Q})$ is infinite-dimensional.

This greatly contrasts with the situation for the stable mapping class group, whose odd-dimensional homology vanishes, and whose even-dimensional homology has finite rank (see [MW]).

A stability conjecture for $H_*(\mathcal{I}_{g,1}, \mathbb{Q})$. The stability of the homology classes we construct, together with the presence of nontrivial $\mathrm{Sp}(2g, \mathbb{Z})$–modules in $H_i(\mathcal{I}_{g,1}, \mathbb{Q})$, shows that the classical kind of homological stability, satisfied for example by $\mathrm{GL}_n, \mathrm{Out}(F_n)$, and the mapping class group, does not hold for the Torelli group. However, our results provide evidence for a new kind of "representation-theoretic stability", which we now describe.

We begin with the simplest, quickest-to-state form of our conjecture. When we want to emphasize the group that acts, we will denote by $V(\lambda)_{2g}$ the irreducible $\mathrm{Sp}(2g, \mathbb{Z})$–representation with highest weight vector λ.

Conjecture 1.6 (Representation stability, I)**.** *The homology of the Torelli group is* representation stable *with respect to g: for each $i \geq 1$ and each g sufficiently large (depending on i), we have that the* $\mathrm{Sp}(2g, \mathbb{Z})$*–module $H_i(\mathcal{I}_{g,1}, \mathbb{Q})$ contains the representation $V(\lambda)_{2g}$ with some multiplicity $0 \leq m \leq \infty$ if and only if for each $h \geq g$ the* $\mathrm{Sp}(2h, \mathbb{Z})$*–module $H_i(\mathcal{I}_{h,1}, \mathbb{Q})$ contains the representation $V(\lambda)_{2h}$ with multiplicity m, and similarly for $\mathcal{I}_{g,*}$ and $\mathcal{I}_g$.*

Applying a result of Kawazumi–Morita [KM, Theorem 5.5], it can be deduced that the truth of this conjecture for $\mathcal{I}_{g,*}$ is equivalent to the truth of the conjecture for $\mathcal{I}_g$. We expect that the conjecture for $\mathcal{I}_{g,1}$ is similarly equivalent.

Morita has conjectured [Mo99, Conjecture 3.4] that the Sp–invariant stable cohomology of $\mathcal{I}_{g,1}$ is generated by the even Miller–Morita–Mumford classes. Morita's Conjecture would immediately imply the special case of Conjecture 1.6 when $V(\lambda)$ is the trivial representation.

We would like to refine Conjecture 1.6 by giving a more direct comparison of the homology of different Torelli groups. Of course we cannot ask for an isomorphism of $H_i(\mathcal{I}_{g,1},\mathbb{Q})$ and $H_i(\mathcal{I}_{h,1},\mathbb{Q})$ as modules since the first is an $\mathrm{Sp}(2g,\mathbb{Z})$–module and the second is an $\mathrm{Sp}(2h,\mathbb{Z})$–module. However, there are meaningful injectivity and surjectivity statements one can ask for, as we will see in Conjecture 1.7 below.

Our main conjecture makes predictions about the finite-dimensional part of $H_i(\mathcal{I}_{g,1},\mathbb{Q})$. We define the *finite-dimensional homology* $H_i(\mathcal{I}_{g,1},\mathbb{Q})^{\mathrm{fd}}$ to be the subspace of $H_i(\mathcal{I}_{g,1},\mathbb{Q})$ consisting of those vectors whose $\mathrm{Sp}(2g,\mathbb{Z})$–orbit spans a finite-dimensional vector space.

Conjecture 1.7 (Representation stability, II)**.** *For each $i \geq 1$ and each g sufficiently large (depending on i), the following hold:*

Finite-dimensionality: *The natural map $i_*\colon H_i(\mathcal{I}_{g,1},\mathbb{Q})^{\mathrm{fd}} \to H_i(\mathcal{I}_{g+1,1},\mathbb{Q})$ induced by the inclusion $i\colon \mathcal{I}_{g,1} \to \mathcal{I}_{g+1,1}$ has image contained in $H_i(\mathcal{I}_{g+1,1},\mathbb{Q})^{\mathrm{fd}}$.*

Injectivity: *The natural map $i_*\colon H_i(\mathcal{I}_{g,1},\mathbb{Q})^{\mathrm{fd}} \to H_i(\mathcal{I}_{g+1,1},\mathbb{Q})$ is injective.*

Surjectivity: *The span of the $\mathrm{Sp}(2g+2,\mathbb{Z})$–orbit of $i_*(H_i(\mathcal{I}_{g,1},\mathbb{Q})^{\mathrm{fd}})$ equals all of $H_i(\mathcal{I}_{g+1},\mathbb{Q})^{\mathrm{fd}}$.*

Rationality: *Every irreducible $\mathrm{Sp}(2g,\mathbb{Z})$–subrepresentation in $H_i(\mathcal{I}_{g,1},\mathbb{Q})^{\mathrm{fd}}$ is the restriction of an irreducible $\mathrm{Sp}(2g,\mathbb{Q})$–representation.*

Type preserving: *For any representation $V(\lambda)_{2g} \subset H_i(\mathcal{I}_{g,1},\mathbb{Q})^{\mathrm{fd}}$, the span of the $\mathrm{Sp}(2g+2,\mathbb{Z})$–orbit of $V(\lambda)_{2g}$ is isomorphic to $V(\lambda)_{2g+2}$.*

Remarks.

1. A form of the Margulis Superrigidity Theorem (see [Ma], Theorem VIII.B) gives that any finite-dimensional representation (over $\mathbb{C}$) of $\mathrm{Sp}(2g,\mathbb{Z})$ either (virtually) extends to a (rational) representation of $\mathrm{Sp}(2g,\mathbb{R})$ or factors through a finite group[1]. The "rationality" statement of Conjecture 1.7 is meant to rule out the latter possibility for subrepresentations of $H_i(\mathcal{I}_{g,1},\mathbb{Q})^{\mathrm{fd}}$.

2. It is possible to embed the $\mathrm{Sp}(2g,\mathbb{Q})$–module $V(\lambda_i)_{2g}$ into the $\mathrm{Sp}(2g+2,\mathbb{Q})$–module $V(\lambda_{i+1})_{2g+2}$ so that the $\mathrm{Sp}(2g+2,\mathbb{Q})$–span of the image is all of $V(\lambda_{i+1})_{2g+2}$, and similarly for other pairs of irreducible representations. The "type preserving" statement in Conjecture 1.7 is meant to rule out this type of phenomenon.

3. Theorem 1.2 shows that the "stable range" in Conjecture 1.7, meaning the smallest g for which $H_i(\mathcal{I}_{g,1},\mathbb{Q})^{\mathrm{fd}}$ stabilizes, must be at least i.

4. Mess [Me] proved that $H_1(\mathcal{I}_{2,1},\mathbb{Q})$ contains an infinite-dimensional, irreducible permutation $\mathrm{Sp}(4,\mathbb{Z})$–module. Similarly, the classes in $H_{3g-2}(\mathcal{I}_{g,1},\mathbb{Q})$ found by Bestvina–Bux–Margalit [BBM] span an infinite-dimensional, permutation $\mathrm{Sp}(2g,\mathbb{Z})$–module. Neither of these is "stable" in g; one might hope that stably, such representations do not arise, and all irreducible Sp–submodules of $H_i(\mathcal{I}_{g,1},\mathbb{Q})$ are finite-dimensional for $g \gg i$.

[1] One can also use the solution to the congruence subgroup property for $\mathrm{Sp}(2g,\mathbb{Z}), g > 1$ here; see [BMS].

5. Conjecture 1.6 and Conjecture 1.7 would give an affirmative answer to Question 7.9 of Hain–Looijenga [HL], which asked "Is $H^k(T_g)$ expressible as an $\mathrm{Sp}_{2g}(\mathbb{Z})$–module in a way that is independent of g if g is large enough?"

Evidence. As mentioned above, Theorem 1.2 provides evidence in every dimension for Conjectures 1.6 and 1.7. Both conjectures are true in dimension 1, by Johnson's computation that $H_1(\mathcal{I}_{g,1}) \approx H_1(\mathcal{I}_{g,*}) \approx V(\lambda_3) \oplus V(\lambda_1)$ and $H_1(\mathcal{I}_g) \approx V(\lambda_3)$. Other than this, very little is known about the homology of the Torelli group. However, in low dimensions we have work of Hain, who found a large subspace of $H_2(\mathcal{I}_g)$, and Sakasai, who found a large subspace of $H_3(\mathcal{I}_g)$. We describe their methods and state their results in Section 5.3; the resulting subspaces satisfy Conjecture 1.7.

Since this paper was first distributed, Boldsen–Dollerup [BD] have proved that the surjectivity condition in Conjecture 1.7 holds for $H_2(\mathcal{I}_{g,1};\mathbb{Q})$ as long as $g \geq 6$.

In the paper [CF] we situate these conjectures in the much broader framework of a general theory of "representation stability".

Outline of paper. In §2 we outline our general approach to computing the τ_i, and explicitly work out τ_0 and τ_1 as a warmup. In §3 we give two ways of computing τ_i. We first show how to compute the image under τ_i of the "product" of a cycle supported on a subsurface with a bounding pair map. We then give a vanishing result for cycles built from gluing subsurfaces along a pair-of-pants. We then apply these tools in order to compute both vanishing and nonvanishing results for τ_i of abelian cycles in $H_i(\mathcal{I}_{g,*})$. Section 4 gives a computation of τ_i on cycles in $H_i(\mathcal{I}_{g,*})$ that are surface bundles over certain tori in $H_{i-2}(\mathcal{I}_g)$. This computation reveals the phenomenon of an even/odd dichotomy for the nonvanishing/vanishing of cycles; in particular we obtain many new nonzero classes in $H_i(\mathcal{I}_{g,*})$. In §5 we use all the computations above to complete the proofs of Theorem 1.1 and Theorem 1.2. We conclude by explaining how theorems of Hain and Sakasai give further evidence for Conjecture 1.6 and Conjecture 1.7.

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2 Setup and first examples

In this section, we outline the framework of the computations in this paper. We then compute τ_0 and τ_1 as the simplest examples of our methods. For the rest of the paper, all homology groups are taken with coefficients in $\mathbb{Q}$, although the reader may just as well imagine the coefficients are $\mathbb{Z}$ if preferred.

Bundles representing homology classes. As mentioned in the introduction, the bundle $S_{g,*} \to \mathcal{T}^*_{g,*} \xrightarrow{\pi} \mathcal{T}_{g,*}$ described in (2) is the *universal* genus g surface bundle equipped with a section and a trivialization of its fiberwise homology. (All the surface bundles we consider in this paper will be endowed with such a trivialization, namely an identification of the homology of each fiber with $H_1(S_g)$.) This means that for any base B, there is a bijective correspondence between such bundles over B up to isomorphism and maps $f\colon B \to \mathcal{T}_{g,*}$ up to homotopy; the correspondence is induced by pulling back the universal bundle along the map f.

Given a homology class $\sigma \in H_i(\mathcal{T}_{g,*})$, we say that a bundle $S_{g,*} \to E \to B$ and a homology class $x \in H_i(B)$ *represent* σ if the induced homology class $f_*(x) \in H_i(\mathcal{T}_{g,*})$ is equal to σ. It is sometimes mentally simplifying to assume that B is a closed manifold, which can be done as follows. Thom [Th, Theorem II.29] proved that every homology class in a closed orientable manifold has an integral multiple which can be represented by (the fundamental class of) a closed submanifold. This can be strengthened to show that every homology class in any CW complex has an odd integral multiple which can be represented by a closed submanifold, see e.g. Conner [Co, Corollary 15.3]. Thus every homology class $\sigma \in H_i(\mathcal{T}_{g,*})$ has a multiple represented by the fundamental class $[B] \in H_i(B)$ for a bundle $S_{g,*} \to E^{i+2} \to B^i$ of closed manifolds. Although this assumption is not logically necessary for our arguments, we let it influence us by often referring to the representing homology class as $[B] \in H_i(B)$.

