Some problems on mapping class groups and moduli space

Benson Farb *

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Abstract

This paper presents a number of problems about mapping class groups and moduli space. The paper will appear in the book *Problems on Mapping Class Groups and Related Topics*, ed. by B. Farb, *Proc. Symp. Pure Math.* series, Amer. Math. Soc.

Contents

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1 Introduction

This paper contains a biased and personal list of problems on mapping class groups of surfaces. One of the difficulties in this area has been that there have not been so many easy problems. One of my goals here is to formulate a number of problems for which it seems that progress is possible. Another goal is the formulation of problems which will force us to penetrate more deeply into the structure of mapping class groups. Useful topological tools have been developed, for example the Thurston normal form, boundary theory, the reduction theory for subgroups, and the geometry and topology of the complex of curves. On the other hand there are basic problems which seem beyond the reach of these methods. One of my goals here is to pose problems whose solutions might require new methods.

1.1 Universal properties of Mod_g and $\mathcal{M}_g$

Let Σ_g denote a closed, oriented surface of genus g, and let Mod_g denote the group of homotopy classes of orientation-preserving homeomorphisms of Σ_g. The mapping class group Mod_g, along with its variations, derives much of its importance from its universal properties. Let me explain this for $g \geq 2$. In this case, a classical result of Earle-Eells gives that the identity component $\mathrm{Diff}^0(\Sigma_g)$ of $\mathrm{Diff}^+(\Sigma_g)$ is contractible. Since $\mathrm{Mod}_g = \pi_0 \mathrm{Diff}^+(\Sigma_g)$ by definition, we have a homotopy equivalence of classifying spaces:

$$\mathrm{BDiff}^+(\Sigma_g) \simeq \mathrm{BMod}_g \tag{1}$$

Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces. The group Mod_g acts properly discontinuously on the Teichmüller space Teich_g of marked, genus g Riemann surfaces. Since Teich_g is contractible it follows that $\mathcal{M}_g$ is a $K(\mathrm{Mod}_g, 1)$ space, i.e. it is homotopy equivalent to the spaces in (1). From these considerations it *morally* follows that, for any topological space B, we have the following bijections:

$$\left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of } \Sigma_g\text{-bundles over}\\ B\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Homotopy classes}\\ \text{of}\\ \text{maps } B \to \mathcal{M}_g\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Conjugacy classes}\\ \text{of representations}\\ \rho : \pi_1 B \to \mathrm{Mod}_g\end{array}\right\} \tag{2}$$

I use the term "morally" because (2) is not exactly true as stated. For example, one can have two nonisomorphic Σ_g bundles over S^1 with finite monodromy and with classifying maps $f : S^1 \to \mathrm{Mod}_g$ having the same image, namely a single point. The problem here comes from the torsion in Mod_g. This torsion prevents $\mathcal{M}_g$ from being a manifold; it is instead an orbifold, and so we need to work in the category of orbifolds. This is a nontrivial issue which requires care. There are two basic fixes to this problem. First, one can simply replace $\mathcal{M}_g$ in (2) with the classifying space BMod_g. Another option is to replace Mod_g in (2) with any torsion-free subgroup $\Gamma < \mathrm{Mod}_g$ of finite index. Then Γ acts freely on Teich_g and the corresponding finite cover of $\mathcal{M}_g$ is a manifold. In this case (2) is true as stated. This torsion subtlety will usually not have a major impact on our discussion, so we will for the most part ignore it. This is fine on the level of rational homology since the homotopy equivalences described above induce isomorphisms:

$$H^*(\mathcal{M}_g, \mathbf{Q}) \approx H^*(\mathrm{BDiff}^+(\Sigma_g), \mathbf{Q}) \approx H^*(\mathrm{Mod}_g, \mathbf{Q}) \tag{3}$$

There is a unique complex orbifold structure on $\mathcal{M}_g$ with the property that these bijections carry over to the holomorphic category. This means that the manifolds are complex manifolds, the bundles are non-isotrivial (i.e. the holomorphic structure of the fibers is not locally constant, unless the map $B \to \mathcal{M}_g$ is trivial), and the maps are holomorphic. For the third entry of (2), one must restrict to such conjugacy classes with holomorphic representatives; many conjugacy classes do not have such a representative.

For $g \geq 3$ there is a canonical Σ_g-bundle $\mathcal{U}_g$ over $\mathcal{M}_g$, called the *universal curve* (terminology from algebraic geometry), for which the (generic) fiber over any $X \in \mathcal{M}_g$ is the Riemann surface X. The first bijection of (2) is realized concretely using $\mathcal{U}_g$. For example given any smooth $f : B \to \mathcal{M}_g$, one simply pulls back the bundle $\mathcal{U}_g$ over f to give a Σ_g-bundle over B. Thus $\mathcal{M}_g$ plays the same role for surface bundles as the (infinite) Grassmann manifolds play for vector bundles. Again one needs to be careful about torsion in Mod_g here, for example by passing to a finite cover of $\mathcal{M}_g$. For $g = 2$ there are more serious problems.

An important consequence is the following. Suppose one wants to associate to every Σ_g-bundle a (say integral) cohomology class on the base of that bundle, so that this association is *natural*, that is, it is preserved under pullbacks. Then each such cohomology class must be the pullback of some element of $H^*(\mathcal{M}_g, \mathbf{Z})$. In this sense the classes in $H^*(\mathcal{M}_g, \mathbf{Z})$ are universal. After circle bundles, this is the next simplest nonlinear bundle theory. Unlike circle bundles, this study connects in a fundamental way to algebraic geometry, among other things.

Understanding the sets in (2) is interesting even in the simplest cases.

Example 1.1. (**Surface bundles over S^1**). Let $B = S^1$. In this case (2) states that the classfication of Σ_g-bundles over S^1, up to bundle isomorphism, is equivalent to the classification of elements of Mod_g up to conjugacy. Now, a fixed 3-manifold may fiber over S^1 in infinitely many different ways, although there are finitely many fiberings with fiber of fixed genus. Since it is possible to compute these fiberings[1], the homeomorphism problem for 3-manifolds fibering over S^1 can easily be reduced to solving the conjugacy problem for Mod_g. This was first done by Hemion [He].

Example 1.2. (**Arakelov-Parshin finiteness**). Now let $B = \Sigma_h$ for some $h \geq 1$, and consider the sets and bijections of (2) in the holomorphic category. The Geometric Shafarevich Conjecture, proved by Arakelov and Parshin, states that these sets are finite, and that the holomorphic map in each nontrivial homotopy class is unique. As beautifully explained in [Mc1], from this result one can derive (with a branched cover trick of Parshin) finiteness theorems for rational points on algebraic varieties over function fields.

Remark on universality. I would like to emphasize the following point. While the existence of characteristic classes associated to *every* Σ_g-bundle is clearly the first case to look at, it seems that requiring such a broad form of universality is too constraining. One reflection of this is the paucity of cohomology of $\mathcal{M}_g$, as the Miller-Morita-Mumford conjecture (now theorem due to Madsen-Weiss [MW]) shows. One problem is that the requirement of naturality for *all* monodromies simply kills what are otherwise natural and common classes. Perhaps more natural would be to consider the characteristic classes for Σ_g-bundles with torsion-free monodromy. This would lead one to understand the cohomology of various finite index subgroups of Mod_g.

Another simple yet striking example of this phenomenon is Harer's theorem that

$$H^2(\mathcal{M}_g, \mathbf{Q}) = \mathbf{Q}$$

In particular the signature cocycle, which assigns to every bundle $\Sigma_g \to M^4 \to B$ the signature $\sigma(M^4)$, is (up to a rational multiple) the only characteristic class in dimension 2. When the monodromy representation $\pi_1 B \to \mathrm{Mod}_g$ lies in the (infinite index) Torelli subgroup $\mathcal{I}_g < \mathrm{Mod}_g$ (see below), σ is always zero, and so is useless. However, there are infinitely many homotopy types of surface bundles M^4 over surfaces with $\sigma(M^4) = 0$; indeed such families of examples can

[1] This is essentially the fact that the Thurston norm is computable.

be taken to have monodromy in $\mathcal{I}_g$. We note that there are no known elements of $H^*(\mathrm{Mod}_g, \mathbf{Q})$ which restrict to a nonzero element of $H^*(\mathcal{I}_g, \mathbf{Q})$.

We can then try to find, for example, characteristic classes for Σ_g-bundles with monodromy lying in $\mathcal{I}_g$, and it is not hard to prove that these are just pullbacks of classes in $H^*(\mathcal{I}_g, \mathbf{Z})$. In dimensions one and two, for example, we obtain a large number of such classes (see [Jo1] and [BFa2], respectively).

I hope I have provided some motivation for understanding the cohomology of subgroups of Mod_g. This topic is wide-open; we will discuss a few aspects of it below.

Three general problems. Understanding the theory of surface bundles thus leads to the following basic general problems.

1. For various finitely presented groups Γ, classify the representations $\rho : \Gamma \to \mathrm{Mod}_g$ up to conjugacy.

2. Try to find analytic and geometric structures on $\mathcal{M}_g$ in order to apply complex and Riemannian geometry ideas to constrain possibilities on such ρ.

3. Understand the cohomology algebra $H^*(\Gamma, K)$ for various subgroups $\Gamma < \mathrm{Mod}_g$ and various modules K, and find topological and geometric interpretations for its elements.

We discuss below some problems in these directions in §2, §4 and §5, respectively. One appealing aspect of such problems is that attempts at understanding them quickly lead to ideas from combinatorial group theory, complex and algebraic geometry, the theory of dynamical systems, low-dimensional topology, symplectic representation theory, and more. In addition to these problems, we will be motivated here by the fact that Mod_g and its subgroups provide a rich and important collection of examples to study in combinatorial and geometric group theory; see §3 below.

Remark on notational conventions. We will usually state conjectures and problems and results for Mod_g, that is, for closed surfaces. This is simply for convenience and simplicity; such conjectures and problems should always be considered as posed for surfaces with boundary and punctures, except perhaps for some sporadic, low-genus cases. Similarly for other subgroups such as the Torelli group $\mathcal{I}_g$. Sometimes the extension to these cases is straight-forward, but sometimes it isn't, and new phenomena actually arise.

1.2 The Torelli group and associated subgroups

One of the recurring objects in this paper will be the Torelli group. We now briefly describe how it fits in to the general picture.

Torelli group. Algebraic intersection number gives a symplectic form on $H_1(\Sigma_g, \mathbf{Z})$. This form is preserved by the natural action of Mod_g. The *Torelli group* $\mathcal{I}_g$ is defined to be the kernel of this action. We then have an exact sequence

$$1 \to \mathcal{I}_g \to \mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbf{Z}) \to 1 \tag{4}$$

The genus g *Torelli space* is defined to be the quotient of Teich_g by $\mathcal{I}_g$. Like $\mathcal{M}_g$, this space has the appropriate universal mapping properties. However, the study of maps into Torelli space is precisely *complementary* to the theory of holomorphic maps into $\mathcal{M}_g$, as follows. Any holomorphic map $f : B \to \mathcal{M}_g$ with $f_*(B) \subseteq \mathcal{I}_g$, when composed with the (holomorphic) period mapping $\mathcal{M}_g \to \mathcal{A}_g$ (see §4.4 below), lifts to the universal cover $\widetilde{\mathcal{A}_g}$, which is the Siegel upper half-space (i.e. the symmetric space $\mathrm{Sp}(2g, \mathbf{R}) / \mathrm{SU}(g)$). Since the domain is compact, the image of this holomorphic lift is constant. Hence f is constant.

The study of $\mathcal{I}_g$ goes back to Nielsen (1919) and Magnus (1936), although the next big breakthrough came in a series of remarkable papers by Dennis Johnson in the late 1970's (see [Jo1] for a summary). Still, many basic questions about $\mathcal{I}_g$ remain open; we add to the list in §5 below.

Group generated by twists about separating curves. The *group generated by twists about separating curves*, denoted $\mathcal{K}_g$, is defined to be the subgroup Mod_g generated by the (infinitely many) Dehn twists about separating (i.e. bounding) curves in Σ_g. The group $\mathcal{K}_g$ is sometimes called the *Johnson kernel* since Johnson proved that $\mathcal{K}_g$ is precisely the kernel of the so-called Johnson homomorphism. This group is a featured player in the study of the Torelli group. Its connection to 3-manifold theory begins with Morita's result that every integral homology 3-sphere comes from removing a handlebody component of some Heegaard embedding $h : \Sigma_g \hookrightarrow S^3$ and gluing it back to the boundary Σ_g by an element of $\mathcal{K}_g$. Morita then proves the beautiful result that for a fixed such h, taking the Casson invariant of the resulting 3-manifold actually gives a *homomorphism* $\mathcal{K}_g \to \mathbf{Z}$. This is a starting point for Morita's analysis of the Casson invariant; see [Mo1] for a summary.

Johnson filtration. We now describe a filtration of Mod_g which generalizes (4). This filtration has become a basic object of study.