Parametrized Abel–Jacobi maps. Given a bundle $S_{g,*} \to E \to B$ and homology class $[B] \in H_i(B)$ representing $\sigma \in H_i(\mathcal{T}_{g,*}) \approx H_i(\mathcal{I}_{g,*})$, we can use the bundle $E \to B$ to compute the Johnson invariant $\tau_i(\sigma)$, as follows. The globalized Abel–Jacobi map (4) restricts to a map $J_E\colon E \to T^{2g}$ defined up to homotopy. We remark that the target should be thought out of not just as a torus T^{2g}, but as a $K(H_\mathbb{Z}, 1)$, so that choosing a basis for $H_\mathbb{Z}$ gives corresponding coordinates on T^{2g}.

We call the map $J_E\colon E \to T^{2g}$ a *parametrized Abel–Jacobi map*, since on each fiber it restricts to a map homotopic to the classical Abel–Jacobi map. Since T^{2g} is aspherical, it is determined (up to homotopy) by the induced map on fundamental group, which is determined by the following two properties:

1. On the fiber S_g the map J_E induces the abelianization $\pi_1(S_g) \to H_\mathbb{Z}$.

2. On the image of the section $B \to E$ the map J_E is constant.

In this situation, the preimage of $[B]$ in E is a class $[E] \in H_{i+2}(E)$. Then the Johnson invariant τ_i can be computed as follows:

$$\tau_i(\sigma) = (J_E)_*[E] \in H_{i+2}(T^{2g}) \approx \textstyle\bigwedge^{i+2} H \tag{8}$$

The key to our computations in this paper is to find convenient models for $E \to B$ and for the parametrized Abel–Jacobi map J_E so that $(J_E)_*[E]$ can be calculated explicitly.

Computing $\boldsymbol{\tau_0}$. The intersection form on $H = H_1(S)$ can be represented by an element $\omega \in \bigwedge^2 H$; if $a_1, b_1, \ldots, a_g, b_g$ is a symplectic basis, we have

$$\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g.$$

Since τ_0 is a map from $H_0(\mathcal{I}_{g,*}) \approx \mathbb{Q}$, it is determined by the image of the generator.

Proposition 2.1. *The image of the generator under $\tau_0\colon H_0(\mathcal{I}_{g,*}) \to \bigwedge^2 H$ is ω.*

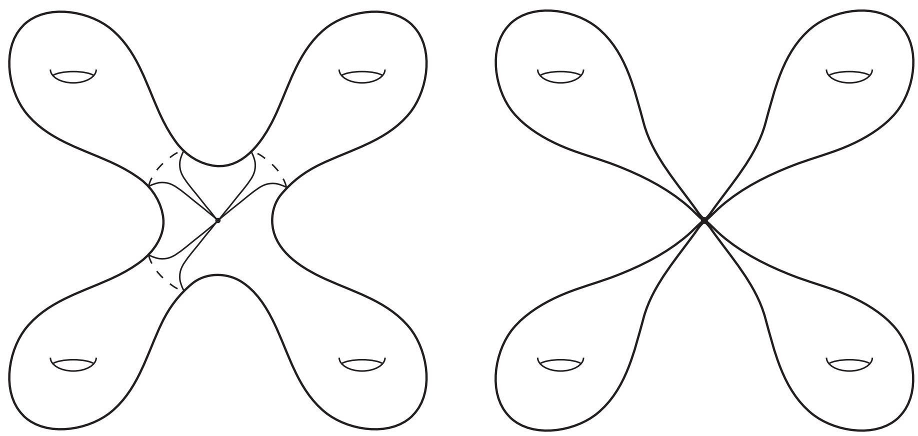


Figure 1: The surface S_g and its quotient $Y = \bigvee T_i$.

Proof. Since the generator of $H_0(\mathcal{I}_{g,*})$ is induced by the inclusion of a point, we see that the image of τ_0 is equal to $j_*[\Sigma_g] \in H_2(T^{2g}) = \bigwedge^2 H$, the image of the fundamental class $[\Sigma_g]$ under the Abel–Jacobi map, which we now compute.

We begin by giving an explicit construction of a map j homotopic to the Abel–Jacobi map that will be useful for our purposes. We will sometimes refer to such a map j as *an Abel–Jacobi map*, since it is uniquely defined only up to homotopy. First, let $Y = \bigvee_{i=1}^{g} T_i$ be the wedge of 2–dimensional tori. There is a natural quotient map $S_g \to Y$, obtained for example by contracting a graph as in Figure 1. In any torus, specifying k distinct coordinates determines a k–dimensional subspace homeomorphic to a torus T^k. The coordinates of T^{2g} are labeled by the symplectic basis $a_1, b_1, \ldots, a_g, b_g$. Identify the ith torus T_i with the torus $T^2 \subset T^{2g}$ determined by the a_i and b_i coordinates. These identifications agree at the origins of the tori T_i, and thus induce an inclusion $Y = \bigvee T_i \hookrightarrow T^{2g}$. The composition $j \colon S_g \to Y \to T^{2g}$ is homotopic to the Abel–Jacobi map; to see this, it is enough to observe that the generators a_i and b_i of $\pi_1(S_g)$ are taken to the corresponding elements of $\pi_1(T^{2g}) = H_{\mathbb{Z}}$.

Finally, we must find $j_*[S_g] \in H_2(T^{2g})$. Under the quotient map $S_g \to Y = \bigvee T_i$, the fundamental class $[S_g] \in H_2(S_g)$ is sent to $\sum [T_i] \in H_2(Y)$. Then T_i is included as the torus determined by the a_i and b_i coordinates; under the natural isomorphism $H_k(T^{2g}) \approx \bigwedge^k H$, this torus represents $a_i \wedge b_i$. Thus we have

$$j_* \colon [S_g] \mapsto \sum [T_i] \mapsto \sum a_i \wedge b_i = \omega$$

as claimed. □

Computing τ_1. In [Jo80], Johnson used the action of $\mathcal{I}_{g,*}$ on the second universal 2–step nilpotent quotient of $\pi_1(S_{g,*})$ to define in a purely algebraic way an $\mathrm{Sp}(2g, \mathbb{Z})$–equivariant homomorphism $\tau_J : \mathcal{I}_{g,*} \to \bigwedge^3 H$ which is now called the *Johnson homomorphism.*

Recall that a *bounding pair map* in $\mathcal{I}_{g,*}$ is a composition of two Dehn twists $T_\alpha T_\beta^{-1}$, where α and β are nonhomotopic, homologous, disjoint nonseparating simple closed curves. For any bounding pair map, up to homeomorphism of S_g, the curves α and β are of the form depicted in Figure 2a. Let S' be the component of $S_g \setminus (\alpha \cup \beta)$ not containing the basepoint, and fix $1 < k \leq g$ so that S' has genus $k-1$. Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H_1(S_g)$ with the property that α (oriented with S' on the left) is homologous to a_k, and so that $\{a_1, b_1, \ldots, a_{k-1}, b_{k-1}, a_k\}$ descends to a basis for $H_1(S')$. Johnson showed in [Jo80] that

$$\tau_J(f) = (a_1 \wedge b_1 + \cdots + a_{k-1} \wedge b_{k-1}) \wedge a_k. \tag{9}$$

The following proposition was stated by Johnson in [Jo83] as the motivation for investigating the maps τ_i. Hain gave a proof in [Ha97] using the work of Sullivan and the cup product structure on the cohomology of mapping tori. Our proof is elementary, and more importantly, it can be generalized to higher-dimensional cycles. Indeed, the ideas introduced in this proof will appear throughout Sections 3 and 4.

Proposition 2.2 (Johnson, Hain [Ha97])**.** *The map $\tau_1 \colon H_1(\mathcal{I}_{g,*}) \to \bigwedge^3 H$ coincides with the Johnson homomorphism τ_J.*

Proposition 2.2 can be thought of as a "parametrized" version of the proof of Proposition 2.1 above.

Proof. Building on work of Birman [Bi] and Powell [Po], Johnson proved in [Jo79] that $\mathcal{I}_{g,*}$ is generated by bounding pair maps for $g \geq 3$; see Hatcher–Margalit [HM] for a modern proof. (For $g = 2$ separating twists are also necessary; however τ_J is known to vanish on separating twists, and τ_1 vanishes on separating twists by Proposition 3.6.) Thus it suffices to check that τ_1 coincides on bounding pair maps with Johnson's map τ_J.

To compute $\tau_1(f)$, we first find a bundle $S_{g,*} \to E \to S^1$ representing $[f] \in H_1(\mathcal{I}_{g,*})$. The natural choice is the mapping torus $S_{g,*} \to M_f \to S^1$, which can be defined as the quotient

$$M_f = S_g \times [0,1]/(f(p), 0) \sim (p, 1).$$

The image of the basepoint $* \in S_g$ in each fiber gives a section of this bundle.

To describe the parametrized Abel–Jacobi map $J = J_{M_f} \colon M_f \to T^{2g}$, we will define J on the cylinder $S_g \times [0,1]$ in such a way that it descends to $M_f = S_g \times [0,1]/\sim$. One obvious first approach is to define J on the fiber $S_g \times \{0\}$ just by the Abel–Jacobi map j. The identification $\sim$ then forces the restriction of J to $S_g \times \{1\}$ to be $j \circ f$. We might naively try to define J simply by interpolating between j and $j \circ f$:

$$J(p,t) \overset{?}{=} (1-t) \cdot j(p) + t \cdot j \circ f(p) \tag{10}$$

But j and $j \circ f$ take values in the torus T^{2g}, so the first term $(1-t) \cdot j(p)$, for example, is not well-defined. However, we can accomplish this idea as follows. Since $f \in \mathcal{I}_{g,*}$, the two maps $j \circ f$ and j induce the same map on the fundamental group and thus are homotopic. Equivalently, their pointwise difference $j \circ f - j$ is homotopically trivial as a map $S_g \to T^{2g}$. We may thus take a lift $\delta \colon S_g \to \mathbb{R}^{2g}$ of $j \circ f - j$; that is, the unique map satisfying $\delta(*) = 0$ and

$$j \circ f - j = \delta \bmod \mathbb{Z}^{2g}.$$

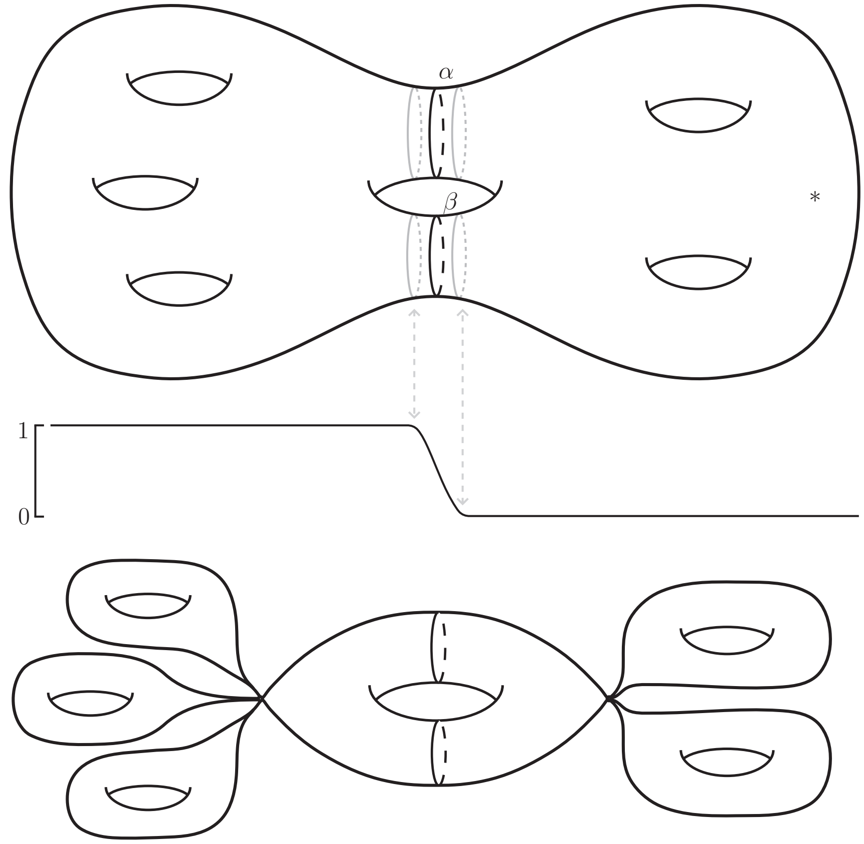


Figure 2: **a.** The surface $S_{g,*}$ and the bounding pair f. **b.** The a_k component of δ. **c.** The quotient $Y = \bigcup T_i$. The torus T_k is in the middle, the tori T_i for $i < k$ are on the left, and the tori T_i for $i > k$ are on the right.