For a group Γ we inductively define $\Gamma_0 := \Gamma$ and $\Gamma_{i+1} = [\Gamma, \Gamma_i]$. The chain of subgroups $\Gamma \supseteq \Gamma_1 \supseteq \cdots$ is the *lower central series* of Γ. The group Γ/Γ_i is i-step nilpotent; indeed Γ/Γ_i has the universal property that any homomorphism from Γ to any i-step nilpotent group factors through Γ/Γ_i. The sequence $\{\Gamma/\Gamma_i\}$ can be thought of as a kind of Taylor series for Γ.

Now let $\Gamma := \pi_1 \Sigma_g$. It is a classical result of Magnus that $\bigcap_{i=1}^{\infty} \Gamma_i = 1$, that is Γ is *residually nilpotent*. Now Mod_g acts by outer automorphisms on Γ, and each of the subgroups Γ_i is clearly characteristic. We may thus define for each $k \geq 0$:

$$\mathcal{I}_g(k) := \ker(\mathrm{Mod}_g \to \mathrm{Out}(\Gamma/\Gamma_k)) \tag{5}$$

That is, $\mathcal{I}_g(k)$ is just the subgroup of Mod_g acting trivially on the k^{th} nilpotent quotient of $\pi_1\Sigma_g$. Clearly $\mathcal{I}_g(1) = \mathcal{I}_g$; Johnson proved that $\mathcal{I}_g(2) = \mathcal{K}_g$. The sequence $\mathcal{I}_g = \mathcal{I}_g(1) \supset \mathcal{I}_g(2) \supset \cdots$ is called the *Johnson filtration*; it has also been called the *relative weight filtration*. This sequence forms a (but not *the*) lower central series for $\mathcal{I}_g$. The fact that $\bigcap_{i=1}^{\infty}\Gamma_i = 1$ easily implies that the Aut versions of the groups defined in (5) have trivial intersection. The stronger fact that $\bigcap_{i=1}^{\infty}\mathcal{I}_g(i) = 1$, so that $\mathcal{I}_g$ is residually nilpotent, is also true, but needs some additional argument (see [BL]).

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2 Subgroups and submanifolds: Existence and classification

It is a natural problem to classify subgroups H of Mod_g. By classify we mean to give an effective list of isomorphism or commensurability types of subgroups, and also to determine all embeddings of a given subgroup, up to conjugacy. While there are some very general structure theorems, one sees rather quickly that the problem as stated is far too ambitous. One thus begins with the problem of finding various invariants associated to subgroups, by which we mean invariants of their isomorphism type, commensurability type, or more extrinsic invariants which depend on the embedding $H \to \mathrm{Mod}_g$. One then tries to determine precisely which values of a particular invariant can occur, and perhaps even classify those subgroups having a given value of the invariant. In this section we present a selection of such problems.

Remark on subvarieties. The classification and existence problem for subgroups and submanifolds of $\mathcal{M}_g$ can be viewed as algebraic and topological analogues of the problem, studied by algebraic geometers, of understanding (complete) subvarieties of $\mathcal{M}_g$. There is an extensive literature on this problem; see, e.g., [Mor] for a survey. To give just one example of the type of problem studied, let

$$c_g := \max\{\dim_{\mathbf{C}}(V) : V \text{ is a complete subvariety of } \mathcal{M}_g\}$$

The goal is to compute c_g. This is a kind of measure of where $\mathcal{M}_g$ sits between being affine (in which case c_g would be 0) and projective (in which case c_g would equal $\dim_{\mathbf{C}}(\mathcal{M}_g) = 3g-3$). While $\mathcal{M}_2$ is affine and so $c_2 = 0$, it is known for $g \geq 3$ that $1 \leq c_g < g-1$; the lower bound is given by construction, the upper bound is a well-known theorem of Diaz.

2.1 Some invariants

The notion of *relative ends* $\text{ends}(\Gamma, H)$ provides a natural way to measure the "codimension" of a subgroup H in a group Γ. To define $e(\Gamma, H)$, consider any proper, connected metric space X on which Γ acts properly and cocompactly by isometries. Then $\text{ends}(\Gamma, H)$ is defined to be the number of ends of the quotient space X/H.

Question 2.1 (Ends spectrum). *What are the possibile values of* $\text{ends}(\text{Mod}_g, H)$ *for finitely-generated subgroups* $H < \text{Mod}_g$*?*

It is well-known that the moduli space $\mathcal{M}_g$ has one end. The key point of the proof is that the complex of curves is connected. This proof actually gives more: any cover of $\mathcal{M}_g$ has one end; see, e.g., [FMa]. However, I do not see how this fact directly gives information about Question 2.1.

Commensurators. Asking for two subgroups of a group to be conjugate is often too restrictive a question. A more robust notion is that of commensurability. Subgroups Γ_1, Γ_2 of a group H are *commensurable* if there exists $h \in H$ such that $h\Gamma_1 h^{-1} \cap \Gamma_2$ has finite index in both $h\Gamma_1 h^{-1}$ and in Γ_2. One then wants to classify subgroups up to commensurability; this is the natural equivalence relation one studies in order to coarsify the relation of "conjugate" to ignore finite index information. The primary commensurability invariant for subgroups $\Gamma < H$ is the *commensurator of* Γ *in* H, denoted $\text{Comm}_H(\Gamma)$, defined as:

$$\text{Comm}_H(\Gamma) := \{h \in H : h\Gamma h^{-1} \cap \Gamma \text{ has finite index in both } \Gamma \text{ and } h\Gamma h^{-1}\}$$

The commensurator has most commonly been studied for discrete subgroups of Lie groups. One of the most striking results about commensurators, due to Margulis, states that if Γ is an irreducible lattice in a semisimple[2] Lie group H then $[\text{Comm}_H(\Gamma) : \Gamma] = \infty$ if and only if Γ is arithmetic. In other words, it is precisely the arithmetic lattices that have infinitely many "hidden symmetries".

Problem 2.2. *Compute* $\text{Comm}_{\text{Mod}_g}(\Gamma)$ *for various subgroups* $\Gamma < \text{Mod}_g$.

Paris-Rolfsen and Paris (see, e.g., [Pa]) have proven that most subgroups of Mod_g stabilizing a simple closed curve, or coming from the mapping class group of a subsurface of S, are self-commensurating in Mod_g. Self-commensurating subgroups, that is subgroups $\Gamma < H$ with $\text{Comm}_H(\Gamma) = \Gamma$, are particularly important since the finite-dimensional unitary dual of Γ injects into the unitary dual of H; in other words, any unitary representation of H induced from a finite-dimensional irreducible unitary representation of Γ must itself be irreducible.

Volumes of representations. Consider the general problem of classifying, for a fixed finitely generated group Γ, the set

$$\mathcal{X}_g(\Gamma) := \text{Hom}(\Gamma, \text{Mod}_g)/\text{Mod}_g$$

[2]By *semisimple* we will always mean linear semisimple with no compact factors.

of conjugacy classes of representations $\rho : \Gamma \to \mathrm{Mod}_g$. Here the representations ρ_1 and ρ_2 are conjugate if $\rho_1 = C_h \circ \rho_2$, where $C_h : \mathrm{Mod}_g \to \mathrm{Mod}_g$ is conjugation by some $h \in \mathrm{Mod}_g$. Suppose $\Gamma = \pi_1 X$ where X is, say, a smooth, closed n-manifold. Since $\mathcal{M}_g$ is a classifying space for Mod_g, we know that for each for each $[\rho] \in \mathcal{X}_g(\Gamma)$ there exists a smooth map $f : X \to \mathcal{M}_g$ with $f_* = \rho$, and that f is unique up to homotopy.

Each n-dimensional real cocycle ξ on $\mathcal{M}_g$ then gives a well-defined invariant

$$\nu_\xi : \mathcal{X}_g(\Gamma) \to \mathbf{R}$$

defined by

$$\nu_\xi([\rho]) := \int_X f^* \xi$$

It is clear that $\nu_\xi([\rho])$ does not depend on the choices, and indeed depends only on the cohomology class of ξ. As a case of special interest, let X be a $2k$-dimensional manifold and let ω_{WP} denote the Weil-Petersson symplectic form on $\mathcal{M}_g$. Define the *complex k-volume* of $\rho : \pi_1 X \to \mathrm{Mod}_g$ to be

$$\mathrm{Vol}_k([\rho]) := \int_X f^* \omega_{\mathrm{WP}}^k$$

Problem 2.3 (Volume spectrum)**.** *Determine for each $1 \leq k \leq 3g-3$ the image of $\mathrm{Vol}_k : \mathcal{X}_g(\Gamma) \to \mathbf{R}$. Determine the union of all such images as Γ ranges over all finitely presented groups.*

It would also be interesting to pose the same problem for representations with special geometric constraints, for example those with holomorphic or totally geodesic (with respect to a fixed metric) representatives. In particular, how do such geometric properties constrain the set of possible volumes? Note that Mirzakhani [Mir] has given recursive formulas for the Weil-Petersson volumes of moduli spaces for surfaces with nonempty totally geodesic boundary.

Invariants from linear representations. Each linear representation $\psi : \mathrm{Mod}_g \to \mathrm{GL}_m(\mathbf{C})$ provides us with many invariants for elements of $\mathcal{X}_g(\Gamma)$, simply by composition with ψ followed by taking any fixed class function on $\mathrm{GL}_m(\mathbf{C})$. One can obtain commensurability invariants for subgroups of Mod_g this way as well. While no *faithful* ψ is known for $g \geq 3$ (indeed the existence of such a ψ remains a major open problem) , there are many such ψ which give a great deal of information. Some computations using this idea can be found in [Su2]. I think further computations would be worthwhile.

2.2 Lattices in semisimple Lie groups

While Ivanov proved that Mod_g is not isomorphic to a lattice in a semisimple Lie group, a recurring theme has been the comparison of algebraic properties of Mod_g with such lattices and geometric/topological properties of moduli space $\mathcal{M}_g$ with those of locally symmetric orbifolds. The question (probably due to Ivanov) then arose: "Which lattices Γ are subgroups of Mod_g?"

This question arises from a slightly different angle under the algebro-geometric guise of studying locally symmetric subvarieties of moduli space; see [Ha1]. The possibilities for such Γ are highly constrained; theorems of Kaimanovich-Masur, Farb-Masur and Yeung (see [FaM, Ye]), give the following.

Theorem 2.4. *Let Γ be an irreducible lattice in a semisimple Lie group $G \neq \mathrm{SO}(m,1), \mathrm{SU}(n,1)$ with $m \geq 2, n \geq 1$. Then any homomorphism $\rho : \Gamma \to \mathrm{Mod}_g$ with $g \geq 1$ has finite image.*

Theorem 2.4 does not extend to the cases $G = \mathrm{SO}(m,1)$ and $G = \mathrm{SU}(n,1)$ in general since these groups admit lattices Γ which surject to $\mathbf{Z}$. Now let us restrict to the case of *injective* ρ, so we are asking about which lattices Γ occur as subgroups of Mod_g. As far as I know, here is what is currently known about this question:

1. (Lattices $\Gamma < \mathrm{SO}(2,1)$): Each such Γ has a finite index subgroup which is free or $\pi_1\Sigma_h$ for some $h \geq 2$. These groups are plentiful in Mod_g and are discussed in more detail in §2.3 below.

2. (Lattices $\Gamma < \mathrm{SO}(3,1)$): These exist in $\mathrm{Mod}_{g,1}$ for $g \geq 2$ and in Mod_g for $g \geq 4$, by the following somewhat well-known construction. Consider the Birman exact sequence

$$1 \to \pi_1\Sigma_g \to \mathrm{Mod}_{g,1} \xrightarrow{\pi} \mathrm{Mod}_g \to 1 \tag{6}$$

   Let $\phi \in \mathrm{Mod}_g$ be a pseudo-Anosov homeomorphism, and let $\Gamma_\phi < \mathrm{Mod}_{g,1}$ be the pullback under π of the cyclic subgroup of Mod_g generated by ϕ. The group Γ_ϕ is isomorphic to the fundamental group of a Σ_g bundle over S^1, namely the bundle obtained from $\Sigma_g \times [0,1]$ by identifying $(x,0)$ with $(\phi(x),1)$. By a deep theorem of Thurston, such manifolds admit a hyperbolic metric, and so Γ_ϕ is a cocompact lattice in $\mathrm{SO}(3,1) = \mathrm{Isom}^+(\mathbf{H}^3)$. A branched covering trick of Gonzalez-Diaz and Harvey (see [GH]) can be used to find Γ_ϕ as a subgroup of Mod_h for appropriate $h \geq 4$. A variation of the above can be used to find nonuniform lattices in $\mathrm{SO}(3,1)$ inside Mod_g for $g \geq 4$.

3. (Cocompact lattices $\Gamma < \mathrm{SO}(4,1)$): Recently John Crisp and I [CF] found one example of a cocompact lattice $\Gamma < \mathrm{SO}(4,1) = \mathrm{Isom}^+(\mathbf{H}^4)$ which embeds in Mod_g for all sufficiently large g. While we only know of one such Γ, it has infinitely many conjugacy classes in Mod_g. The group Γ is a right-angled Artin group, which is commensurable with a group of reflections in the right-angled 120-cell in $\mathbf{H}^4$.

4. (Noncocompact lattices $\Gamma < \mathrm{SU}(n,1), n \geq 2$): These Γ have nilpotent subgroups which are not virtually abelian. Since every nilpotent, indeed solvable subgroup of Mod_g is virtually abelian, Γ is not isomorphic to any subgroup of Mod_g.

Hence the problem of understanding which lattices in semisimple Lie groups occur as subgroups of Mod_g comes down to the following.

Question 2.5. *Does there exist some* $\mathrm{Mod}_g, g \geq 2$ *that contains a subgroup* Γ *isomorphic to a cocompact (resp. noncocompact) lattice in* $\mathrm{SO}(m,1)$ *with* $m \geq 5$ *(resp.* $m \geq 4$*)? a cocompact lattice in* $\mathrm{SU}(n,1), n \geq 2$*? Must there be only finitely many conjugacy classes of any such fixed* Γ *in* Mod_g*?*

In light of example (3) above, I would like to specifically ask: can Mod_g contain infinitely many isomorphism types of cocompact lattices in $\mathrm{SO}(4,1)$?

Note that when Γ is the fundamental group of a (complex) algebraic variety V, then it is known that there can be at most finitely many representations $\rho : \Gamma \to \mathrm{Mod}_g$ which have *holomorphic* representatives, by which we mean the unique homotopy class of maps $f : V \to \mathcal{M}_g$ with $f_* = \rho$ contains a holomorphic map. This result follows from repeatedly taking hyperplane sections and finally quoting the result for (complex) curves. The result for these is a theorem of Arakelov-Parshin (cf. Example 1.1 above, and §2.3 below.)

For representations which do not *a priori* have a holomorphic representative, one might try to find a harmonic representative and then to prove a Siu-type rigidity result to obtain a holomorphic representative. One difficulty here is that it is not easy to find harmonic representatives, since (among other problems) every loop in $\mathcal{M}_g, g \geq 2$ can be freely homotoped outside every compact set. For recent progress, however, see [DW] and the references contained therein.

2.3 Surfaces in moduli space

Motivation for studying representations $\rho : \pi_1\Sigma_h \to \mathrm{Mod}_g$ for $g, h \geq 2$ comes from many directions. These include: the analogy of Mod_g with Kleinian groups (see, e.g., [LR]); the fact that such subgroups are the main source of locally symmetric families of Riemann surfaces (see §2.2 above, and [Ha1]) ; and their appearance as a key piece of data in the topological classification of surface bundles over surfaces (cf. (2) above).

Of course understanding such ρ with holomorphic representatives is the Arakelov-Parshin Finiteness Theorem discussed in Example 1.1 above. Holomorphicity is a key feature of this result. For example, one can prove (see, e.g., [Mc1]) that there are finitely many such ρ with a holomorphic representative by finding a *Schwarz Lemma for* $\mathcal{M}_g$: any holomorphic map from a compact hyperbolic surface into $\mathcal{M}_g$ endowed with the Teichmüller metric is distance decreasing. The finiteness is also just not true without the holomorphic assumption (see Theorem 2.7 below). We therefore want to recognize when a given representation has a holomorphic representative.

Problem 2.6 (Holomorphic representatives)**.** *Find an algorithm or a group-theoretic invariant which determines or detects whether or not a given representation* $\rho : \pi_1\Sigma_h \to \mathrm{Mod}_g$ *has a holomorphic representative.*

Note that a necessary, but not sufficient, condition for a representation $\rho : \pi_1\Sigma_h \to \mathrm{Mod}_g$ to be holomorphic is that it be *irreducible*, i.e. there is no essential isotopy class of simple closed curve

α in Σ_g such that $\rho(\pi_1\Sigma_h)(\alpha) = \alpha$. I believe it is not difficult to give an algorithm to determine whether or not any given ρ is irreducible or not.

We would like to construct and classify (up to conjugacy) such ρ. We would also like to compute their associated invariants, such as

$$\nu(\rho) := \int_{\Sigma_h} f^*\omega_{\rm WP}$$

where $\omega_{\rm WP}$ is the Weil-Petersson 2-form on $\mathcal{M}_g$, and where $f : \Sigma_h \to \mathcal{M}_g$ is any map with $f_* = \rho$. This would give information on the signatures of surface bundles over surfaces, and also on the Gromov co-norm of $[\omega_{\rm WP}] \in H^*(\mathcal{M}_g, \mathbf{R})$.

The classification question is basically impossible as stated, since e.g. surface groups surject onto free groups, so it is natural to first restrict to injective ρ. Using a technique of Crisp-Wiest, J. Crisp and I show in [CF] that irreducible, injective ρ are quite common.

Theorem 2.7. *For each $g \geq 4$ and each $h \geq 2$, there are infinitely many Mod_g-conjugacy classes of injective, irreducible representations $\rho : \pi_1\Sigma_h \to \mathrm{Mod}_g$. One can take the images to lie inside the Torelli group $\mathcal{I}_g$. Further, for any $n \geq 1$, one can take the images to lie inside the subgroup of* Mod_g *generated by $n^{\rm th}$ powers of all Dehn twists.*

One can try to use these representations, as well as those of [LR], to give new constructions of surface bundles over surfaces with small genus base and fiber and nonzero signature. The idea is that Meyer's signature cocycle is positively proportional to the Weil-Petersson 2-form ω_{WP}, and one can actually explicitly integrate the pullback of ω_{WP} under various representations, for example those glued together using Teichmüller curves.

Note that Theorem 2.7 provides us with infinitely many topological types of surface bundles, each with irreducible, faithful monodromy, all having the same base and the same fiber, and all with signature zero.

2.4 Normal subgroups

It is a well-known open question to determine whether or not Mod_g contains a normal subgroup H consisting of all pseudo-Anosov homeomorphisms. For genus $g = 2$ Whittlesey [Wh] found such an H; it is an infinitely generated free group. As far as I know this problem is still wide open.

Actually, when starting to think about this problem I began to realize that it is not easy to find *finitely generated* normal subgroups of Mod_g which are not commensurable with either $\mathcal{I}_g$ or Mod_g. There are many normal subgroups of Mod_g which are not commensurable to either $\mathcal{I}_g$ or to Mod_g, most notably the terms $\mathcal{I}_g(k)$ of the Johnson filtration for $k \geq 2$ and the terms of the lower central series of $\mathcal{I}_g$. However, the former are infinitely generated and the latter are likely to be infinitely generated; see Theorem 5.4 and Conjecture 5.5 below.

Question 2.8 (Normal subgroups)**.** *Let Γ be a finitely generated normal subgroup of* Mod_g*, where $g \geq 3$. Must Γ be commensurable with* Mod_g *or with $\mathcal{I}_g$?*

One way of constructing *infinitely generated* normal subgroups of Mod_g is to take the group generated by the n^{th} powers of all Dehn twists. Another way is to take the normal closure N_ϕ of a single element $\phi \in \mathrm{Mod}_g$. It seems unclear how to determine the algebraic structure of these N_ϕ, in particular to determine whether N_ϕ is finite index in Mod_g, or in one of the $\mathcal{I}_g(k)$. The following is a basic test question.

Question 2.9. *Is it true that, given any pseudo-Anosov $\phi \in \mathrm{Mod}_g$, there exists $n = n(\phi)$ such that the normal closure of ϕ^n is free?*

Gromov discovered the analogous phenomenon for elements of hyperbolic type inside nonelementary word-hyperbolic groups; see [Gro], Theorem 5.3.E.

One should compare Question 2.8 to the Margulis Normal Subgroup Theorem (see, e.g. [Ma]), which states that if Λ is any irreducible lattice in a real, linear semisimple Lie group with no compact factors and with $\mathbf{R}$-rank at least 2, then any (not necessarily finitely generated) normal subgroup of Λ is finite and central or has finite index in Λ. Indeed, we may apply this result to analyzing normal subgroups Γ of Mod_g. For $g \geq 2$ the group $\mathrm{Sp}(2g, \mathbf{Z})$ satisfies the hypotheses of Margulis's theorem, and so the image $\pi(\Gamma)$ under the natural representation $\pi : \mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbf{Z})$ is normal, hence is finite or finite index. This proves the following.

Proposition 2.10 (Maximality of Torelli)**.** *Any normal subgroup Γ of Mod_g containing $\mathcal{I}_g$ is commensurable either with Mod_g or with $\mathcal{I}_g$.*

Proposition 2.10 is a starting point for trying to understand Question 2.8. Note too that Mess [Me] proved that the group $\mathcal{I}_2$ is an infinitely generated free group, and so it has no finitely generated normal subgroups. Thus we know that if Γ is any finitely generated normal subgroup of Mod_2, then $\pi(\Gamma)$ has finite index in $\mathrm{Sp}(4, \mathbf{Z})$. This in turn gives strong information about Γ; see [Fa3].

One can go further by considering the Malcev Lie algebra $\mathfrak{t}_g$ of $\mathcal{I}_g$, computed by Hain in [Ha3]; cf. §5.4 below. The normal subgroup $\Gamma \cap \mathcal{I}_g$ of $\mathcal{I}_g$ gives an $\mathrm{Sp}(2g, \mathbf{Z})$-invariant subalgebra $\mathfrak{h}$ of $\mathfrak{t}_g$. Let $H = H_1(\Sigma_g, \mathbf{Z})$. The Johnson homomorphism $\tau : \mathcal{I}_g \to \wedge^3 H/H$ is equivariant with respect to the action of $\mathrm{Sp}(2g, \mathbf{Z})$, and $\wedge^3 H/H$ is an irreducible $\mathrm{Sp}(2g, \mathbf{Z})$-module. It follows that the first quotient in the lower central series of $\Gamma \cap \mathcal{I}_g$ is either trivial or is all of $\wedge^3 H/H$. With more work, one can extend the result of Proposition 2.10 from $\mathcal{I}_g$ to $\mathcal{K}_g = \mathcal{I}_g(2)$; see [Fa3].

Theorem 2.11 (Normal subgroups containing $\mathcal{K}_g$)**.** *Any normal subgroup Γ of Mod_g containing $\mathcal{K}_g$ is commensurable to $\mathcal{K}_g$, $\mathcal{I}_g$ or Mod_g.*

One can continue this "all or nothing image" line of reasoning to deeper levels of $\mathfrak{t}_g$. Indeed, I believe one can completely reduce the classification of normal subgroups of Mod_g, at least those that contain some $\mathcal{I}_g(k)$, to some symplectic representation theory problems, such as the following.

Problem 2.12. *For $g \geq 2$, determine the irreducible factors of the graded pieces of the Malcev Lie algebra $\mathfrak{t}_g$ of $\mathcal{I}_g$ as Sp-modules.*

While Hain gives in [Ha3] an explicit and reasonably simple presentation for $\mathfrak{t}_g$ when $g > 6$, Problem 2.12 still seems to be an involved problem in classical representation theory.

As Chris Leininger pointed out to me, all of the questions above have natural "virtual versions". For example, one can ask about the classification of normal subgroups of finite index subgroups of Mod_g. Another variation is the classification of *virtually normal* subgroups of Mod_g, that is, subgroups whose normalizers in Mod_g have finite index in Mod_g.

2.5 Numerology of finite subgroups

The Nielsen Realization Theorem, due to Kerckhoff, states that any finite subgroup $F < \mathrm{Mod}_g$ can be realized as a group of automorphisms of some Riemann surface X_F. Here by *automorphism* group we mean group of (orientation-preserving) isometries in some Riemannian metric, or equivalently in the hyperbolic metric, or equivalently the group of biholomorphic automorphisms of X_F. An easy application of the uniformization theorem gives these equivalences. It is classical that the automorphism group $\mathrm{Aut}(X)$ of a closed Riemann surface of genus $g \geq 2$ is finite. Thus the study of finite subgroups of $\mathrm{Mod}_g, g \geq 2$ reduces to the study of automorphism groups of Riemann surfaces.

Let $N(g)$ denote the largest possible order of $\mathrm{Aut}(X)$ as X ranges over all genus g surfaces. Then for $g \geq 2$ it is known that

$$8(g+1) \leq N(g) \leq 84(g-1) \tag{7}$$

the lower bound due to Accola and Maclachlan (see, e.g., [Ac]); the upper due to Hurwitz. It is also known that each of these bounds is achieved for infinitely many g and is not achieved for infinitely many g. There is an extensive literature seeking to compute $N(g)$ and variations of it. The most significant achievement in this direction is the following result of M. Larsen [La].