For convenience, we will take j to be an Abel–Jacobi map chosen so that the only coordinate of δ which is nonzero is that corresponding to a_k, and in that coordinate δ is of the form shown in Figure 2b. In particular $\delta(p)$ only depends on the "horizontal" coordinate of p in the depiction in Figure 2a.

One way to ensure this is as follows. The twists T_α and T_β are supported on annular neighborhoods N_α and N_β of α and β respectively. Identify these with $S^1 \times [0,1]$ so that $T_\alpha(\theta,t) = (\theta+t,t)$ on N_α and $T_\beta^{-1}(\theta,t) = (\theta-t,t)$ on N_β. We define j to be zero on $S_g \setminus (N_\alpha \cup N_\beta)$; on N_α and on N_β the a_k coordinate of j is given by θ, and all other coordinates are zero. Since $j = j \circ f$ outside $N_\alpha \cup N_\beta$, the function δ is constant there. On N_α the a_k coordinate of $j \circ f - j$ is given by t, and similarly on N_β by $-t$. Thus δ has the properties claimed above.

Now we may define $J \colon M_f \to T^{2g}$ by

$$J(p,t) = j(p) + t \cdot \delta(p).$$

Substituting $j \circ f - j = \delta \bmod \mathbb{Z}^{2g}$, we see that this definition realizes the idea set out in (10) above. The bounding pair map f does not factor through a wedge of tori, but it does factor through the space $Y = \bigcup T_i$ depicted in Figure 2c, which is the union of tori T_i meeting pairwise in at most 1 point. We may assume that the sympletic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ was chosen so that $\{a_i, b_i\}$ descends to a basis for $H_1(T_i)$ for each $1 \le i \le g$. It is easy to see that j factors through $S_g \to Y$ as well, and thus so does δ. From our explicit formula for J, we see that J factors through the space

$$Z := Y \times [0,1]/(f(p),0) \sim (p,1),$$

which fibers as a bundle $Y \to Z \to S^1$. Just as Y is the union of tori, we see that Z is the union of torus bundles $T_i \to Z_i \to S^1$ meeting pairwise in at most a circle.The quotient $M_f \to Z$ maps the fundamental class $[M_f]$ to $\sum [Z_i]$. Thus it remains to understand $J_*[Z_i]$.

Note that f is supported on the torus T_k; it follows that for $i \neq k$ the torus bundle $T_i \to Z_i \to S^1$ is in fact a product $Z_i \approx T_i \times S^1$. For $i \neq k$, since j and $j \circ f$ agree on T_i, we have that δ is constant on T_i. Since δ is as depicted in Figure 2b, we have that the a_k component of δ is 1 on T_i for $i < k$ and is 0 on T_i for $i > k$. Thus when $i < k$, the restriction of J to $Z_i \approx T_i \times S^1$ is given by

$$J(p,t) = j(p) + (0, \ldots, 0, t, 0, \ldots, 0).$$

This is just the inclusion of $T^3 \subset T^{2g}$ determined by the a_i, b_i, and a_k coordinates; in particular, we have

$$J_*[Z_i] = a_i \wedge b_i \wedge a_k \text{ for } \; i < k.$$

When $i > k$, we have that $J(p,t) = j(p)$, so the image of J is contained in the 2–dimensional subspace determined by a_i and b_i. Since $H_3(T^2) \subset H_3(T^{2g})$ is trivial, we have $J_*[Z_i] = 0$ for $i > k$. Finally, on T_k the function δ is nonconstant; however, the images of both j and δ are contained in the 2–dimensional subspace determined by a_k and b_k. The same is thus true of the image of J, so $J_*[Z_k] = 0$ as well. We conclude that

$$J_* \colon [M_f] \mapsto \sum_{i=1}^{g} [Z_i] \mapsto \sum_{i<k} a_i \wedge b_i \wedge a_k = \tau_J(f),$$

as desired. □

Andy Putman has pointed out that one can view the idea of this proof as "moving the cycle represented by M_f to the boundary of Torelli space", where the computation is easier to verify; from this viewpoint, moving to the boundary of Torelli space corresponds to the degeneration of S_g to the union-of-tori $Y = \bigcup T_i$.

3 Tools for computing τ_i

In this section we provide two of our main tools for computing τ_i, and we use them to compute τ_i on abelian cycles. It will be convenient for us to state our results for the case of surfaces with boundary, namely for the map $\widehat{\tau_i}$ mentioned in the introduction. For simplicity of notation we will call this map τ_i as well.

Product with a bounding pair map. Our first proposition gives a method to bootstrap up homology classes which can be detected using τ_i. Let $S_{g,1} \hookrightarrow S_{g+1,1}$ be the standard inclusion, inducing an inclusion $\mathcal{I}_{g,1} \hookrightarrow \mathcal{I}_{g+1,1}$. Let $f = T_\alpha T_\beta^{-1}$ be a bounding pair map supported in the complement of $S_{g,1}$, and let a be the common homology class of α and β (oriented with $S_{g,1}$ on the left). Then we have a natural map

$$\cdot \times f \colon H_i(\mathcal{I}_{g,1}) \to H_{i+1}(\mathcal{I}_{g+1,1})$$

given by the Gysin homomorphism $H_i(\mathcal{I}_{g,1}) \to H_{i+1}(\mathcal{I}_{g,1} \times \langle f \rangle)$ followed by the inclusion $\mathcal{I}_{g,1} \times \langle f \rangle \to \mathcal{I}_{g+1,1}$.

Proposition 3.1. *Let f be as above. For any $\sigma \in H_i(\mathcal{I}_{g,1})$ we have*

$$\tau_{i+1}(\sigma \times f) = \tau_i(\sigma) \wedge a.$$

Note that Proposition 2.2 can be deduced from Proposition 2.1 by applying Proposition 3.1.

Proof. Let $S_{g,1} \to E' \to B$ be a bundle with a homology class $[B] \in H_i(B)$ representing $\sigma \in H_i(\mathcal{I}_{g,1})$. There is an associated bundle $S_{g,*} \to E \to B$ representing $\sigma \in H_i(\mathcal{I}_{g,*})$. Recall that by (8), $\tau_i(\sigma)$ is the image of $[E]$ under the parametrized Abel–Jacobi map $J_E \colon E \to T^{2g}$.

Similarly, there is a bundle $S_{g+1,*} \to \overline{E} \to B \times S^1$ representing

$$[B] \times [S^1] \mapsto \sigma \times f \in H_{i+1}(\mathcal{I}_{g+1,*}).$$

Here $[B] \times [S^1] \in H_{i+1}(B \times S^1)$ is the class corresponding to $[B] \otimes [S^1] \in H_i(B) \otimes H_1(S^1)$ under the Künneth formula; the preimage of $[B] \times [S^1]$ is a class denoted $[\overline{E}] \in H_{i+3}(\overline{E})$.

To compute $\tau_{i+1}(\sigma \times f)$, we need to explicitly describe the space $\overline{E}$. By $S_{1,1,*}$ we mean a surface of genus 1 with one boundary component and a separate marked point. We can glue E' to the trivial bundle $S_{1,1,*} \times B$ fiberwise along their common boundary component $S^1 \times B$. Now let

$$\overline{E} = (E' \cup (S_{1,1,*} \times B)) \times [0,1] / \sim,$$

where the identification is given by:

$$\begin{aligned} (e,0) &\sim (e,1) && \text{for } e \in E' \\ \big((f(p),b),0\big) &\sim \big((p,b),1\big) && \text{for } (p,b) \in S_{1,1,*} \times B \end{aligned}$$

Note that $\overline{E}$ naturally has the structure of a bundle

$$S_{g+1,*} \to \overline{E} \to B \times S^1.$$

Over $B \subset B \times S^1$, this bundle restricts to

$$S_{g+1,*} \to E' \cup (S_{1,1,*} \times B) \to B,$$

which represents $[B] \mapsto \iota(\sigma) \in H_i(\mathcal{I}_{g+1,*})$. Over $S^1 \subset B \times S^1$, it restricts to the mapping torus $S_{g+1,*} \to M_f \to S^1$ of f, which represents $[S^1] \mapsto [f] \in H_1(\mathcal{I}_{g+1,*})$. It follows that $\overline{E} \to B \times S^1$ represents $[B] \times [S^1] \mapsto \sigma \times f \in H_{i+1}(\mathcal{I}_{g+1,*})$, as desired.

Now we construct the parametrized Abel–Jacobi map $J_{\overline{E}}\colon \overline{E} \to T^{2g+2}$. The quotient $S_{g+1,*} \to S_g \vee S_{1,*}$ induces a quotient $\overline{E} \to Z$, where Z is a bundle $S_g \vee S_{1,*} \to Z \to B \times S^1$. Note that Z is the union of two subspaces: the first a bundle $S_g \to Z_1 \to B \times S^1$ and the second a bundle $S_{1,*} \to Z_2 \to B \times S^1$, meeting in a codimension 2 subspace homeomorphic to $B \times S^1$. By examination, we see that Z_1 is in fact simply $S_g \to E \times S^1 \to B \times S^1$, and that Z_2 is simply $S_{1,*} \to B \times M_f \to B \times S_1$. In particular, the quotient $\overline{E} \to Z$ maps

$$[\overline{E}] \quad \mapsto \quad [E] \times [S^1] \ + \ [B] \times [M_f] \quad \in H_{i+3}(Z).$$

We will define $J_{\overline{E}}$ by defining it on the pieces $E \times S^1$ and $M_f \times B$ of the quotient space Z. Let $J_E\colon E \to T^{2g}$ be a parametrized Abel–Jacobi map for E. Let $j\colon S_{1,*} \to T^2$ be an Abel–Jacobi map, and as above let $\delta\colon S_{1,*} \to \mathbb{R}^2$ be the map defined by the conditions that $\delta(*) = 0$ and $j \circ f - j = \delta \bmod \mathbb{Z}^2$. Assume that we have chosen a basis for $H_1(S_{g+1}, *)$ so that $a = a_{g+1}$. We define the parametrized Abel–Jacobi map $J_{\overline{E}}\colon Z \to T^{2g+2} = T^{2g} \times T^2$ by

$$\begin{aligned} (e,t) &\mapsto \big(J_E(e),\ (t,0)\big) && \text{for } (e,t) \in E \times S^1 \\ (b,(p,t)) &\mapsto \big(0,\ j(p) + t\delta(p)\big) && \text{for } (b,(p,t)) \in B \times M_f \end{aligned}$$

On the intersection $(E \times S^1) \cap (B \times M_f)$ we have $J_E(e) = 0$, while $j(p) = 0$ and $\delta(p) = (1,0)$ (this can be checked as in the proof of Proposition 2.2); thus the resulting map $J_{\overline{E}}$ is well-defined. To see that $J_{\overline{E}}$ is a parametrized Abel–Jacobi map, we consider the restriction to a fiber and to the section. On the section, which is contained in $B \times M_f$, we have

$$(b,(*,t)) \mapsto (0, j_1(*) + t\delta(*), j_2(*)) = 0$$

as desired. Restricted to a fiber $S_{g+1,*}$ of $\overline{E}$, the map $J_{\overline{E}}$ factors through $S_g \vee S_{1,*}$. On the first component the map is $(J_E, 0, 0)$, which induces the abelianization; on the second component we have $(0, J_{M_f})$, which does the same. Thus $J_{\overline{E}}\colon \overline{E} \to T^{2g+2}$ is the desired parametrized Abel–Jacobi map.