Theorem 2.13 (Larsen)**.** *Let H denote the set of integers $g \geq 2$ such that there exists at least one compact Riemann surface X_g of genus g with $|\mathrm{Aut}(X_g)| = 84(g-1)$. Then the series $\sum_{g \in H} g^{-s}$ converges absolutely for $\Re(s) > 1/3$ and has a singularity at $s = 1/3$.*

In particular, the g for which $N(g) = 84(g-1)$ occur with the same frequency as perfect cubes. It follows easily from the Riemann-Hurwitz formula that the bound $84(g-1)$ is achieved precisely for those surfaces which isometrically (orbifold) cover the $(2,3,7)$ orbifold. Thus the problem of determining which genera realize the $84(g-1)$ bound comes down to figuring out the possible (finite) indices of subgroups, or what is close to the same thing finite quotients, of the $(2,3,7)$ orbifold group. Larsen's argument uses in a fundamental way the classification of finite simple groups.

Problem 2.14. *Give a proof of Theorem 2.13 which does not depend on the classification of finite simple groups.*

To complete the picture, one would like to understand the frequency of those g for which the lower bound in (7) occurs.

Problem 2.15 (Frequency of low symmetry). *Let H denote the set of integers $g \geq 2$ such that $N(g) = 8(g+1)$. Find the s_0 for which the series $\sum_{g \in H} g^{-s}$ converges absolutely for the real part $\Re(s)$ of s satisfying $\Re(s) > s_0$, and has a singularity at $s = s_0$.*

There are various refinements and variations on Problem 2.15. For example, Accola proves in [Ac] that when g is divisible by 3, then $N(g) \geq 8(g+3)$, with the bound attained infinitely often. One can try to build on this for other g, and can also ask for the frequency of this occurence.

One can begin to refine Hurwitz's Theorem by asking for bounds of orders of groups of automorphisms which in addition satisfy various algebraic constraints, such as being nilpotent, being solvable, being a p-group, etc. There already exist a number of theorems of this sort. For example, Zomorrodian [Zo] proved that if $\mathrm{Aut}(X_g)$ is nilpotent then it has order at most $16(g-1)$, and if this bound is attained then $g-1$ must be a power of 2. One can also ask for lower bounds in this context. As these kinds of bounds are typically attained and not attained for infinitely many g, one then wants to solve the following.

Problem 2.16 (Automorphism groups with special properties). *Let P be a property of finite groups, for example being nilpotent, solvable, or a p-group. Prove a version of Larsen's theorem which counts those g for which the upper bound of $|\mathrm{Aut}(X_g)|$ is realized for some X_g with $Aut(X_g)$ having P. Similarly for lower bounds. Determine the least g for which each given bound is realized.*

Many of the surfaces realizing the extremal bounds in all of the above questions are *arithmetic*, that is they are quotients of $\mathbf{H}^2$ by an arithmetic lattice. Such lattices are well-known to have special properties, in particular they have a lot of symmetry. On the other hand arithmetic surfaces are not typical. Thus to understand the "typical" surface with symmetry, the natural problem is the following.

Problem 2.17 (Nonarithmetic extremal surfaces). *Give answers to all of the above problems on automorphisms of Riemann surfaces for the collection of* non-arithmetic *surfaces. For example find bounds on orders of automorphism groups which are nilpotent, solvable, p-groups, etc. Prove that these bounds are sharp for infinitely many g. Determine the frequency of those g for which such bounds are sharp. Determine the least genus for which the bounds are sharp.*

The model result for these kind of problems is the following.

Theorem 2.18 (Belolipetsky [Be]). *Let X_g be any non-arithmetic Riemann surface of genus $g \geq 2$. Then*

$$|\mathrm{Aut}(X_g)| \leq \frac{156}{7}(g-1)$$

Further, this bound is sharp for infinitely many g; the least such is $g = 50$.

The key idea in the proof of Theorem 2.18 is the following. One considers the quotient orbifold $X_g/\operatorname{Aut}(X_g)$, and computes via the Riemann-Hurwitz formula that it is the quotient of $\mathbf{H}^2$ by a triangle group. Lower bounds on the area of this orbifold give upper bounds on $|\operatorname{Aut}(X_g)|$; for example the universal lower bound of $\pi/21$ for the area of every 2-dimensional hyperbolic orbifold gives the classical Hurwitz $84(g-1)$ theorem. Now Takeuchi classified all arithmetic triangle groups; they are given by a finite list. One can then use this list to refine the usual calculations with the Riemann-Hurwitz formula to give results such as Theorem 2.18; see [Be]. This idea should also be applicable to Problem 2.17.

I would like to mention a second instance of the theme of playing off algebraic properties of automorphism groups versus the numerology of their orders. One can prove that the maximal possible order of an automorphism of a Riemann surface X_g is $4g+2$. This bound is easily seen to be achieved for each $g \geq 2$ by considering the rotation of the appropriately regular hyperbolic $(4g+2)$-gon. Kulkarni [Ku] proved that there is a *unique* Riemann surface W_g admitting such an automorphism. Further, he proved that $\operatorname{Aut}(W_g)$ is cyclic, and he gave the equation describing W_g as an algebraic curve.

Problem 2.19 (Canonical basepoints for $\mathcal{M}_g$)**.** *Find other properties of automorphisms or automorphism groups that determine a unique point of $\mathcal{M}_g$. For example, is there a unique Riemann surface of genus $g \geq 2$ whose automorphism group is nilpotent, and is the largest possible order among nilpotent automorphism groups of genus g surfaces?*

A *Hurwitz surface* is a hyperbolic surface attaining the bound $84(g-1)$. As the quotient of such a surface by its automorphism group is the $(2,3,7)$ orbifold, which has a unique hyperbolic metric, it follows that for each $g \geq 2$ there are finitely many Hurwitz surfaces. As these are the surfaces of maximal symmetry, it is natural to ask precisely how many there are.

Question 2.20 (Number of Hurwitz surfaces)**.** *Give a formula for the number of Hurwitz surfaces of genus g. What is the frequency of those g for which there is a unique Hurwitz surface?*

3 Combinatorial and geometric group theory of Mod_g

Ever since Dehn, Mod_g has been a central example in combinatorial and geometric group theory. One reason for this is that Mod_g lies at a gateway: on one side are matrix groups and groups naturally equipped with a geometric structure (e.g. hyperbolic geometry); on the other side are groups given purely combinatorially. In this section we pose some problems in this direction.

3.1 Decision problems and almost convexity

Word and conjugacy problems. Recall that the *word problem* for a finitely presented group Γ asks for an algorithm which takes as input any word w in a fixed generating set for Γ, and as output tells whether or not w is trivial. The *conjugacy problem* for Γ asks for an algorithm which takes as input two words, and as output tells whether or not these words represent conjugate elements of Γ.

There is some history to the word and conjugacy problems for Mod_g, beginning with braid groups. These problems have topological applications; for example the conjugacy problem for Mod_g is one of the two main ingredients one needs to solve the homeomorphism problem for 3-manifolds fibering over the circle[3]. Lee Mosher proved in [Mos] that Mod_g is automatic. From this it follows that there is an $O(n^2)$-time algorithm to solve the word problem for Mod_g; indeed there is an $O(n^2)$-time algorithm which puts each word in a fixed generating set into a unique normal form. However, the following is still open.

Question 3.1 (Fast word problem)**.** *Is there a sub-quadratic time algorithm to solve the word problem in* Mod_g*?*

One might guess that $n \log n$ is possible here, as there is such an algorithm for certain relatively (strongly) hyperbolic groups (see [Fa3]), and mapping class groups are at least weakly hyperbolic, as proven by Masur-Minsky (Theorem 1.3 of [MM1]).

The conjugacy problem for Mod_g is harder. The original algorithm of Hemion [He] seems to give no reasonable (even exponential) time bound. One refinement of the problem would be to prove that Mod_g is *biautomatic*. However, even a biautomatic structure gives only an exponential time algorithm to solve the conjugacy problem. Another approach to solving the conjugacy problem is the following.

Problem 3.2 (Conjugator length bounds)**.** *Prove that there exist constants C, K, depending only on S, so that if $u, v \in \mathrm{Mod}_g$ are conjugate, then there exists $g \in \mathrm{Mod}_g$ with $||g|| \leq K \max\{||u||, ||v||\} + C$ so that $u = gvg^{-1}$.*

Masur-Minsky ([MM2], Theorem 7.2) solved this problem in the case where u and v are pseudo-Anosov; their method of hierarchies seems quite applicable to solving Problem 3.2 in the general case. While interesting in its own right, even the solution to Problem 3.2 would not answer the following basic problem.

Problem 3.3 (Fast conjugacy problem)**.** *Find a polynomial time algorithm to solve the conjugacy problem in* Mod_g*. Is there a quadratic time algorithm, as for the word problem?*

As explained in the example on page 4, a solution to Problem 3.3 would be a major step in finding a polynomial time algorithm to solve the homeomorphism problem for 3-manifolds that fiber over the circle.

Almost convexity. In [Ca] Cannon initiated the beautiful theory of almost convex groups. A group Γ with generating set S is *almost convex* if there exists $C > 0$ so that for each $r > 0$, and

[3] The second ingredient, crucial but ignored by some authors, is the computability of the Thurston norm.

for any two points $x, y \in \Gamma$ on the sphere of radius r in Γ with $d(x, y) = 2$, there exists a path γ of length at most C connecting x to y and lying completely inside the ball of radius r in Γ. There is an obvious generalization of this concept from groups to spaces. One strong consequence of the almost convexity of Γ is that for such groups one can recursively build the Cayley graph of Γ near any point x in the n-sphere of Γ by only doing a local computation involving elements of Γ lying (universally) close to x; see [Ca]. In particular one can build each n-ball, $n \geq 0$, and so solve the word problem in an efficient way.

Question 3.4 (Almost convexity)**.** *Does there exist a finite generating set for* Mod_g *for which it is almost convex?*

One would also like to know the answer to this question for various subgroups of Mod_g. Here is a related, but different, basic question about the geometry of Teichmüller space.

Question 3.5. *Is* $\mathrm{Teich}(\Sigma_g)$*, endowed with the Teichmüller metric, almost convex?*

Note that Cannon proves in [Ca] that fundamental groups of closed, negatively curved manifolds are almost convex with respect to any generating set. He conjectures that this should generalize both to the finite volume case and to the nonpositively curved case.

3.2 The generalized word problem and distortion

In this subsection we pose some problems relating to the ways in which subgroups embed in Mod_g.

Generalized word problem. Let Γ be a finitely presented group with finite generating set S, and let H be a finitely generated subgroup of Γ. The *generalized word problem*, or GWP, for H in Γ, asks for an algorithm which takes as input an element of the free group on S, and as output tells whether or not this element represents an element of H. When H is the trivial subgroup then this is simply the word problem for Γ. The group Γ is said to have *solvable generalized word problem* if the GWP is solvable for every finitely generated subgroup H in Γ.

The generalized word problem, also called the *membership, occurrence* or *Magnus problem*, was formulated by K. Mihailova [Mi] in 1958, but special cases had already been studied by Nielsen [Ni] and Magnus [Ma]. Mihailova [Mi] found a finitely generated subgroup of a product $F_m \times F_m$ of free groups which has unsolvable generalized word problem.

Now when $g \geq 2$, the group Mod_g contains a product of free groups $F_m \times F_m$ (for any $m > 0$). Further, one can find copies of $F_m \times F_m$ such that the generalized word problem for these subgroups is solvable inside Mod_g. To give one concrete example, simply divide Σ_g into two subsurfaces S_1 and S_2 which intersect in a (possibly empty) collection of curves. Then take, for each $i = 1, 2$, a pair f_i, g_i of independent pseudo-Anosov homeomorphisms. After perhaps taking powers of f_i and g_i if necessary, the group generated by $\{f_1, g_1, f_2, g_2\}$ will be the group we require; one can pass to finite index subgroups if one wants $m > 2$. The point here is that such subgroups are not distorted (see below), as they are convex cocompact in the subgroups $\mathrm{Mod}(S_i)$ of Mod_g, and these in turn are not distorted in Mod_g. It follows that the generalized word problem for Mihailova's subgroup $G < F_m \times F_m$ is not solvable in Mod_g.

Problem 3.6 (Generalized word problem)**.** *Determine the subgroups H in Mod_g for which the generalized word problem is solvable. Give efficient algorithms to solve the generalized word problem for these subgroups. Find the optimal time bounds for such algorithms.*

Of course this problem is too broad to solve in complete generality; results even in special cases might be interesting. Some solutions to Problem 3.6 are given by Leininger-McReynolds in [LM]. We also note that Bridson-Miller (personal communication), extending an old result of Baumslag-Roseblade, have proven that any product of finite rank free groups has solvable generalized word problem with respect to any *finitely presented* subgroup. In light of this result, it would be interesting to determine whether or not there is a finitely presented subgroup of Mod_g with respect to which the generalized word problem is not solvable.