It remains to compute $(J_{\overline{E}})_*([E] \times [S^1])$ and $(J_{\overline{E}})_*([B] \times [M_f])$. The restriction of $J_{\overline{E}}$ to $E \times S^1$ is of the form $J_E \times (t,0)$; it is then immediate that

$$(J_{\overline{E}})_*([E] \times [S^1]) = (J_E)_*([E]) \wedge a.$$

The image of $J_{\overline{E}}$ restricted to $B \times M_f$ is contained in the 2–dimensional subtorus determined by the last two coordinates, and is thus trivial in $H_{i+3}(T^{2g+2})$. It follows that

$$\tau_{i+1}(\sigma \times f) = (J_{\overline{E}})_*[\overline{E}] = (J_E)_*[E] \wedge a + 0 = \tau_i(\sigma) \wedge a$$

as desired. □

The pair-of-pants product. Our second kind of computation of τ_i is a vanishing result. To state it in a general form, we make the following definition. There is a natural inclusion $S_{g,1} \sqcup S_{h,1} \to S_{g+h,1}$ defined by gluing two surfaces $S_{g,1}$ and $S_{h,1}$ to a pair-of-pants $S_{0,3}$ along their boundary components, producing a surface homeomorphic to $S_{g+h,1}$, as depicted in Figure 3.

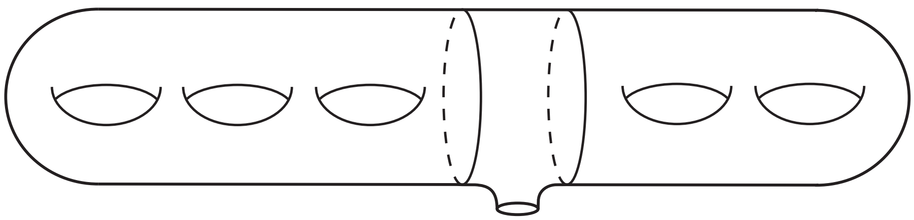


Figure 3: Gluing $S_{3,1}$ to $S_{2,1}$ to produce $S_{5,1}$.

This inclusion induces a map

$$\mathcal{I}_{g,1} \times \mathcal{I}_{h,1} \to \mathcal{I}_{g+h,1} \tag{11}$$

The *pair-of-pants product*

$$H_i(\mathcal{I}_{g,1}) \times H_j(\mathcal{I}_{h,1}) \to H_{i+j}(\mathcal{I}_{g+h,1})$$

is the map obtained by composing the Künneth map $H_i(\mathcal{I}_{g,1}) \times H_j(\mathcal{I}_{h,1}) \to H_{i+j}(\mathcal{I}_{g,1} \times \mathcal{I}_{h,1})$ with the map on homology induced by (11). Given $\sigma \in H_i(\mathcal{I}_{g,1})$ and $\eta \in H_j(\mathcal{I}_{h,1})$, we denote their pair-of-pants product by $\sigma \times \eta \in H_{i+j}(\mathcal{I}_{g+h,1})$.

Proposition 3.2. *Given $\sigma \in H_i(\mathcal{I}_{g,1})$ and $\eta \in H_j(\mathcal{I}_{h,1})$ with $i, j \geq 1$, we have $\tau_{i+j}(\sigma \times \eta) = 0$.*

Proof. Let $S_{g,1} \to E'_\sigma \to B_\sigma$ represent $[B_\sigma] \mapsto \sigma \in H_i(\mathcal{I}_{g,1})$, with $S_{g,*} \to E_\sigma \to B_\sigma$ representing $[B_\sigma] \mapsto \sigma \in H_i(\mathcal{I}_{g,*})$; similarly define $S_{g,1} \to E'_\eta \to B_\eta$ and $S_{g,*} \to E_\eta \to B_\eta$. Let $[E_\sigma] \in H_{i+2}(E_\sigma)$ be the preimage of $[B_\sigma]$ in E_σ, and similarly for $[E_\eta] \in H_{j+2}(E_\eta)$. Let the bundle $S_{g+h,*} \to \overline{E} \to B_\sigma \times B_\eta$ represent $[B_\sigma] \times [B_\eta] \mapsto \sigma \times \eta \in H_{i+j}(\mathcal{I}_{g+h,*})$. The restriction of $\overline{E}$ to B_σ is the bundle obtained by identifying $S_{g,1} \to E'_\sigma \to B_\sigma$ with the trivial bundle $S_{h,1,*} \to S_{h,1,*} \times B_\sigma \to B_\sigma$ along their mutual boundary component $S^1 \times B_\sigma$; a similar observation applies to the restriction to B_η.

The quotient $S_{g+h,*} \to S_g \vee S_h$ induces a quotient $\overline{E} \to Z$, where Z is a bundle $S_g \vee S_h \to Z \to B_\sigma \times B_\eta$. The point where the two surfaces intersect gives a basepoint for $S_g \vee S_h$, and taking this point in each fiber yields a section of Z. Note that Z is the union of two subspaces: the first a bundle $S_g \to Z_1 \to B_\sigma \times B_\eta$, and the second a bundle $S_h \to Z_2 \to B_\sigma \times B_\eta$. By inspection, we see that Z_1 is just $S_g \to E_\sigma \times B_\eta \to B_\sigma \times B_\eta$, and similarly Z_2 is $S_h \to B_\sigma \times E_\eta \to B_\sigma \times B_\eta$. The quotient $\overline{E} \to Z$ maps

$$[\overline{E}] \quad \mapsto \quad [E_\sigma] \times [B_\eta] \; + \; [B_\sigma] \times [E_\eta] \quad \in H_{i+j+2}(Z).$$

The parametrized Abel–Jacobi map $J_{\overline{E}}\colon Z \to T^{2g+2h}$ can be defined on $E_\sigma \times B_\eta$ by $J_{E_\sigma} \times 0$, and on $B_\sigma \times E_\eta$ by $0 \times J_{E_\eta}$. It is easy to check that this is well-defined, and that it induces the appropriate map on fundamental group. From this formula, we see that the image under $J_{\overline{E}}$ of the first piece $E_\sigma \times B_\eta$ is contained in the image of J_{E_σ}, which has dimension at most $i+2$. Thus $[E_\sigma] \times [B_\eta] \in H_{i+j+2}(E_\sigma \times B_\eta)$ is mapped to zero in $H_{i+j+2}(T^{2g+2h})$. The same applies to the second piece $B_\sigma \times E_\eta$, and so we have

$$\tau_{i+j}(\sigma \times \eta) = (J_{\overline{E}})_*[\overline{E}] = (J_{\overline{E}})_*([E_\sigma] \times [B_\eta]) + (J_{\overline{E}})_*([B_\sigma] \times [E_\eta]) = 0 + 0 = 0$$

as desired. □

Abelian cycles. A collection of commuting elements $f_1, \ldots, f_d$ of a group G induces a map $\mathbb{Z}^d \to G$; we denote the image of the fundamental class $[\mathbb{Z}^d] \in H_d(\mathbb{Z}^d, \mathbb{Q})$ in $H_d(G, \mathbb{Q})$ by $\{f_1, \ldots, f_d\}$. This is called an *abelian cycle* in $H_d(G, \mathbb{Q})$. Proposition 3.1 and Proposition 3.2 can be used to compute τ_i on certain abelian cycles.

Definition 3.3. Let $f_1, \ldots, f_k$ be a collection of bounding pair maps on $S_{g,*}$, with f_i being the twist about α_i composed with the inverse twist about β_i. Recall that the curves α_i, β_i are assumed to be nonseparating. We say that this collection is *truly nested* if

1. the curves α_i are pairwise non-homologous, and
2. after possibly re-ordering $\{f_i\}$, the union $\alpha_j \cup \beta_j$ separates the basepoint from $\alpha_i \cup \beta_i$ whenever $i < j$.

An easy induction shows that these conditions force the curves α_i and β_i to be in one of the "standard configurations", a representative example of which is given in Figure 4a. Note that, by this definition, a single bounding pair map is truly nested. For further examples, the collections depicted in Figures 5a, 6, 7, and 9 are truly nested, while the collection depicted in Figure 8 is not. We assume that any truly nested collection has been reordered so that the second condition above holds.

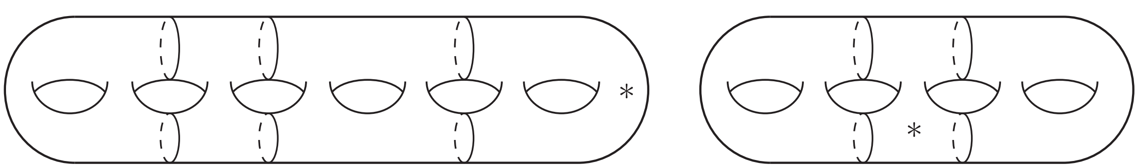


Figure 4: The first collection is truly nested; the second collection is not.

For any bounding pair f_i, let c_i be the common homology class of α_i and β_i. If a collection is truly nested (and ordered as above), then consider the "farthest" subsurface cut off, namely the component of $S_{g,*} \setminus (\alpha_1 \cup \beta_1)$ not containing the basepoint. Choose S_0 to be a surface with one boundary component, contained in the "farthest" subsurface, of maximal genus. Let $\omega_0 \in \bigwedge^2 H_1(S_0) \subset \bigwedge^2 H_1(S_{g,*})$ be a symplectic form for $H_1(S_0)$. Note that the subspace $H_1(S_0)$ is not uniquely determined. However, the following theorem holds regardless of the choice of S_0.

Theorem 3.4. *Any truly nested collection of bounding pair maps determines a nonzero abelian cycle. More precisely, with notation as above, the image under* τ_k *of the abelian cycle* $\{f_1, \ldots, f_k\}$ *is*

$$\tau_k(\{f_1, \ldots, f_k\}) = \omega_0 \wedge c_1 \wedge \cdots \wedge c_k.$$

Proof. Let S_0 be as above. For $1 \le i \le k$, sequentially choose S_i to be a subsurface of $S_{g,1}$ with one boundary component and maximal genus subject to the condition that S_i contains $\alpha_j \cup \beta_j$ for all $j \le i$, S_i contains S_{i-1}, and S_i is disjoint from $\alpha_j \cup \beta_j$ for all $j > i$. The existence of such subsurfaces S_i follows from the assumption that the collection is truly nested. Note that S_k is the whole surface $S_{g,1}$.