Distortion and quasiconvexity. There is a refinement of Problem 3.6. Let H be a finitely generated subgroup of a finitely generated group Γ. Fix finite generating sets on both H and Γ. This choice gives a *word metric* on both H and Γ, where $d_\Gamma(g,h)$ is defined to be the minimal number of generators of Γ needed to represent gh^{-1}. Let $\mathbf{N}$ denote the natural numbers. We say that a function $f : \mathbf{N} \longrightarrow \mathbf{N}$ is a *distortion function* for H in Γ if for every word w in the generators of Γ, if w represents an element $\overline{w} \in H$ then

$$d_H(1,\overline{w}) \leq f(d_\Gamma(1,\overline{w}))$$

In this case we also say that "H has distortion $f(n)$ in Γ." It is easy to see that the growth type of f, i.e. polynomial, exponential, etc., does not depend on the choice of generators for either H or Γ. It is also easy to see that $f(n)$ is constant if and only if H is finite; otherwise f is at least linear. It is proved in [Fa1] that, for a group Γ with solvable word problem, the distortion of H in Γ is recursive if and only if H has solvable generalized word problem in Γ. For some concrete examples, we note that the center of the 3-dimensional integral Heisenberg group has quadratic distortion; and the cyclic group generated by b in the group $\langle a, b : aba^{-1} = b^2\rangle$ has exponential distortion since $a^n b a^{-n} = b^{2^n}$.

Problem 3.7 (Distortion)**.** *Find the possible distortions of subgroups in Mod_g. In particular, compute the distortions of $\mathcal{I}_g$. Determine the asymptotics of the distortion of $\mathcal{I}_g(k)$ as $k \to \infty$. Is there a subgroup $H < \mathrm{Mod}_g$ that has precisely polynomial distortion of degree $d > 1$?*

There are some known results on distortion of subgroups in Mod_g. Convex cocompact subgroups (in the sense of [FMo]) have linear distortion in Mod_g; there are many such examples where H is a free group. Abelian subgroups of Mod_g have linear distortion (see [FLMi]), as do subgroups corresponding to mapping class groups of subsurfaces (see [MM2, Ham]).

I would guess that $\mathcal{I}_g$ has exponential distortion in Mod_g. A first step to the question of how the distortion of $\mathcal{I}_g(k)$ in Mod_g behaves as $k \to \infty$ would be to determine the distortion of $\mathcal{I}_g(k+1)$ in $\mathcal{I}_g(k)$. The "higher Johnson homomorphisms" (see, e.g., [Mo3]) might be useful here.

A stronger notion than linear distortion is that of quasiconvexity. Let $S = S^{-1}$ be a fixed generating set for a group Γ, and let $\pi : S^* \to \Gamma$ be the natural surjective homomorphism from the free monoid on S to Γ sending a word to the group element it represents. Let $\sigma : \Gamma \to S^*$ be a (perhaps multi-valued) section of π; that is, σ is just a choice of paths in Γ from the origin to each $g \in \Gamma$. We say that a subgroup $H < \Gamma$ is *quasiconvex* (with respect to σ) if there exists $K > 0$ so that for each $h \in H$, each path $\sigma(h)$ lies in the K-neighborhood of H in Γ. Quasiconvexity is a well-known and basic notion in geometric group theory. It is easy to see that if H is quasiconvex with respect to some collection of quasigeodesics then H has linear distortion in Mod_g (see [Fa1]).

Problem 3.8. *Determine which subgroups of* Mod_g *are quasiconvex with respect to some collection of geodesics.*

This question is closely related to, but different than, the question of *convex cocompactness* of subgroups of Mod_g, as defined in [FMo], since the embedding of Mod_g in Teich_g via any orbit is exponentially distorted, by [FLMi].

3.3 Decision problems for subgroups

As a collection of groups, how rich and varied can the set of subgroups of Mod_g be? One instance of this general question is the following.

Question 3.9. *Does every finitely presented subgroup* $H < \text{Mod}_g$ *have solvable conjugacy problem? is it combable? automatic?*

Note that every finitely-generated subgroup of a group with solvable word problem has solvable word problem. The same is not true for the conjugacy problem: there are subgroups of $\text{GL}(n, \mathbf{Z})$ with unsolvable conjugacy problem; see [Mi].

It is not hard to see that Mod_g, like $\text{GL}(n, \mathbf{Z})$, has finitely many conjugacy classes of finite subgroups. However, we pose the following.

Problem 3.10. *Find a finitely presented subgroup* $H < \text{Mod}_g$ *for which there are infinitely many conjugacy classes of finite subgroups in* H*.*

The motivation for this problem comes from a corresponding example, due to Bridson [Br], of such an H in $\text{GL}(n, \mathbf{Z})$. One might to solve Problem 3.10 by extending Bridson's construction to $\text{Sp}(2g, \mathbf{Z})$, pulling back such an H, and also noting that the natural map $\text{Mod}_g \to \text{Sp}(2g, \mathbf{Z})$ is injective on torsion.

Another determination of the variety of subgroups of Mod_g is the following. Recall that the *isomorphism problem* for a collection $\mathcal{S}$ of finitely presented groups asks for an algorithm which takes as input two presentations for two elements of $\mathcal{S}$ and as output tells whether or not those groups are isomorphic.

Question 3.11 (Isomorphism problem for subgroups)**.** *Is the isomorphism problem for the collection of finitely presented subgroups of* Mod_g *solvable?*

Note that the isomorphism problem is not solvable for the collection of all finitely generated linear groups, nor is it even solvable for the collection of finitely generated subgroups of $\mathrm{GL}(n, \mathbf{Z})$.

There are many other algorithmic questions one can ask; we mention just one more.

Question 3.12. *Is there an algorithm to decide whether or not a given subgroup* $H < \mathrm{Mod}_g$ *is freely indecomposable? Whether or not* H *splits over* $\mathbf{Z}$*?*

3.4 Growth and counting questions

Recall that the *growth series* of a group Γ with respect to a finite generating set S is defined to be the power series

$$f(z) = \sum_{i=0}^{\infty} c_i z^i \tag{8}$$

where c_i denotes the cardinality $\#B_\Gamma(i)$ of the ball of radius i in Γ with respect to the word metric induced by S. We say that Γ has *rational growth* (with respect to S) if f is a rational function, that is the quotient of two polynomials. This is equivalent to the existence of a linear recurrence relation among the c_i; that is, there exist $m > 0$ real numbers $a_1, \ldots, a_m \geq 0$ so that for each r:

$$c_r = a_1 c_{r-1} + \cdots + a_m c_{r-m}$$

Many groups have rational growth with respect to various (sometimes every) generating sets. Examples include word-hyperbolic groups, abelian groups, and Coxeter groups. See, e.g., [Harp2] for an introduction to the theory of growth series.

In [Mos], Mosher constructed an automatic structure for Mod_g. This result suggests that the following might have a positive answer.

Question 3.13 (Rational growth)**.** *Does* Mod_g *have rational growth function with respect to some set of generators? with respect to every set of generators?*

Of course one can also ask the same question for any finitely generated subgroup of Mod_g, for example $\mathcal{I}_g$. Note that the existence of an automatic structure is not known to imply rationality of growth (even for one generating set); one needs in addition the property that the automatic structure consist of geodesics. Unfortunately Mosher's automatic structure does not satisfy this stronger condition.

It is natural to ask for other recursive patterns in the Cayley graph of Mod_g. To be more precise, let P denote a property that elements of Mod_g might or might not have. For example, P might be the property of being finite order, of lying in a fixed subgroup $H < \mathrm{Mod}_g$, or of being pseudo-Anosov. Now let

$$c_P(r) = \#\{B_{\mathrm{Mod}_g}(r) \cap \{x \in \mathrm{Mod}_g : x \text{ has P}\}\}$$

We now define the *growth series for the property* P, with respect to a fixed generating set for Γ, to be the power series

$$f_P(z) = \sum_{i=0}^{\infty} c_P(i) z^i$$

Question 3.14 (Rational growth for properties)**.** *For which properties* P *is the function* f_P *is rational?*

Densities. For any subset $S \subset \mathrm{Mod}_g$, it is natural to ask how common elements of S are in Mod_g. There are various ways to interpret this question, and the answer likely depends in a strong way on the choice of interpretation[4]. One way to formalize this is via the *density* $d(S)$ of S in Mod_g, where

$$d(S) = \lim_{r \to \infty} \frac{\#[B(r) \cap S]}{\#B(r)} \tag{9}$$

where $B(r)$ is the number of elements of Mod_g in the ball of radius r, with respect to a fixed set of generators. While for subgroups $H < \mathrm{Mod}_g$ the number $d(H)$ itself may depend on the choice of generating sets for H and Mod_g, it is not hard to see that the (non)positivity of $d(H)$ does not depend on the choices of generating sets.

As the denominator and (typically) the numerator in (9) are exponential, one expects that $d(S) = 0$ for most S. Thus is it natural to replace both the numerator and denominator of (9) with their logarithms; we denote the corresponding limit as in (9) by $d_{\log}(S)$, and we call this the *logarithmic density* of S in Mod_g.

The following is one interpretation of a folklore conjecture.

Conjecture 3.15 (Density of pseudo-Anosovs)**.** *Let* $\mathcal{P}$ *denote the set of pseudo-Anosov elements of* Mod_g*. Then* $d(\mathcal{P}) = 1$.

J. Maher [Mah] has recently proven that a random walk on Mod_g lands on a pseudo-Anosov element with probability tending to one as the length of the walk tends to infinity. I. Rivin [Ri] has proven that a random (in a certain specific sense) element of Mod_g is pseudo-Anosov by proving a corresponding result for $\mathrm{Sp}(2g, \mathbf{Z})$. While the methods in [Mah] and [Ri] may be relevant, Conjecture 3.15 does not seem to follow directly from these results. As another test we pose the following.

Conjecture 3.16. $d(\mathcal{I}_g) = 0$.

[4] For a wonderful discussion of this kind of issue, see Barry Mazur's article [Maz].

Even better would be to determine $d_{\log}(\mathcal{I}_g)$. Conjecture 3.16 would imply that $d(\mathcal{I}_g(m)) = 0$ for each $m \geq 2$. It is not hard to see that $\mathcal{I}_g(m)$ has exponential growth for each $g \geq 2, m \geq 1$, and one wants to understand how the various exponential growth rates compare to each other. In other words, one wants to know how common an occurence it is, as a function of k, for an element of Mod_g to act trivially on the first k terms of the lower central series of $\pi_1\Sigma_g$.

Problem 3.17 (Logarithmic densities of the Johnson filtration)**.** *Determine the asymptotics of* $d_{\log}(\mathcal{I}_g(m))$ *both as* $g \to \infty$ *and as* $m \to \infty$.

Indeed, as far as I know, even the asymptotics of the (logarithmic) density of the kth term of the lower central series of $\pi_1\Sigma_g$ in $\pi_1\Sigma_g$ as $k \to \infty$ has not been determined.

Entropy. The *exponential growth rate* of a group Γ with respect to a finite generating set S is defined as

$$w(\Gamma, S) := \lim_{r\to\infty} (B_r(\Gamma, S))^{1/r}$$

where $B_r(\Gamma, S)$ denotes the cardinality of the r-ball in the Cayley graph of Γ with respect to the generating set S; the limit exists since β is submultiplicative. The *entropy* of Γ is defined to be

$$\mathrm{ent}(\Gamma) = \inf\{\log w(\Gamma, S) : S \text{ is finite and generates } \Gamma\}$$

Among other things, the group-theoretic entropy of $\mathrm{ent}(\pi_1 M)$ of a closed, Riemannian manifold M gives a lower bound for (the product of diameter times) both the volume growth entropy of M and the topological entropy of the geodesic flow on M. See [Harp1] for a survey.

Eskin-Mozes-Oh [EMO] proved that nonsolvable, finitely-generated linear groups Γ have positive entropy. Since it is classical that the action of Mod_g on $H_1(\Sigma_g, \mathbf{Z})$ gives a surjection $\mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbf{Z})$, it follows immediately that $\mathrm{ent}(\mathrm{Mod}_g) > 0$. This method of proving positivity of entropy fails for the Torelli group $\mathcal{I}_g$ since it is in the kernel of the standard symplectic representation of Mod_g. However, one can consider the action of $\mathcal{I}_g$ on the homology of the universal abelian cover of Σ_g, considered as a (finitely generated) module over the corresponding covering group, to find a linear representation of $\mathcal{I}_g$ which is not virtually solvable. This is basically the *Magnus representation.* Again by Eskin-Mozes-Oh we conclude that $\mathrm{ent}(\mathcal{I}_g) > 0$.

Problem 3.18. *Give explicit upper and lower bounds for* $\mathrm{ent}(\mathrm{Mod}_g)$*. Compute the asymptotics of* $\mathrm{ent}(\mathrm{Mod}_g)$ *and of* $\mathrm{ent}(\mathcal{I}_g)$ *as* $g \to \infty$*. Similarly for* $\mathrm{ent}(\mathcal{I}_g(k))$ *as* $k \to \infty$.

4 Problems on the geometry of $\mathcal{M}_g$

It is a basic question to understand properties of complex analytic and geometric structures on $\mathcal{M}_g$, and how these structures constrain, and are constrained by, the global topology of $\mathcal{M}_g$. Such structures arise frequently in applications. For example one first tries to put a geometric structure on $\mathcal{M}_g$, such as that of a complex orbifold or of a negatively curved Riemannian manifold. Once this is done, general theory for such structures (e.g. Schwarz Lemmas or fixed point theorems) can then be applied to prove hard theorems. Arakelov-Parshin Rigidity and Nielsen Realization are two examples of this. In this section we pose a few problems about the topology and geometry of $\mathcal{M}_g$.

4.1 Isometries

Royden's Theorem [Ro] states that when $g \geq 3$, every isometry of Teichmüller space Teich_g, endowed with the Teichmüller metric d_{Teich}, is induced by an element of Mod_g; that is

$$\mathrm{Isom}(\mathrm{Teich}_g, d_{\mathrm{Teich}}) = \mathrm{Mod}_g$$

Note that d_{Teich} comes from a non-Riemannian Finsler metric, namely a norm on the cotangent space at each point $X \in \mathrm{Teich}_g$. This cotangent space can be identified with the space $Q(X)$ of holomorphic quadratic differentials on X.

I believe Royden's theorem can be generalized from the Teichmüller metric to all metrics.

Conjecture 4.1 (Inhomogeneity of all metrics)**.** *Let* Teich_g *denote the Teichmüller space of closed, genus $g \geq 2$ Riemann surfaces. Let h be any Riemannian metric (or any Finsler metric with some weak regularity conditions) on* Teich_g *which is invariant under the action of the mapping class group* Mod_g*, and for which this action has finite covolume. Then* $\mathrm{Isom}(\mathrm{Teich}_g, h)$ *is discrete; even better, it contains* Mod_g *as a subgroup of index $C = C(g)$.*

Royden's Theorem is the special case when h is the Teichmüller metric (Royden gets $C = 2$ here). A key philosophical implication of Conjecture 4.1 is that the mechanism behind the inhomogeneity of Teichmüller space is due not to fine regularity properties of the unit ball in $Q(X)$ (as Royden's proof suggests), but to the global topology of moduli space. This in turn is tightly controlled by the structure of Mod_g. As one piece of evidence for Conjecture 4.1, I would like to point out that it would follow if one could extend the main theorem of [FW1] from the closed to the finite volume case.

In some sense looking at Mod_g-invariant metrics seems too strong, especially since Mod_g has torsion. Perhaps, for example, the inhomogeneity of Teich_g is simply caused by the constraints of the torsion in Mod_g. Sufficiently large index subgroups of Mod_g are torsion free. Thus one really wants to strengthen Conjecture 4.1 by replacing Mod_g by any finite index subgroup H, and by replacing the constant $C = C(g)$ by a constant $C = C(g, [\mathrm{Mod}_g : H])$. After this one can explore metrics invariant by much smaller subgroups, such as $\mathcal{I}_g$, and at least hope for discreteness of the corresponding isometry group (as long as the subgroup is sufficiently large).

If one can prove the part of Conjecture 4.1 which gives discreteness of the isometry group of any Mod_g-invariant metric on Teich_g, one can approach the stronger statement that $C_g = 1$ or $C_g = 2$ as follows. Take the quotient of Teich_g by any group Λ of isometries properly containing Mod_g.

By discreteness of Λ, the quotient Teich_g/Λ is a smooth orbifold which is finitely orbifold-covered by $\mathcal{M}_g$.

Conjecture 4.2 ($\mathcal{M}_g$ is maximal)**.** *For $g \geq 3$ the smooth orbifold $\mathcal{M}_g$ does not finitely orbifold-cover any other smooth orbifold.*

A much stronger statement, which may be true, would be to prove that if N is any finite cover of $\mathcal{M}_g$, then the only orbifolds which N can orbifold cover are just the covers of $\mathcal{M}_g$. Here is a related basic topology question about $\mathcal{M}_g$.

Question 4.3. *Let Y be any finite cover of $\mathcal{M}_g$, and let $f : Y \to Y$ be a finite order homeomorphism. If f is homotopic to the identity, must f equal the identity?*

4.2 Curvature and Q-rank

Nonpositive sectional curvature. There has been a long history of studying metrics and curvature on $\mathcal{M}_g$ [5]; see, e.g., [BrF, LSY, Mc2]. A recurring theme is to find aspects of negative curvature in $\mathcal{M}_g$. While $\mathcal{M}_g$ admits no metrics of negative curvature, even in a coarse sense, if $g \geq 2$ (see [BrF]), the following remains a basic open problem.

Conjecture 4.4 (Nonpositive curvature)**.** *For $g \geq 2$ the orbifold $\mathcal{M}_g$ admits no complete, finite volume Riemannian metric with nonpositive sectional curvatures uniformly bounded away from $-\infty$.*

One might be more ambitious in stating Conjecture 4.4, by weakening the finite volume condition, by dropping the uniformity of the curvature bound, or by extending the statement from $\mathcal{M}_g$ to any finite cover of $\mathcal{M}_g$ (or perhaps even to certain infinite covers). It would also be interesting to extend Conjecture 4.4 beyond the Riemannian realm to that of CAT(0) metrics; see, e.g., [BrF] for a notion of finite volume which extends to this context.

In the end, it seems that we will have to make do with various relative notions of nonpositive or negative curvature, as in [MM1, MM2], or with various weaker notions of nonpositive curvature, such as holomorphic, Ricci, or highly singular versions (see, e.g.,[LSY]), or isoperimetric type versions such a Kobayashi or Kahler hyperbolicity (see [Mc2]). Part of the difficulty with trying to fit $\mathcal{M}_g$ into the "standard models" seems to be the topological structure of the cusp of $\mathcal{M}_g$.

Scalar curvature and Q-rank. Let S be a genus g surface with n punctures, and let $\mathcal{M}(S)$ denote the corresponding moduli space. We set

$$d(S) = 3g - 3 + n$$

The constant $d(S)$ is fundamental in Teichmüller theory: it is the complex dimension of the Teichmüller space $\mathrm{Teich}(S)$; it is also the number of curves in any pair-of-pants decomposition

[5] As $\mathcal{M}_g$ is an orbifold, technically one studies Mod_g-invariant metrics on the Teichmüller space Teich_g.

of S. While previous results have concentrated on sectional and holomorphic curvatures, Shmuel Weinberger and I have recently proven the following; see [FW2].

Theorem 4.5 (Positive scalar curvature)**.** *Let M be any finite cover of $\mathcal{M}(S)$. Then M admits a complete, finite-volume Riemannian metric of (uniformly bounded) positive scalar curvature if and only if $d(S) \geq 3$.*

The analogous statement was proven for locally symmetric arithmetic manifolds $\Gamma\backslash G/K$ by Block-Weinberger [BW], where $d(S)$ is replaced by the $\mathbf{Q}$-rank of Γ. When $d(S) \geq 3$, the metric on M has the quasi-isometry type of a ray, so that it is not quasi-isometric to the Teichmüller metric on $\mathcal{M}_g$. It seems likely that this is not an accident.

Conjecture 4.6. *Let S be any surface with $d(S) \geq 1$. Then M does not admit a finite volume Riemannian metric of (uniformly bounded) positive scalar curvature in the quasi-isometry class of the Teichmüller metric.*

The analogue of Conjecture 4.6 in the $\Gamma\backslash G/K$ case was proven by S. Chang in [Ch]. The same method of proof as in [Ch] should reduce Conjecture 4.6 to the following discussion, which seems to be of independent interest, and which came out of discussions with H. Masur.

What does $\mathcal{M}_g$, endowed with the Teichmüller metric d_{Teich}, look like from far away? This can be formalized by Gromov's notion of *tangent cone at infinity*:

$$\mathrm{Cone}(\mathcal{M}_g) := \lim_{n\to\infty} (\mathcal{M}_{\mathrm{g}}, \frac{1}{n} d_{\mathrm{Teich}}) \tag{10}$$

where the limit is taken in the sense of Gromov-Hausdorff convergence of pointed metric spaces; here we have fixed a basepoint in $\mathcal{M}_g$ once and for all. This limit is easily shown to make sense and exist in our context. To state our conjectural answer as to what $\mathrm{Cone}(\mathcal{M}_g)$ looks like, we will need the complex of curves on Σ_g. Recall that the *complex of curves* $\mathcal{C}_g$ for $g \geq 2$ is the simplicial complex with one vertex for each nontrivial, nonperipheral isotopy class of simple closed curves on Σ_g, and with a k-simplex for every $(k+1)$-tuple of such isotopy classes for which there are mutually disjoint representatives. Note that Mod_g acts by simplicial automorphisms on $\mathcal{C}_g$.

Conjecture 4.7 ($\mathbf{Q}$-rank of moduli space)**.** $\mathrm{Cone}(\mathcal{M}_g)$ *is homeomorphic to the (open) cone on the quotient $\mathcal{C}_g/\mathrm{Mod}_g$* [6]*.*

One can pose a stronger version of Conjecture 4.7 that predicts the precise bilipschitz type of the natural metric on $\mathrm{Cone}(\mathcal{M}_g)$; an analogous statement for quotients $\Gamma\backslash G/K$ of symmetric spaces by lattices was proven by Hattori [Hat]. H. Masur and I have identified the right candidate for a coarse fundamental domain needed to prove Conjecture 4.7; its description involves certain length inequalities analogous to those on roots defining Weyl chambers. Further, the (conjectured) dimensions of the corresponding tangent cones are $\mathbf{Q}$-rank(Γ) and $d(S)$, respectively. Thus we propose the following additions to the list of analogies between arithmetic lattices and Mod_g.

[6]Note: This statement is a slight cheat; the actual version requires the language of orbi-complexes.

arithmetic lattices	Mod_g
$\mathbf{Q}$-rank(Γ)	$d(S)$
root lattice	{simple closed curves}
simple roots	top. types of simple closed curves
$\mathrm{Cone}(\Gamma\backslash G/K)$	$\mathrm{Cone}(\mathcal{M}_{\mathrm{g}})$

As alluded to above, Conjecture 4.7 should imply, together with the methods in [Ch], the second statement of Conjecture 4.5.

4.3 The Kahler group problem

Complete Kahler metrics on $\mathcal{M}_g$ with finite volume and bounded curvatures have been found by Cheng-Yau, McMullen and others (see, e.g., [LSY] for a survey). The following conjecture, however, is still not known. I believe this is a folklore conjecture.

Conjecture 4.8 (Mod$_g$ is Kahler). *For $g \geq 3$, the group* Mod_g *is a* Kahler group, *i.e. it is isomorphic to the fundamental group of a compact Kahler manifold.*

It was shown in [Ve] that Mod_2 is not a Kahler group. This is proven by reducing (via finite extensions) to the pure braid group case; these groups are not Kahler since they are iterated extensions of free groups by free groups.

A natural place to begin proving Conjecture 4.8 is the same strategy that Toledo uses in [To] for nonuniform lattices in $\mathrm{SU}(n,1), n \geq 3$. The main point is the following. One starts with a smooth open variety V and wants to prove that $\pi_1 V$ is a Kahler group. The first step is to find a compactification $\overline{V}$ of V which is projective, and for which $\overline{V} - V$ has codimension *at least* 3. This assumption guarantees that the intersection of the generic 2-plane P in projective space with $\overline{V}$ misses V. The (weak) Lefschetz Theorem then implies that the inclusion $i : \overline{V} \cap P \hookrightarrow V$ induces an isomorphism on fundamental groups, thus giving that $\pi_1 V$ is a Kahler group.

One wants to apply this idea to the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of moduli space $\mathcal{M}_g$. This almost works, except that there is a (complex) codimension *one* singular stratum of $\overline{\mathcal{M}_g}$, so that the above does not apply. Other compactifications of $\mathcal{M}_g$ are also problematic in this regard.

What about the Torelli group $\mathcal{I}_g$? This group, at least for $g \geq 6$, is not known to violate any of the known constraints on Kahler groups. Most notably, Hain [Ha3] proved the deep result that for $g \geq 6$ the group $\mathcal{I}_g$ has a quadratically presented Malcev Lie algebra; this is one of the more subtle properties posessed by Kahler groups. Note Akita's theorem (Theorem 5.13 above) that the classifying space of $\mathcal{I}_g, g \geq 7$ does not have the homotopy type of a finite complex shows that these $\mathcal{I}_g$ are not fundamental groups of closed *aspherical* manifolds. There are of course Kahler groups (e.g. finite Kahler groups) with this property. In contrast to Conjecture 4.8, I have recently proven [Fa3] the following.