Let $1 \in H_0(\mathcal{I}(S_0))$ be a generator. By Proposition 2.1, $\tau_0(1) = \omega_0$. For each $0 \le i \le k$, the abelian cycle $\{f_1, \ldots, f_i\}$ may be considered as an element of $H_i(\mathcal{I}(S_i))$. We show by induction that $\tau_i(\{f_1, \ldots, f_i\}) = \omega \wedge c_1 \wedge \cdots \wedge c_i$. The abelian cycle $\{f_1, \ldots, f_{i+1}\}$ can be written as the cross product $\{f_1, \ldots, f_i\} \times f_{i+1}$ in the sense of Proposition 3.1, followed by the map to $H_{i+1}(S_{i+1})$ induced by the inclusion of the subsurface. The inductive step follows by applying Proposition 3.1. □

Conversely, we have the following.

Theorem 3.5. *If* $f_1, \ldots, f_k$ *is a collection of commuting bounding pair maps that is not truly nested, then*

$$\tau_k(\{f_1, \ldots, f_k\}) = 0.$$

Proof. A collection which is not truly nested must fail either the first or second condition in Definition 3.3.

Case I. We prove the following (*a priori* stronger) claim: if the homology classes of the curves $\alpha_1, \ldots, \alpha_k$ are not linearly independent, then $\tau_i(\{f_1, \ldots, f_k\}) = 0$. Let m be the rank of the span of the homology classes $c_1, \ldots, c_k$, and let $\gamma_1, \ldots, \gamma_{2k}$ be the α_i and β_i, ordered arbitrarily. We will prove below that it is possible to choose curves $\delta_1, \ldots, \delta_{2g}$ with the following properties:

1. their homology classes $d_1, \ldots, d_{2g}$ are a symplectic basis for $H_1(S_g)$, so that $(d_i, d_{g+i}) = 1$ for $1 \le i \le g$;
2. for $i \le m$ each curve δ_i is one of the γ_j;
3. the span of $\langle d_1, \ldots, d_m \rangle$ is the span of $\langle c_1, \ldots, c_k \rangle$;
4. the curve δ_i is disjoint from all the curves γ_j for all i except $g+1 \le i \le g+m$.

Given such a collection $\delta_1, \ldots, \delta_{2g}$, we construct an Abel–Jacobi map $j\colon (S_g, *) \to (T^{2g}, 0)$ supported on a neighborhood of the union $\delta_1 \cup \cdots \cup \delta_{2g}$. One way to do this is to choose 1–forms θ_i dual to δ_i and supported in a small neighborhood. Then j is defined by:

$$j(p) = \left(\int_*^p \theta_{g+1}\, , \ \ldots\, , \ \int_*^p \theta_{2g}\, , \int_*^p -\theta_1\, , \ \ldots\, , \ \int_*^p -\theta_g \right)$$

By transversality, we may assume that the curves δ_i intersect at most pairwise; it follows that the image $j(S_g)$ is contained in the 2–skeleton of T^{2g}.

Consider the bundle $S_{g,*} \to E \to T^k$ representing $\{f_1, \ldots, f_k\} \in H_k(\mathcal{I}_{g,*})$. As in the proof of Proposition 2.2, we may use the map j to construct a parametrized Abel–Jacobi map $J_E \colon E \to T^{2g}$. The disjointness properties of δ_i imply that for each ℓ, $j \circ f_\ell - j$ is nonzero only in the components determined by $d_1, \ldots, d_m$. It follows from the construction of J that the image $J_E(E)$ is contained in the finite union of the tori (of dimension at most $m+2$) determined by the components $d_1, \ldots, d_m$ together with at most two other basis elements d_i and d_j. This subcomplex of T^{2g} has dimension $m+2$; since $m < k$, it follows that $(J_E)_*[E] = 0$ in $H_{k+2}(T^{2g})$.

We now show how to find such a collection δ_i. We first find $\delta_{m+1}, \ldots, \delta_g$ and $\delta_{g+m+1}, \ldots, \delta_{2g}$ as follows. Consider again the complement $S_{g,1} \setminus \bigcup \gamma_i$. Each component of the complement is a surface of some genus $g_j \geq 0$; we may easily find g_j pairs of curves on this subsurface, each pair intersecting in one point, and whose homology classes are a symplectic basis for the subspace they span. The claim is that doing so on each complementary subsurface yields $g - m$ such pairs. By collapsing to a point the genus 1 subsurface which is a regular neighborhood of such a pair, we may assume that each complementary subsurface has genus 0; to prove the claim, we need to prove that $m = g$ under this assumption. Consider the functionals $H_1(S_{g,1}) \to \mathbb{Q}$ given by intersection with each of the γ_i. The space of functionals spanned by this collection has rank m. But if the complementary components have genus 0, their homology is spanned by the homology of their boundary components. Then Mayer–Vietoris implies that the mutual kernel of all these functionals is generated by the boundary components γ_i, and thus has rank m. We conclude that $H_1(S_{g,1})$ has rank $2m$; this verifies the claim, and so we have $g - m$ pairs of curves, which we take as $\delta_{m+1}, \ldots, \delta_g$ and $\delta_{g+m+1}, \ldots, \delta_{2g}$. At this point it is easy to choose $\delta_1, \ldots, \delta_m$ and $\delta_g, \ldots, \delta_{g+m}$. For the former, we choose any m curves from the γ_j whose homology classes are linearly independent to be $\delta_1, \ldots, \delta_m$. Now the only condition on the remaining curves is that their homology classes should make $d_1, \ldots, d_{2g}$ a symplectic basis, so we may choose $\delta_{g+m+1}, \ldots, \delta_{2g}$ arbitrarily subject to this condition. This completes the proof in the first case.

Case II. Now consider the case when the second condition is violated. We explain first the case when no bounding pair separates the basepoint from the others. Consider the component C of $S_{g,1} \setminus \bigcup \gamma_i$ which is adjacent to the boundary component. The boundary of C consists of curves α_i or β_j (plus $\partial S_{g,1}$), and under our assumptions it contains curves from at least two bounding pairs. There must be some bounding pair f_i so that C contains both α_i and β_i; otherwise, without loss of generality the boundary of C would consist of $\alpha_1, \ldots, \alpha_j$ for some j, plus $\partial S_{g,1}$. But then the homology classes of these curves would be linearly dependent, and this case has already been dealt with. Thus C contains both α_i and β_i for some i, and so there is a separating curve γ in C cutting off exactly α_i and β_i. Extend this arbitrarily to a pair-of-pants $S_{0,3}$ contained in C having both γ and $\partial S_{g,1}$ as boundary components.

Note that $S_{g,1} \setminus S_{0,3}$ has two components $S_{h,1}$ and $S_{g-h,1}$, each of which contains at least one bounding pair. Relabeling, we may assume that $f_1, \ldots, f_j$ are contained in $S_{h,1}$ and $f_{j+1}, \ldots, f_k$ are contained in $S_{g-h,1}$ for $0 < j < k$. Then the abelian cycle $\{f_1,\ldots,f_k\} \in H_k(\mathcal{I}_{g,1})$ is obtained as the pair-of-pants product of $\{f_1,\ldots,f_j\} \in H_j(\mathcal{I}_{h,1})$ and $\{f_{j+1},\ldots,f_k\} \in H_{k-j}(\mathcal{I}_{g-h,1})$. Applying Proposition 3.1, we conclude that $\tau_k(\{f_1,\ldots,f_k\}) = 0$.

In general such a configuration will be present, but not necessarily adjacent to the basepoint. We attempt to order the bounding pairs inductively as f_k, f_{k-1}, etc., so that for each i the union $\alpha_i \cup \beta_i$ separates the basepoint from all bounding pairs not yet labeled. Since the collection is not truly nested, at some point we cannot continue this process; we are left with some subset $\{f_1,\ldots,f_\ell\}$ which cannot be so ordered. Let S_ℓ be a subsurface with one boundary component and maximal genus subject to the condition that S_ℓ contains $\alpha_i \cup \beta_i$ if $i \leq \ell$ and is disjoint from $\alpha_i \cup \beta_i$ for $i > \ell$. Then $\{f_1,\ldots,f_\ell\} \in H_\ell(\mathcal{I}(S_\ell))$ is as discussed in the previous two paragraphs, and so $\tau_\ell(\{f_1,\ldots,f_\ell\}) = 0$. Now just as in the proof of Theorem 3.4, we may filter $S_{g,1}$ by nested subsurfaces S_i for $\ell \leq i \leq k$ with S_i containing $\alpha_j \cup \beta_j$ iff $j \leq i$. As before, $\{f_1,\ldots,f_{i+1}\}$ is the cross product $\{f_1,\ldots,f_i\} \times f_{i+1}$, so applying Proposition 3.1, we have by induction

$$\tau_{i+1}(\{f_1,\ldots,f_{i+1}\}) = \tau_i(\{f_1,\ldots,f_i\}) \wedge c_i = 0 \wedge c_i = 0. \qquad \square$$

Separating twists. We can try to generalize these techniques beyond bounding pair maps. In general, given $f \in \mathcal{I}_{g,*}$ and $\sigma \in H_i(\mathcal{I}_{g,*})$, we cannot form $\sigma \times f \in H_{i+1}(\mathcal{I}_{g,*})$. However, consider the inclusion of the centralizer $C_{\mathcal{I}}(f)$ into $\mathcal{I}_{g,*}$; if σ is represented by some $\widetilde{\sigma} \in H_i(C_{\mathcal{I}}(f))$, we can consider $\widetilde{\sigma} \times f \in H_{i+1}(C_{\mathcal{I}}(f) \times \langle f\rangle)$ and define its image to be $\sigma \times f \in H_{i+1}(\mathcal{I}_{g,*})$. Of particular importance is the case when f is a twist T_γ about a separating curve. However, unlike bounding pair maps, separating twists do not produce nontrivial abelian cycles with respect to τ_i.

Proposition 3.6. *Let T_γ be the Dehn twist about a separating curve γ, and let $\sigma \in H_i(\mathcal{I}_{g,*})$ be such that $\sigma \times T_\gamma$ is well-defined. Then $\tau_{i+1}(\sigma \times T_\gamma) = 0$.*

Proof. Let $S_{g,*} \to E \to B$ represent $[B] \mapsto \sigma$. By assumption we may assume that the classifying map factors through $C_{\mathcal{I}}(f)$, so the entire image of $\pi_1(B) \to \mathcal{I}_{g,*}$ fixes the curve γ. Thus by fiberwise contracting γ to a point, we have the quotient $E \to Y$, where Y fibers as

$$S_h \vee S_{g-h} \to Y \to B$$

for some $1 \leq h < g$. This is the union of two subspaces, $S_h \to Y_1 \to B$ and $S_{g-h} \to Y_2 \to B$. Since γ is separating, we may start with an Abel–Jacobi map $j\colon S_{g,*} \to T^{2g}$ so that γ is mapped to 0, so we can find a parametrized Abel–Jacobi map $J_E\colon E \to T^{2g}$ which factors through Y.