Theorem 4.9. *$\mathcal{I}_g$ is not a Kahler group for any $g \geq 2$.*

Denoting the symplectic representation by $\pi : \mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbf{Z})$, the next question is to ask which of the groups $\pi^{-1}(\mathrm{Sp}(2k, \mathbf{Z}))$ for $1 \leq k \leq 2g$ interpolating between the two extremes $\mathcal{I}_g$ and Mod_g are Kahler groups. My only guess is that when $k = 1$ the group is not Kahler.

4.4 The period mapping

To every Riemann surface $X \in \mathrm{Teich}_g$ one attaches its *Jacobian* $\mathrm{Jac}(X)$, which is the quotient of the dual $(\Omega^1(X))^* \approx \mathbf{C}^g$ of the space of holomorphic 1-forms on X by the lattice $H_1(X, \mathbf{Z})$, where $\gamma \in H_1(X, \mathbf{Z})$ is the linear functional on $\Omega^1(X)$ given by $\omega \mapsto \int_\gamma \omega$. Now $\mathrm{Jac}(X)$ is a complex torus, and Riemann's period relations show that $\mathrm{Jac}(X)$ is also an algebraic variety, i.e. $\mathrm{Jac}(X)$ is an *abelian variety.* The algebraic intersection number on $H_1(X, \mathbf{Z})$ induces a symplectic form on $\mathrm{Jac}(X)$, which can be thought of as the imaginary part of a positive definite Hermitian form on $\mathbf{C}^g$. This extra bit of structure is called a *principal polarization* on the abelian variety $\mathrm{Jac}(X)$.

The space $\mathcal{A}_g$ of all g-dimensional (over $\mathbf{C}$) principally polarized abelian varieties is parameterized by the locally symmetric orbifold $\mathrm{Sp}(2g, \mathbf{Z}) \backslash \mathrm{Sp}(2g, \mathbf{R})/\mathrm{U}(g)$. The *Schottky problem*, one of the central classical problems of algebraic geometry, asks for the image of the *period mapping*

$$\Psi : \mathcal{M}_g \to \mathcal{A}_g$$

which sends a surface X to its Jacobian $\mathrm{Jac}(X)$. In other words, the Schottky problem asks: which principally polarized abelian varieties occur as Jacobians of some Riemann surface? Torelli's Theorem states that Ψ is injective; the image $\Psi(\mathcal{M}_g)$ is called the *period locus.* The literature on this problem is vast (see, e.g., [D] for a survey), and goes well beyond the scope of the present paper.

Inspired by the beautiful paper [BS] of Buser-Sarnak, I would like to pose some questions about the period locus from a different (and probably nonstandard) point of view. Instead of looking for precise algebraic equations describing $\Psi(\mathcal{M}_g)$, what if we instead try to figure out how to tell whether or not it contains a given torus, or if we try to describe what the period locus *roughly* looks like? Let's make these questions more precise.

The data determining a principally polarized abelian variety is not combinatorial, but is a matrix of real numbers. However, one can still ask for algorithms involving such data by using the *complexity theory over* $\mathbf{R}$ developed by Blum-Shub-Smale; see, e.g., [BCSS]. Unlike classical complexity theory, here one assumes that real numbers can be given and computed precisely, and develops algorithms, measures of complexity, and the whole theory under this assumption. In the language of this theory we can then pose the following.

Problem 4.10 (Algorithmic Schottky problem)**.** *Give an algorithm, in the sense of complexity theory over $\mathbf{R}$, which takes as input a $2g \times 2g$ symplectic matrix representing a principally polarized abelian variety, and as output tells whether or not that torus lies in the period locus.*

One might also fix some $\epsilon = \epsilon(g)$, and then ask whether or not a given principally polarized abelian variety lies within ϵ (in the locally symmetric metric on $\mathcal{A}_g$) of the period locus.

It should be noted that S. Grushevsky [Gr] has made the KP-equations solution to the Schottky problem effective in an algebraic sense. This seems to be different than what we have just discussed, though.

We now address the question of what the Schottky locus looks like from far away. To make this precise, let $\mathrm{Cone}(\mathcal{A}_g)$ denote the tangent cone at infinity (defined in (10) above) of the locally symmetric Riemannian orbifold $\mathcal{A}_g$. Hattori [Hat] proved that $\mathrm{Cone}(\mathcal{A}_g)$ is homeomorphic to the open cone on the quotient of the Tits boundary of the symmetric space $\mathrm{Sp}(2g, \mathbf{R})/\mathrm{U}(g)$; indeed it is isometric to a Weyl chamber in the symmetric space, which is just a Euclidean sector of dimension g.

Problem 4.11 (Coarse Schottky problem)**.** *Describe, as a subset of a g-dimensional Euclidean sector, the subset of* $\mathrm{Cone}(\mathcal{A}_g)$ *determined by the Schottky locus in $\mathcal{A}_g$.*

Points in $\mathrm{Cone}(\mathcal{A}_g)$ are recording how the relative sizes of basis vectors of the tori are changing; it is precisely the "skewing parameters" that are being thrown away. It doesn't seem unreasonable to think that much of the complexity in describing the Schottky locus is coming precisely from these skewing parameters, so that this coarsification of the Schottky problem, unlike the classical version, may have a reasonably explicit solution.

There is a well-known feeling that the Schottky locus is quite distorted in $\mathcal{A}_g$. Hain and Toledo (perhaps among others) have posed the problem of determining the second fundamental form of the Schottky locus, although they indicate that this would be a rather difficult computation. We can coarsify this question by extending the definition of distortion of subgroups given in Subsection 3.2 above to the context of subspaces of metric spaces. Here the distortion of a subset S in a metric space Y is defined by comparing the restriction of the metric d_Y to S versus the induced path metric on S.

Problem 4.12 (Distortion of the Schottky locus)**.** *Compute the distortion of the Schottky locus in $\mathcal{A}_g$.*

A naive guess might be that it is exponential.

5 The Torelli group

Problems about the Torelli group $\mathcal{I}_g$ have a special flavor of their own. As one passes from Mod_g to $\mathcal{I}_g$, significant and beautiful new phenomena occur. One reason for the richness of this theory is that the standard exact sequence

$$1 \to \mathcal{I}_g \to \mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbf{Z}) \to 1$$

gives an action $\psi : \mathrm{Sp}(2g, \mathbf{Z}) \to \mathrm{Out}(\mathcal{I}_g)$, so that any natural invariant attached to $\mathcal{I}_g$ comes equipped with an $\mathrm{Sp}(2g, \mathbf{Z})$-action. The most notable examples of this are the cohomology algebra $H^*(\mathcal{I}_g, \mathbf{Q})$ and the Malcev Lie algebra $\mathcal{L}(\mathcal{I}_g) \otimes \mathbf{Q}$, both of which become $\mathrm{Sp}(2g, \mathbf{Q})$-modules, allowing for the application of symplectic representation theory. See, e.g., [Jo1, Ha3, Mo4] for more detailed explanations and examples.

In this section I present a few of my favorite problems. I refer the reader to the work of Johnson, Hain and Morita for other problems about $\mathcal{I}_g$; see, e.g., [Mo1, Mo3].

5.1 Finite generation problems

For some time it was not known if the group $\mathcal{K}_g$ generated by Dehn twists about bounding curves was equal to, or perhaps a finite index subgroup of $\mathcal{I}_g$, until Johnson found the *Johnson homomorphism* τ and proved exactness of:

$$1 \to \mathcal{K}_{\mathrm{g}} \to \mathcal{I}_g \xrightarrow{\tau} \wedge^3 H/H \to 1 \tag{11}$$

where $H = H_1(\Sigma_g; \mathbf{Z})$ and where the inclusion $H \hookrightarrow \wedge^3 H$ is given by the map $x \mapsto x \wedge \hat{i}$, where $\hat{i}$ is the intersection from $\hat{i} \in \wedge^2 H$. Recall that a *bounding pair map* is a composition $T_a \circ T_b^{-1}$ of Dehn twists about *bounding pairs*, i.e. pairs of disjoint, nonseparating, homologous simple closed curves $\{a, b\}$. Such a homeomorphism clearly lies in $\mathcal{I}_g$. By direct computation Johnson shows that the τ-image of such a map is nonzero, while $\tau(\mathcal{K}_g) = 0$; proving that $\ker(\tau) = \mathcal{K}_g$ is much harder to prove.

Johnson proved in [Jo2] that $\mathcal{I}_g$ is finitely generated for all $g \geq 3$. McCullough-Miller [McM] proved that $\mathcal{K}_2$ is not finitely generated; Mess [Me] then proved that $\mathcal{K}_2$ is in fact a free group of infinite rank, generated by the set of symplectic splittings of $H_1(\Sigma_2, \mathbf{Z})$. The problem of finite generation of $\mathcal{K}_g$ for all $g \geq 3$ was recently solved by Daniel Biss and me in [BF].

Theorem 5.1. *The group $\mathcal{K}_g$ is not finitely generated for any $g \geq 2$.*

The following basic problem, however, remains open (see, e.g., [Mo3], Problem 2.2(ii)).

Question 5.2 (Morita)**.** *Is $H_1(\mathcal{K}_g, \mathbf{Z})$ finitely generated for $g \geq 3$?*

Note that Birman-Craggs-Johnson (see, e.g., [BC, Jo1]) and Morita [Mo2] have found large abelian quotients of $\mathcal{K}_g$.

The proof in [BF] of Theorem 5.1, suggests an approach to answering to Question 5.2. Let me briefly describe the idea. Following the the outline in [McM], we first find an action of $\mathcal{K}_g$ on the first homology of a certain abelian cover Y of Σ_g; this action respects the structure of $H_1(Y, \mathbf{Z})$ as a module over the Galois group of the cover. The crucial piece is that we are able to reduce this to a representation

$$\rho : \mathcal{K}_g \to \mathrm{SL}_2(\mathbf{Z}[t, t^{-1}])$$

on the special linear group over the ring of integral laurent polynomials in one variable. This group acts on an associated Bruhat-Tits-Serre tree, and one can then analyze this action using combinatorial group theory.

One might now try to answer Question 5.2 in the negative by systematically computing more elements in the image of ρ, and then analyzing more closely the action on the tree for SL_2. One potentially useful ingredient is a theorem of Grunewald-Mennike-Vaserstein [GMV] which gives free quotients of arbitrarily high rank for the group $\mathrm{SL}_2(\mathbf{Z}[t])$ and the group $\mathrm{SL}_2(K[s,t])$, where K is an arbitrary finite field.

Since we know for $g \geq 3$ that $\mathcal{I}_g$ is finitely generated and $\mathcal{K}_g$ is not, it is natural to ask about the subgroups interpolating between these two. To be precise, consider the exact sequence (11). Corresponding to each subgroup $L < \wedge^3 H/H$ is its pullback $\pi^{-1}(L)$. The lattice of such subgroups L can be thought of as a kind of interpolation between $\mathcal{I}_g$ and $\mathcal{K}_g$.

Problem 5.3 (Interpolations). *Let $g \geq 3$. For each subgroup $L < \wedge^3 H/H$, determine whether or not $\pi^{-1}(L)$ is finitely generated.*

As for subgroups deeper down than $\mathcal{K}_g = \mathcal{I}_g(2)$ in the Johnson filtration $\{\mathcal{I}_g(k)\}$, we would like to record the following.

Theorem 5.4 (Johnson filtration not finitely generated). *For each $g \geq 3$ and each $k \geq 2$, the group $\mathcal{I}_g(k)$ is not finitely generated.*

Proof. We proceed by induction on k. For $k = 2$ this is just the theorem of [BF] that $\mathcal{K}_g = \mathcal{I}_g(2)$ is not finitely generated for any $g \geq 3$. Now assume the theorem is true for $\mathcal{I}_g(k)$. The *kth Johnson homomorphism* is a homomorphism

$$\tau_g(k) : \mathcal{I}_g(k) \to \mathfrak{h}_g(k)$$

where $\mathfrak{h}_g(k)$ is a certain finitely-generated abelian group, coming from the k^{th} graded piece of a certain graded Lie algebra; for the precise definitions and constructions, see, e.g., §5 of [Mo3]. All we will need is Morita's result (again, see §5 of [Mo3]) that $\ker(\tau_g(k)) = \mathcal{I}_g(k+1)$. We thus have an exact sequence

$$1 \to \mathcal{I}_g(k+1) \to \mathcal{I}_g(k) \to \tau_g(k)(\mathcal{I}_g(k)) \to 1 \qquad (12)$$

Now the image $\tau_g(k)(\mathcal{I}_g(k))$ is a subgroup of the finitely generated abelian group $\mathfrak{h}_g(k)$, and so is finitely generated. If $\mathcal{I}_g(k+1)$ were finitely-generated, then by (12) we would have that $\mathcal{I}_g(k)$ is finitely generated, contradicting the induction hypothesis. Hence $\mathcal{I}_g(k+1)$ cannot be finitely generated, and we are done by induction. $\diamond$

The Johnson filtration $\{\mathcal{I}_g(k)\}$ and the lower central series $\{(\mathcal{I}_g)_k\}$ do not coincide; indeed Hain proved in [Ha3] that there are terms of the former not contained in any term of the latter. Thus the following remains open.