The class $\sigma \times T_\gamma$ is represented by $S_{g,*} \to \overline{E} \to B \times S^1$, where as above $\overline{E} = E \times [0,1]/(T_\gamma(p),0) \sim (p,1)$. As above, $\overline{E}$ descends to a quotient $S_h \vee S_{g-h} \to Z \to B \times S^1$. This is the union of two subspaces, which are easily seen to be products $Y_1 \times S^1$ and $Y_2 \times S^1$. We may define $J_{\overline{E}}\colon Z \to T^{2g}$ on both $Y_1 \times S^1$ and $Y_2 \times S^1$ by $J_E \times 0$. Thus $J_{\overline{E}}$ factors through the $(i+2)$–dimensional complex Y, and so

$$\tau_{i+1}(\sigma \times T_\gamma) = (J_{\overline{E}})_*[\overline{E}] = 0. \qquad \square$$

4 The Gysin homomorphism and τ_i

In this section we show how the Gysin homomorphism can be used to construct nonzero cycles detectable by τ_i. To this end, consider the universal surface bundle

$$1 \to S_g \to \mathcal{T}_{g,*} \xrightarrow{\pi} \mathcal{T}_g \to 1.$$

We then have the Gysin homomorphism $\pi^!\colon H_i(\mathcal{I}_g) \to H_{i+2}(\mathcal{I}_{g,*})$; by precomposing with the map $H_i(\mathcal{I}_{g,*}) \to H_i(\mathcal{I}_g)$ induced by π, we can also consider $\pi^!$ as a map $H_i(\mathcal{I}_{g,*}) \to H_{i+2}(\mathcal{I}_{g,*})$. Composing with τ_{i+2} we obtain

$$\tau_{i+2} \circ \pi^!\colon H_i(\mathcal{I}_{g,*}) \to \bigwedge^{i+4} H.$$

We can use this map to detect new nontrivial cycles in $H_{i+2}(\mathcal{I}_{g,*})$. Let $\{f_1, \dots, f_k\}$ be a truly nested collection of bounding pair maps with homology classes $c_1, \dots, c_k$. As before, consider the component of $S_g \setminus (\alpha_1 \cup \beta_1)$ not containing the basepoint (the "farthest" subsurface), let S_0 be a maximal subsurface with one boundary component, and let ω_0 represent the symplectic form on $H_1(S_0)$. Similarly, consider the component of $S_g \setminus (\alpha_k \cup \beta_k)$ containing the basepoint (the "closest" subsurface), let S^0 be a maximal subsurface with one boundary component, and let ω^0 represent the symplectic form on $H_1(S^0)$. The following theorem holds regardless of the choice of S_0 and S^0.

Theorem 4.1. *Let $k \geq 2$ be even, and let $\{f_1, \dots, f_k\}$ be a truly nested collection of bounding pair maps with homology classes $c_1, \dots, c_k$. Then with ω_0 and ω^0 as above,*

$$\tau_{k+2}\big(\pi^!\{f_1, \dots, f_k\}\big) = 2 \cdot \omega_0 \wedge \omega^0 \wedge c_1 \wedge \cdots \wedge c_k.$$

In contrast, when k is odd, we have the following theorem.

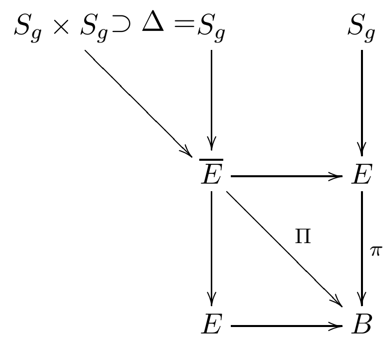
Theorem 4.2. *If k is odd, then $\tau_{k+2} \circ \pi^!$ is the zero map.*

Before proving these theorems, we interpret $\pi^!$ in terms of bundles as above. The composition $\mathcal{T}^*_{g,*} \to \mathcal{T}_{g,*} \to \mathcal{T}_g$ yields (as we will show in the following two paragraphs) a fiber bundle $S_g \times S_g \to \mathcal{T}^*_{g,*} \xrightarrow{\Pi} \mathcal{T}_g$ with associated Gysin homomorphism $\Pi^!\colon H_i(\mathcal{I}_g) \to H_{i+4}(\mathcal{I}^*_{g,*})$. Recall that $J\colon \mathcal{I}^*_{g,*} \to H_{\mathbb{Z}}$ is the homomorphism which is the abelianization on the fiber and trivial on the subgroup $\mathcal{I}_{g,*}$. By definition, we have

$$\tau_{k+2} \circ \pi^! = J_* \circ \Pi^!\colon H_k(\mathcal{I}_g) \to H_{k+4}(\mathcal{I}^*_{g,*}) \to H_{k+4}(H_{\mathbb{Z}}) \approx \bigwedge^{k+4} H.$$

This can be described explicitly in terms of bundles, as follows.

Let $S_g \to E \to B$ represent $[B] \mapsto \sigma \in H_k(\mathcal{I}_g)$, with $[E] \in H_{k+2}(E)$ denoting the preimage of $[B]$. Let $S_g \to \overline{E} \to E$ be the pullback of $S_g \to E \to B$ to E by the map $p\colon E \to B$, and let $[\overline{E}] \in H_{k+4}(\overline{E})$ be the preimage of $[E]$. This pullback consists of pairs of points $(e_1, e_2) \in E \times E$ such that $p(e_1) = p(e_2)$. Thus the "diagonal" consisting of pairs (e, e) gives a section $s\colon E \to \overline{E}$ of the bundle $S_g \to \overline{E} \to E$. In summary, we have the following diagram:



By composing with the map $E \to B$, we can consider $\overline{E}$ as a bundle over B. The fiber F is a bundle-with-section of the form $S_{g,*} \to F \to S_g$. It can be verified that the monodromy $\pi_1(S_g, *) \to \mathrm{Mod}_{g,*}$ is contained in the kernel of the natural map $\mathrm{Mod}_{g,*} \to \mathrm{Mod}_g$ (it is easy to check that this kernel is contained in $\mathcal{I}_{g,*}$). Indeed Birman proved (see, e.g. [FM, Theorem 4.6]) that this map gives an isomorphism

$$\pi_1(S_g, *) \approx \ker(\mathrm{Mod}_{g,*} \to \mathrm{Mod}_g) \tag{12}$$

It thus follows that as a surface bundle, $F \approx S_g \times S_g$. The section s intersects each fiber F in the diagonal $\Delta \subset S_g \times S_g$.

Since $[\overline{E}] = \Pi^!([B])$, we have that $\tau_{k+2}(\pi^!(\sigma))$ is the image of $[\overline{E}]$ under the parametrized Abel–Jacobi map $J_{\overline{E}} \colon \overline{E} \to T^{2g}$, which can be constructed as follows. Let $J_E \colon E \to T^{2g}$ be a parametrized Abel–Jacobi map, and define $J_{\overline{E}} \colon \overline{E} \to T^{2g}$ to be

$$J_{\overline{E}}\big((e_1, e_2)\big) = J_E(e_1) - J_E(e_2).$$

As above, to verify that $J_{\overline{E}}$ is the parametrized Abel–Jacobi map for $\overline{E}$, we need to check that the induced map on fundamental group is trivial when restricted to the section s, and is the abelianization when restricted to a fiber S_g. The former is immediate, since s consists of pairs (e, e). The fiber S_g is the set of pairs $\{(e, e_0)\}$ in $\overline{E}$, for some fixed $e_0 \in E$. The map $e \mapsto (e, e_0)$ identifies this with the fiber of E containing e_0. Note that since J_E is a parametrized Abel–Jacobi map for E, its restriction to a fiber induces the abelianization. Since $J_{\overline{E}}\big((e, e_0)\big) = J_E(e) - J_E(e_0)$, the restriction of $J_{\overline{E}}$ to this fiber S_g is the translate of J_E by the constant $-J_E(e_0)$. Thus when restricted to this fiber, $J_{\overline{E}}$ is homotopic to J_E and thus induces the same map on the fundamental group.

With this description in terms of bundles in hand, we can now prove the theorems stated above.

Proof of Theorem 4.2. The bundle $S_g \times S_g \to \overline{E} \to B$ admits a natural involution $\rho \colon \overline{E} \to \overline{E}$ defined by $\rho\big((e_1, e_2)\big) = (e_2, e_1)$. Note that ρ covers the identity $B \to B$. Restricted to a fiber, this is just the transposition of coordinates $S_g \times S_g \to S_g \times S_g$. Since S_g is even-dimensional, this homeomorphism is orientation-preserving, and so it fixes the fundamental class $[S_g \times S_g] \in H_4(S_g \times S_g)$. Thus by the naturality of the Gysin homomorphism (see e.g. [Mo01, Proposition 4.8(iii)]) we have $\rho_* \circ \Pi^! = \Pi^!$. Define $\nu \colon T^{2g} \to T^{2g}$ to be the map induced by the map $\mathbb{R}^{2g} \to \mathbb{R}^{2g}$ given by $v \mapsto -v$.

Note that $\nu_*\colon H_k(T^{2g}) \to H_k(T^{2g})$ is the identity when k is even, and is minus the identity when k is odd. From the way we constructed $J_{\overline{E}}$, we see that

$$J_{\overline{E}} \circ \rho = \nu \circ J_{\overline{E}}.$$

But now we have

$$(J_{\overline{E}})_* \circ \Pi^! = (J_{\overline{E}})_* \circ \rho_* \circ \Pi^! = \nu_* \circ (J_{\overline{E}})_* \circ \Pi^!$$

Thus when k is odd, we have $(J_{\overline{E}})_* \circ \Pi^! = -(J_{\overline{E}})_* \circ \Pi^!$, which implies

$$\tau_{k+2} \circ \pi^! = (J_{\overline{E}})_* \circ \Pi^! = 0$$

as desired. □

Proof of Theorem 4.1. Let $S_g \to E \to T^k$ be the bundle classifying the abelian cycle $\sigma = \{f_1, \ldots, f_k\} \in H_k(\mathcal{I}_g)$. Form as above the fiber product bundle $S_g \to \overline{E} \to E$ representing $\pi^!(\sigma)$, and view it as a bundle $S_g \times S_g \to \overline{E} \to T^k$. The parametrized Abel–Jacobi map $J_{\overline{E}}\colon \overline{E} \to T^{2g}$ can be defined as in the proof of Proposition 2.2. We construct $\overline{E}$ as

$$S_g \times S_g \times [0,1]^k / \sim$$

where

$$(p, q, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_k) \sim (f_i(p), f_i(q), t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_k).$$

Let $j\colon S_g \to T^{2g}$ be the Abel–Jacobi map, and let $\delta_i\colon S_g \to \mathbb{R}^{2g}$ be the unique map satisfying $j \circ f_i = j + \delta_i \bmod \mathbb{Z}^{2g}$ and $\delta_i(*) = 0$. Then $J_{\overline{E}}$ can now be defined by

$$J_{\overline{E}}\big((p, q, t_1, \ldots, t_k)\big) = j(p) - j(q) + \sum_i t_i\big(\delta_i(p) - \delta_i(q)\big).$$

The definition of δ_i was chosen exactly so that this descends to a map $J_{\overline{E}}\colon \overline{E} \to T^{2g}$.

Any truly nested collection of bounding pairs is, up to homeomorphism, of the form depicted in Figure 5a. The maps f_i factors through a union of 2–dimensional tori $Y = \bigcup T_\ell$, any two of which meet in at most one point, as depicted in Figure 5b. We have a basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for H so that a_ℓ and b_ℓ span the homology of the torus T_ℓ. For each bounding pair map f_i, the homology class c_i of its defining pair of curves is equal to a_{ℓ_i} for some ℓ_i. As in the definition of a truly nested collection, we assume that $\ell_i < \ell_{i'}$ if $i < i'$; for simplicity, we order the T_ℓ so that T_ℓ is separated from the basepoint by T_{ℓ_i} iff $\ell < \ell_i$.

We can choose j so that j, and thus also the δ_i, factors through Y, and furthermore so that as in Proposition 2.2, j and $j \circ f_i$ differ only in the component corresponding to a_{ℓ_i}. The restriction of j to the torus T_ℓ gives an identification with the linear subspace of T^{2g} consisting of the $\langle a_\ell, b_\ell \rangle$ plane; parameterizing T_ℓ by this identification, we have that the restriction of j to T_ℓ is just the inclusion of this subspace.

It follows that $J_{\overline{E}}$ factors through a space Z which fibers as a bundle $Y \times Y \to Z \to T^k$; call the resulting map $J_Z\colon Z \to T^{2g}$. This bundle is the union

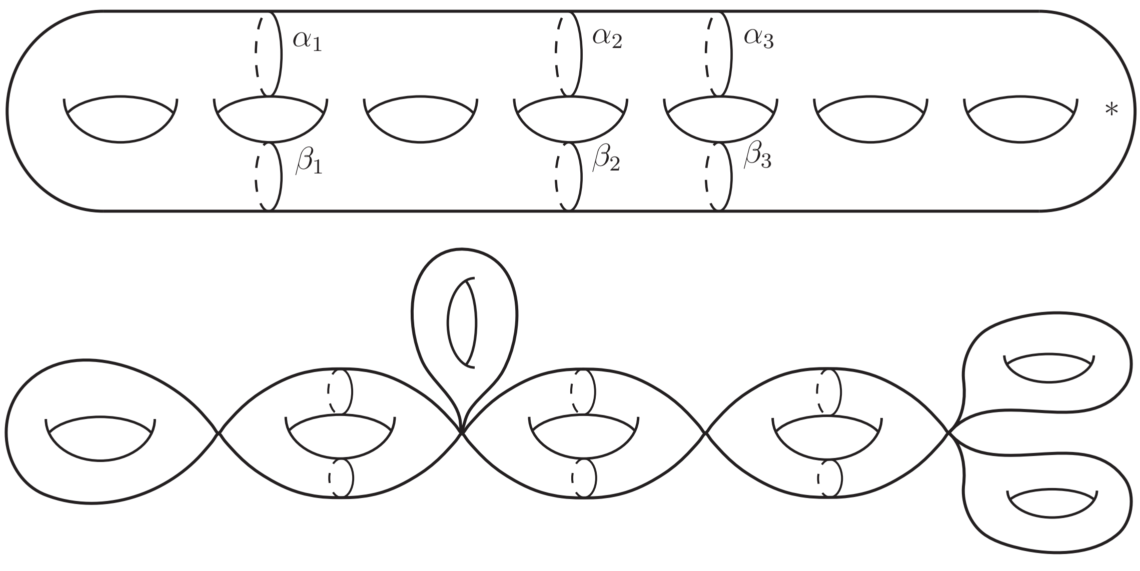


Figure 5: **a.** The bounding pair maps f_i. **b.** The quotient $Y = \bigcup T_\ell$.

of subspaces of the form $T_\ell \times T_{\ell'} \to Z_{\ell,\ell'} \to T^k$; the intersection of two such subspaces has codimension at least 2, corresponding to $T_{\ell_1} \times T_{\ell'} \cap T_{\ell_2} \times T_{\ell'} = * \times T_{\ell'}$ or to $T_{\ell_1} \times T_{\ell'_1} \cap T_{\ell_2} \times T_{\ell'_2} = * \times *$. It follows that the fundamental class $[\overline{E}] \in H_{k+4}(\overline{E})$ projects to the sum of the fundamental classes $\sum[Z_{\ell,\ell'}] \in H_{k+4}(Z)$. Thus to compute $\tau_{k+2}(\pi^!\{f_1,\ldots,f_k\}) = (J_{\overline{E}})_*[\overline{E}]$, it remains to understand $(J_Z)_*[Z_{\ell,\ell'}]$.

Call the subspace $Z_{\ell,\ell'}$ *bad* if either ℓ or ℓ' is equal to ℓ_i for some i; otherwise call $Z_{\ell,\ell'}$ *good*. First, let us check that for bad $Z_{\ell,\ell'}$, we have $(J_Z)_*[Z_{\ell,\ell'}] = 0$. For $(p,q,t_1,\ldots,t_k) \in Z_{\ell,\ell'}$ we have $p \in T_\ell$ and $q \in T_{\ell'}$; thus $j(p)$ and $j(q)$ are contained in the subspace determined by $\langle a_\ell, b_\ell\rangle$ and $\langle a_{\ell'}, b_{\ell'}\rangle$ respectively. Recall that each δ_i is nonzero only in the coordinate corresponding to c_i. Our formula for $J_{\overline{E}}((p,q,t_1,\ldots,t_k))$ thus implies that $J_Z(Z_{\ell,\ell'})$ is contained in the subspace determined by the collection $\langle a_\ell, b_\ell, a_{\ell'}, b_{\ell'}, c_1, \ldots, c_k\rangle$. However, the assumption that $Z_{\ell,\ell'}$ is bad implies that a_ℓ or $a_{\ell'}$ coincides with some c_i. Thus this subspace has dimension at most $k+3$, and so $(J_Z)_*[Z_{\ell,\ell'}] \in H_{k+4}(T^{2g})$ must be zero.

Now we consider the good pieces $Z_{\ell,\ell'}$. Since neither ℓ nor ℓ' is of the form ℓ_i for any i, we have that each map f_i is the identity on T_ℓ and $T_{\ell'}$. It follows that the bundle $T_\ell \times T_{\ell'} \to Z_{\ell,\ell'} \to T^k$ is actually a product $Z_{\ell,\ell'} \approx T_\ell \times T_{\ell'} \times T^k$. Note that since $j \circ f_i = j$ on T_ℓ, we have that each δ_i is constant on T_ℓ and is nonzero only in the component corresponding to c_i. In that component, we have as before that δ_i is either 1 or 0 on T_ℓ, depending on whether the torus T_ℓ is cut off from the basepoint by $\alpha_i \cup \beta_i$ or not. Denoting this number by $n^i_\ell \in \{0,1\}$, we see that n^i_ℓ is 1 if $\ell < \ell_i$ and is 0 if $\ell_i < \ell$.

The restriction of J_Z to $Z_{\ell,\ell'} \approx T_\ell \times T_{\ell'} \times T^k$ may now be read off from the formula for $J_{\overline{E}}$ above. The restriction is a linear map, which can be described on each factor. On the first and second factors, it is the inclusion of T_ℓ as the torus determined by $\langle a_\ell, b_\ell\rangle$ and the inclusion of $T_{\ell'}$ as the torus determined by $\langle a_{\ell'}, b_{\ell'}\rangle$ respectively. Let

$\varepsilon_i \in \{-1, 0, 1\}$ be the number $n^i_\ell - n^i_{\ell'}$. On the third factor T^k, J_Z is the composition of the map $T^k \to T^k$ given by

$$(t_1, \ldots, t_k) \mapsto (\varepsilon_1 t_1, \ldots, \varepsilon_k t_k) \tag{13}$$

with the inclusion of T^k as the torus determined by $\langle c_1, c_2, \ldots, c_k \rangle$. Note that the map $H_k(T^k) \to H_k(T^k)$ induced by (13) is multiplication by $\varepsilon_{\ell,\ell'} = \varepsilon_1 \cdots \varepsilon_k$. From this description we see that the image of the fundamental class $[Z_{\ell,\ell'}]$ under J_Z is

$$(J_Z)_*[Z_{\ell,\ell'}] = \varepsilon_{\ell,\ell'} \cdot a_\ell \wedge b_\ell \wedge a_{\ell'} \wedge b_{\ell'} \wedge c_1 \wedge \cdots \wedge c_k.$$

It thus remains only to understand $\varepsilon_{\ell,\ell'}$. Since n^i_ℓ is 1 if $\ell < \ell_i$ and 0 otherwise, we have that ε_i is 1 if $\ell < \ell_i < \ell'$, -1 if $\ell' < \ell_i < \ell$, and 0 otherwise. Thus $\varepsilon_{\ell,\ell'}$ is nonzero only if we have $\ell < \ell_1 < \cdots < \ell_k < \ell'$ or $\ell' < \ell_1 < \cdots < \ell_k < \ell$. In the former case, each $\varepsilon_i = 1$, so $\varepsilon_{\ell,\ell'} = 1$; in the latter, each $\varepsilon_i = -1$, but since k is even we have $\varepsilon_{\ell,\ell'} = \varepsilon_1 \cdots \varepsilon_k = 1$ again. Note that if k were odd, these terms would instead cancel, yielding another proof of Theorem 4.2 (for the case of abelian cycles of bounding pairs). Combining these cases, we conclude that

$$\begin{aligned} J_{\overline{E}} \colon [\overline{E}] &\mapsto \sum_{\ell,\ell'} [Z_{\ell,\ell'}] \\ &\mapsto \sum_{\ell < \ell_1 < \cdots < \ell_k < \ell'} 2 \cdot a_\ell \wedge b_\ell \wedge a_{\ell'} \wedge b_{\ell'} \wedge c_1 \wedge \cdots \wedge c_k \\ &= 2 \cdot \omega_0 \wedge \omega^0 \wedge c_1 \wedge \cdots \wedge c_k \end{aligned}$$

and thus, as desired,

$$\tau_{k+2}\big(\pi^!\{f_1, \ldots, f_k\}\big) = 2 \cdot \omega_0 \wedge \omega^0 \wedge c_1 \wedge \cdots \wedge c_k. \qquad \square$$

This proof is valid for $k = 0$ as well, except for the last equality above; in this case we instead have just

$$\sum_{\ell < \ell'} 2 \cdot a_\ell \wedge b_\ell \wedge a_{\ell'} \wedge b_{\ell'} = \omega \wedge \omega.$$

Thus if $[S_g] \in H_2(\mathcal{I}_{g,*})$ represents the point-pushing subgroup as in (12), we deduce the following, which is necessary for Corollary 1.3.

Corollary 4.3. $\tau_2([S_g]) = \omega \wedge \omega \in \bigwedge^4 H$.

5 The image and kernel of τ_i

In this section we complete the proofs of Theorem 1.2 and Theorem 1.1.

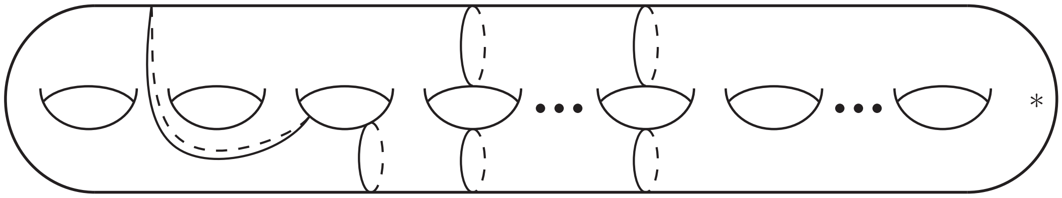


Figure 6: The collection of bounding pairs generating $V(\lambda_i) \oplus V(\lambda_{i+2})$.

5.1 Proof of Theorem 1.2

First, for all $i \le g-2$, we show that $\tau_i(H_i(\mathcal{I}_{g,*}))$ contains $V(\lambda_i) \oplus V(\lambda_{i+2})$. Consider the collection of bounding pairs displayed in Figure 6. Let $\sigma \in H_i(\mathcal{I}_{g,*})$ be the associated abelian cycle; by Theorem 3.4 we have

$$\tau_i(\sigma) = a_1 \wedge b_1 \wedge a_3 \wedge \cdots \wedge a_{i+2}.$$

Recall that there is an Sp–equivariant contraction $C_k \colon \bigwedge^k H \to \bigwedge^{k-2} H$, which was defined in (5). We claim that $\tau_i(\sigma) \in \bigwedge^{i+2} H$ generates $\ker C_i \circ C_{i+2}$ as a module. As previously noted, $\ker C_k \approx V(\lambda_k)$ and thus $\ker C_i \circ C_{i+2} \approx V(\lambda_i) \oplus V(\lambda_{i+2})$, so this will verify this case of the theorem.

First note that $\tau_i(\sigma)$ is contained in $\ker C_i \circ C_{i+2}$; indeed, we have $C_{i+2}(\tau_i(\sigma)) = a_3 \wedge \cdots \wedge a_{i+2}$, which lies in $\ker C_i$. In particular, we see that $\tau_i(\sigma)$ is not contained in $\ker C_{i+2} \approx V(\lambda_{i+2})$. The element

$$\nu := a_1 \wedge (b_1 + a_2) \wedge a_3 \wedge \cdots \wedge a_{i+2}$$

is clearly in the Sp–orbit of $\tau_i(\sigma)$. Thus $\nu - \tau_i(\sigma) = a_1 \wedge a_2 \wedge a_3 \wedge \cdots \wedge a_{i+2}$, which lies in $\ker C_{i+2} \approx V(\lambda_{i+2})$, is in the image of τ_i. We conclude that the Sp–span of $\tau_i(\sigma)$ is contained in $V(\lambda_i) \oplus V(\lambda_{i+2})$ and properly contains $V(\lambda_{i+2})$, and thus since the $V(\lambda_k)$ are irreducible, the Sp–span of $\tau_i(\sigma)$ is $V(\lambda_i) \oplus V(\lambda_{i+2})$.

Note that for $i = g-1$, a similar collection of $g-1$ bounding pairs determines an abelian cycle σ so that $\tau_{g-1}(\sigma) = a_1 \wedge b_1 \wedge a_2 \wedge \cdots \wedge a_g \in \bigwedge^{g+1} H$. The contraction $C_{g+1} \colon \bigwedge^{g+1} H \to \bigwedge^{g-1} H$ is injective [FH, Theorem 17.11]. The image $C_{g+1}(\tau_{g-1}(\sigma)) = a_2 \wedge \cdots \wedge a_g$ clearly generates $V(\lambda_{g-1})$, and so $\tau_{g-1}(H_{g-1}(\mathcal{I}_{g,*})) \supseteq V(\lambda_{g-1})$.

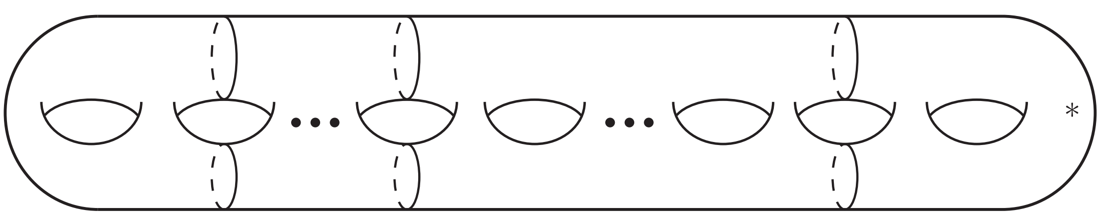


Figure 7: The $i-2$ bounding pairs used to generate $V(\lambda_{i-2})$.

We now show that when $1 \le i \le g$ and i is even, $\tau_i(H_i(\mathcal{I}_{g,*}))$ also contains $V(\lambda_{i-2})$. Take the collection of bounding pairs displayed in Figure 7, and let $\sigma \in H_{i-2}(\mathcal{I}_{g,*})$ be the associated abelian cycle; we will consider $\pi^! \sigma \in H_i(\mathcal{I}_{g,*})$. By Theorem 4.1,

$$\tau_i(\pi^! \sigma) = 2 \cdot a_1 \wedge b_1 \wedge a_g \wedge b_g \wedge a_2 \wedge \cdots \wedge a_{i-2} \wedge a_{g-1}.$$

We claim that this element lies in $\ker C_{i-2} \circ C_i \circ C_{i+2}$, but not $\ker C_i \circ C_{i+2}$, and thus its Sp–span contains $V(\lambda_{i-2})$ as desired. To see this, note that

$$C_{i+2}(\tau_i(\pi^! \sigma)) = 2 \cdot (a_1 \wedge b_1 + a_g \wedge b_g) \wedge a_2 \wedge \cdots \wedge a_{i-2} \wedge a_{g-1},$$

so

$$C_i \circ C_{i+2}(\tau_i(\pi^! \sigma)) = 4 \cdot a_2 \wedge \cdots \wedge a_{i-2} \wedge a_{g-1},$$

which lies in $\ker C_{i-2}$ as claimed.

5.2 Proof of Theorem 1.1

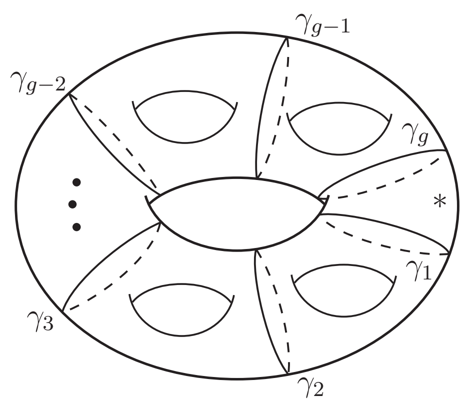


Figure 8: Nonseparating curves; each pair of adjacent curves yields a bounding pair f_k.

Consider the curves $\gamma_1, \ldots, \gamma_g$ displayed in Figure 8. For $1 \leq k < g$, let $f_k = T_{\gamma_{k+1}} T_{\gamma_k}^{-1}$, and for $2 \leq i < g$, let $\sigma \in H_i(\mathcal{I}_{g,*})$ be the abelian cycle $\sigma \coloneqq \{f_1, \ldots, f_i\}$. By Theorem 3.5, $\tau_i(\sigma) = 0$ since the bounding pairs f_k are not truly nested. We will show that σ is nontrivial in $H_i(\mathcal{I}_{g,*})$, proving the theorem.

Recall Johnson's map $\tau_J \colon \mathcal{I}_{g,*} \to \bigwedge^3 H_{\mathbb{Z}}$, which induces

$$(\tau_J)_* \colon H_i(\mathcal{I}_{g,*}) \to H_i(\textstyle\bigwedge^3 H_{\mathbb{Z}}) \approx \textstyle\bigwedge^i(\textstyle\bigwedge^3 H).$$

Let $\iota \colon \mathbb{Z}^i \to \mathcal{I}_{g,*}$ be the inclusion of the subgroup $\langle f_1, \ldots, f_i \rangle$. By definition,

$$(\tau_J)_* \sigma = (\tau_J \circ \iota)_* [\mathbb{Z}^i].$$

From the identification of $H_i(\bigwedge^3 H_{\mathbb{Z}})$ with $\bigwedge^i(\bigwedge^3 H)$, we have that

$$(\tau_J \circ \iota)_* [\mathbb{Z}^i] = (\tau_J)_*[f_1] \wedge \cdots \wedge (\tau_J)_*[f_i].$$

Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H_1(S_g)$ so that $[\gamma_k] = a_g$ for all $1 \leq k < g$, and so that $\{a_k, b_k\}$ gives a basis for the homology of the subsurface cut off by γ_k and γ_{k+1} for each $1 \leq k < g$. By Johnson's computation of τ_J (see (9) in Section 2 above), $(\tau_J)_*[f_k] = a_k \wedge b_k \wedge a_g$, and thus

$$(\tau_J)_*\sigma = \big(a_1 \wedge b_1 \wedge a_g\big) \wedge \cdots \wedge \big(a_i \wedge b_i \wedge a_g\big).$$

This element is nonzero in $\bigwedge^i(\bigwedge^3 H)$ since $\{a_k \wedge b_k \wedge a_g\}_{k=1}^{i}$ is linearly independent in $\bigwedge^3 H$. Thus $\tau_J(\sigma) \neq 0$ and so $\sigma \neq 0$, completing the proof of Theorem 1.1.

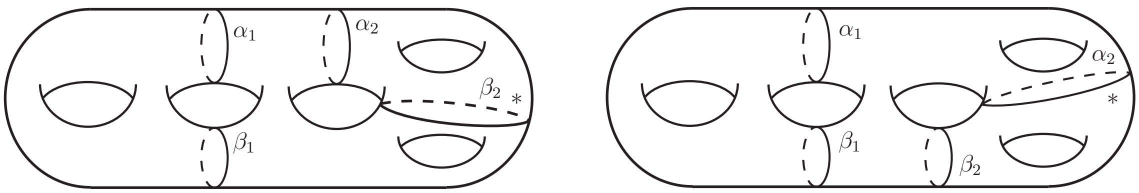


Figure 9: Homologically distinct abelian cycles not distinguishable by τ_i.
a. The collection $\{f_1, f_2\}$. **b.** The collection $\{g_1, g_2\}$.

Remark. We now give another example showing the non-injectivity of τ_i. One notable feature of this example is that we replace τ_J in the proof above by the maps τ_i themselves. For even $i \leq g-2$, let $\{f_k\}$ and $\{g_k\}$ be two truly nested collections of bounding pairs as in Figure 9, so that f_k and g_k are homologous; the collections cut off the same farthest subsurface; but the closest subsurfaces cut off by $\{f_k\}$ and $\{g_k\}$ determine different symplectic forms in $\bigwedge^2 H_1(S_g)$. By Theorem 3.4, $\tau_i(\{f_1, \ldots, f_i\}) = \tau_i(\{g_1, \ldots, g_i\})$. However, Theorem 4.1 shows that $\pi^!(\{f_1, \ldots, f_i\})$ is not equal to $\pi^!(\{g_1, \ldots, g_i\})$, and thus $\{f_1, \ldots, f_i\}$ is not equal to $\{g_1, \ldots, g_i\}$ in $H_i(\mathcal{I}_{g,*})$. Finally, note that we may choose $\{f_i\}$ and $\{g_i\}$ so that $(\tau_J)_*\{f_1, \ldots, f_i\} = (\tau_J)_*\{g_1, \ldots, g_i\}$, so this method yields new elements of $\ker \tau_i$ which cannot be detected by $(\tau_J)_*$.

5.3 Detecting homology using τ_J

In general, computing the image of $(\tau_J)_*$ is very difficult; in particular, by work of Kawazumi–Morita (see [Mo99, §6.4]), a complete solution would resolve the long-standing question of whether the even Morita–Mumford–Miller classes $e_{2i} \in H^{4i}(\mathcal{I}_{g,*})$ are nontrivial. However, in the lowest dimensions, the images have been found explicitly for the related case of closed surfaces. Considering the map

$$(\tau_J)_*\colon H_2(\mathcal{I}_g) \to \textstyle\bigwedge^2\big(\bigwedge^3 H/H\big),$$

Hain [Ha97] found that for $g \geq 6$, the image of $(\tau_J)_*$ is isomorphic to

$$V(\lambda_6) \oplus V(\lambda_4) \oplus V(\lambda_2) \oplus V(\lambda_2 + \lambda_4).$$

Similarly, Sakasai [Sa] found that, up to possibly a factor of $V(\lambda_1) = H$, the image of

$$(\tau_J)_*\colon H_3(\mathcal{I}_g) \to \textstyle\bigwedge^3(\bigwedge^3 H/H)$$

for $g \geq 9$ is isomorphic to

$$\begin{aligned} &V(\lambda_5 + 2\lambda_2) \oplus V(2\lambda_4 + \lambda_1) \oplus V(\lambda_6 + \lambda_3) \oplus V(\lambda_4 + \lambda_3) \\ \oplus &V(\lambda_7 + \lambda_2) \oplus V(\lambda_5 + \lambda_2) \oplus V(\lambda_3 + \lambda_2) \oplus V(\lambda_6 + \lambda_1) \oplus V(\lambda_4 + \lambda_1) \\ \oplus &V(\lambda_9) \oplus V(\lambda_7) \oplus V(\lambda_5) \oplus V(\lambda_5) \oplus V(\lambda_3). \end{aligned}$$

The stability of these decompositions is exactly the behavior predicted by Conjecture 1.6. Hain and Sakasai also compute the decompositions for smaller g, but they do not stabilize until $g \geq 6$ and $g \geq 9$ respectively.

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