Conjecture 5.5. *For each $k \geq 1$, the group $(\mathcal{I}_g)_k$ is not finitely generated.*

Another test of our understanding of the Johnson filtration is the following.

Problem 5.6. *Find $H_1(\mathcal{I}_g(k), \mathbf{Z})$ for all $k \geq 2$.*

Generating sets for $\mathcal{I}_g$. One difficulty in working with $\mathcal{I}_g$ is the complexity of its generating sets: any such set must have at least $\frac{1}{3}[4g^3 - g]$ elements since $\mathcal{I}_g$ has abelian quotients of this rank (see [Jo5], Corollary after Theorem 5). Compare this with Mod_g, which can always be generated by $2g+1$ Dehn twists (Humphries), or even by 2 elements (Wajnryb [Wa])! How does one keep track, for example, of the (at least) 1330 generators for $\mathcal{I}_{10}$? How does one even give a usable naming scheme for working with these? Even worse, in Johnson's proof of finite generation of $\mathcal{I}_g$ (see [Jo2]), the given generating set has $O(2^g)$ elements. The following therefore seems fundamental; at the very least it seems that solving it will require us to understand the combinatorial topology underlying $\mathcal{I}_g$ in a deeper way than we now understand it.

Problem 5.7 (Cubic genset problem)**.** *Find a generating set for $\mathcal{I}_g$ with $O(g^d)$ many elements for some $d \geq 3$. Optimally one would like $d = 3$.*

In fact in §5 of [Jo2], Johnson explicitly poses a much harder problem: for $g \geq 4$ can $\mathcal{I}_g$ be generated by $\frac{1}{3}[4g^3 - g]$ elements? As noted above, this would be a sharp result. Johnson actually obtains this sharp result in genus three, by finding ([Jo2], Theorem 3) an explicit set of 35 generators for $\mathcal{I}_g$. His method of converting his $O(2^g)$ generators to $O(g^3)$ becomes far too unwieldy when $g > 3$.

One approach to Problem 5.7 is to follow the original plan of [Jo2], but using a simpler generating set for Mod_g. This was indeed the motivation for Brendle and me when we found in [BFa1] a generating set for Mod_g consisting of 6 involutions, i.e. 6 elements of order 2. This bound was later improved by Kassabov [Ka] to 4 elements of order 2, at least when $g \geq 7$. Clearly Mod_g is never generated by 2 elements of order two, for then it would be a quotient of the infinite dihedral group, and so would be virtually abelian. Since the current known bounds are so close to being sharp, it is natural to ask for the sharpest bounds.

Problem 5.8 (Sharp bounds for involution generating sets)**.** *For each $g \geq 2$, prove sharp bounds for the minimal number of involutions required to generate* Mod_g*. In particular, for $g \geq 7$ determine whether or not* Mod_g *is generated by* 3 *involutions.*

5.2 Higher finiteness properties and cohomology

While there has been spectacular progress in understanding $H^*(\mathrm{Mod}_g, \mathbf{Z})$ (see [MW]), very little is known about $H^*(\mathcal{I}_g, \mathbf{Z})$, and even less about $H^*(\mathcal{K}_g, \mathbf{Z})$. Further, we do not have answers to the basic finiteness questions one asks about groups.

Recall that the *cohomological dimension* of a group Γ, denoted $\mathrm{cd}(\Gamma)$, is defined to be

$$\mathrm{cd}(\Gamma) := \sup\{i : H^i(\Gamma, V) \neq 0 \text{ for some } \Gamma\text{-module } V\}$$

If a group Γ has a torsion-free subgroup H of finite index, then the *virtual* cohomological dimension of Γ is defined to be $\mathrm{cd}(H)$; Serre proved that this number does not depend on the choice of H. It is a theorem of Harer, with earlier estimates and a later different proof due to Ivanov, that Mod_g has virtual cohomological dimension $4g-5$; see [Iv1] for a summary.

Problem 5.9 (Cohomological Dimension)**.** *Compute the cohomological dimension of $\mathcal{I}_g$ and of $\mathcal{K}_g$. More generally, compute the cohomological dimension of $\mathcal{I}_g(k)$ for all $k \geq 1$.*

Note that the cohomological dimension $\mathrm{cd}(\mathcal{I}_g)$ is bounded above by the (virtual) cohomological dimension of Mod_g, which is $4g-5$. The following is a start on some lower bounds.

Theorem 5.10 (Lower bounds on cd)**.** *For all $g \geq 2$, the following inequalities hold:*

$$\mathrm{cd}(\mathcal{I}_g) \geq \begin{cases} (5g-8)/2 & \text{if } g \text{ is even} \\ \\ (5g-9)/2 & \text{if } g \text{ is odd} \end{cases}$$

$$\mathrm{cd}(\mathcal{K}_g) \geq 2g-3$$

$$\mathrm{cd}(\mathcal{I}_g(k)) \geq g-1 \quad \text{for } k \geq 3$$

Proof. Since for any group Γ with $\mathrm{cd}(\Gamma) < \infty$ we have $\mathrm{cd}(\Gamma) \geq \mathrm{cd}(H)$ for any subgroup $H < \Gamma$, an easy way to obtain lower bounds for $\mathrm{cd}(\Gamma)$ is to find large free abelian subgroups of Γ. To construct such subgroups for $\mathcal{I}_g$ and for $\mathcal{K}_g$, take a maximal collection of mutually disjoint separting curves on Σ_g; by an Euler characteristic argument it is easy to see that there are $2g-3$ of these, and it is not hard to find them. The group generated by Dehn twists on these curves is isomorphic to $\mathbf{Z}^{2g-3}$, and is contained in $\mathcal{K}_g < \mathcal{I}_g$.

For $\mathcal{I}_g$ we obtain the better bounds by giving a slight variation of Ivanov's discussion of *Mess subgroups*, given in §6.3 of [Iv1], adapted so that the constructed subgroups lie in $\mathcal{I}_g$. Let Mod_g^1 denote the group of homotopy classes of orientation-preserving homeomorphisms of the surface Σ_g^1 of genus g with one boundary component, fixing $\partial\Sigma_g^1$ pointwise, up to isotopies which fix $\partial\Sigma_g^1$ pointwise. We then have a well-known exact sequence (see, e.g. [Iv1], §6.3)

$$1 \to \pi_1 T^1\Sigma_g \to \mathrm{Mod}_g^1 \xrightarrow{\pi} \mathrm{Mod}_g \to 1 \tag{13}$$

where $T^1\Sigma_g$ is the unit tangent bundle of Σ_g. Now suppose $g \geq 2$. Let C_2 and C_3 be maximal abelian subgroups of $\mathcal{I}_2$ and $\mathcal{I}_3$, respectively; these have ranks 1 and 3, respectively. We now define C_g inductively, beginning with C_2 if g is even, and with C_3 if g is odd. Let C_g^1 be the pullback

$\pi^{-1}(C_g)$ of C_g under the map π in (13). Note that, since the copy of $\pi_1 T^1\Sigma_g$ in Mod_g^1 is generated by "point pushing" and the twist about $\partial\Sigma_g$, it actually lies in the corresponding Torelli group $\mathcal{I}_g^1$. The inclusion $\Sigma_g^1 \hookrightarrow \Sigma_{g+2}$ induces an injective homomorphism $i : \mathcal{I}_g^1 \hookrightarrow \mathcal{I}_{g+2}$ via "extend by the identity". The complement of Σ_g^1 in Σ_{g+2} clearly contains a pair of disjoint separating curves. Now define C_{g+2} to be the group by the Dehn twists about these curves together with $i(C_g^1)$. Thus $C_{g+2} \approx C_g^1 \times \mathbf{Z}^2$. The same exact argument as in the proof of Theorem 6.3A in [Iv1] gives the claimed answers for $\mathrm{cd}(C_g)$.

Finally, for the groups $\mathcal{I}_g(k)$ with $k, g \geq 3$ we make the following construction. Σ_g admits a homeomorphism f of order $g-1$, given by rotation in the picture of a genus one subsurface V with $g-1$ boundary components, with a torus-with-boundary attached to each component of ∂V. It is then easy to see that there is a collection of $g-1$ mutually disjoint, f-invariant collection of simple closed curves which decomposes Σ_g into a union of $g-1$ subsurfaces $S_1, \ldots, S_{g-1}$, each having genus one and two boundary components, with mutually disjoint interiors.

Each S_i contains a pair of separating curves α_i, β_i with $i(\alpha_i, \beta_i) \geq 2$. The group generated by the Dehn twists about α_i and β_i thus generates a free group L_i of rank 2 (see, e.g. [FMa]). Nonabelian free groups have elements arbitrarily far down in their lower central series. As proven in Lemma 4.3 of [FLM], any element in the kth level of the lower central series for any L_i gives an element γ_i lying in $\mathcal{I}_g(k)$. Since $i(\gamma_i, \gamma_j) = 0$ for each i, j, it follows that the group A generated by Dehn twists about each γ_i is isomorphic to $\mathbf{Z}^{g-1}$. As A can be chosen to lie in in any $\mathcal{I}_g(k)$ with $k \geq 3$, we are done. $\diamond$

Since $\cap_{k=1}^{\infty} \mathcal{I}_g(k) = 0$, we know that there exists $K > 1$ with the property that the cohomological dimension of $\mathcal{I}_g(k)$ is constant for all $k \geq K$. It would be interesting to determine the smallest such K. A number of people have different guesses about what the higher finiteness properties of $\mathcal{I}_g$ should be.

Problem 5.11 (Torelli finiteness)**.** *Determine the maximal number $f(g)$ for which there is a $K(\mathcal{I}_g, 1)$ space with finitely many cells in dimensions $\leq f(g)$.*

Here is what is currently known about Problem 5.11:

1. $f(2) = 0$ since $\mathcal{I}_2$ is not finitely generated (McCullough-Miller [McM]).
2. $f(3) \leq 3$ (Johnson-Millson, unpublished, referred to in [Me]).
3. For $g \geq 3$, combining Johnson's finite generation result [Jo2] and a theorem of Akita (Theorem 5.13 below) gives $1 \leq f(g) \leq 6g-5$.

One natural guess which fits with the (albeit small amount of) known data is that $f(g) = g-2$. As a special case of Problem 5.11, we emphasize the following, which is a folklore conjecture.

Conjecture 5.12. *$\mathcal{I}_g$ is finitely presented for $g \geq 4$.*

One thing we do know is that, in contrast to Mod_g, neither $\mathcal{I}_g$ nor $\mathcal{K}_g$ has a classifying space which is homotopy equivalent to a finite complex; indeed Akita proved the following stronger result.

Theorem 5.13 (Akita [Ak])**.** *For each $g \geq 7$ the vector space $H_*(\mathcal{I}_g, \mathbf{Q})$ is infinite dimensional. For each $g \geq 2$ the vector space $H_*(\mathcal{K}_g, \mathbf{Q})$ is infinite dimensional.*

Unfortunately the proof of Theorem 5.13 does not illuminate the reasons why the theorem is true, especially since the proof is far from constructive. In order to demonstrate this, and since the proof idea is simple and pretty, we sketch the proof.

Proof sketch of Theorem 5.13 for $\mathcal{I}_g$. We give the main ideas of the proof, which is based on a similar argument made for $\mathrm{Out}(F_n)$ by Smillie-Vogtmann; see [Ak] for details and references.

If $\dim_{\mathbf{Q}}(H_*(\mathcal{I}_g, \mathbf{Q})) < \infty$, then the multiplicativity of the Euler characteristic for group extensions, applied to

$$1 \to \mathcal{I}_g \to \mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbf{Z}) \to 1$$

gives that

$$\chi(\mathcal{I}_g) = \chi(\mathrm{Mod}_g)/\chi(\mathrm{Sp}(2g, \mathbf{Z})) \tag{14}$$

Each of the groups on the right hand side of (14) has been computed; the numerator by Harer-Zagier and the denominator by Harder. Each of these values is given as a value of the Riemann zeta function ζ. Plugging in these values into (14) gives

$$\chi(\mathcal{I}_g) = \frac{1}{2-2g} \prod_{k=1}^{g-1} \frac{1}{\zeta(1-2k)} \tag{15}$$

It is a classical result of Hurwitz that each finite order element in Mod_g acts nontrivially on $H_1(\Sigma_g, \mathbf{Z})$; hence $\mathcal{I}_g$ is torsion-free. Thus $\chi(\mathcal{I}_g)$ is an integer. The rest of the proof of the theorem consists of using some basic properties of ζ to prove that the right hand side of (15) is not an integer. $\diamond$

The hypothesis $g \geq 7$ in Akita's proof is used only in showing that the right hand side of (15) is not an integer. This might still hold for $g < 7$.

Problem 5.14. *Extend Akita's result to $2 < g < 7$.*

Since Akita's proof produces no explicit homology classes, the following seems fundamental.

Problem 5.15 (Explicit cycles)**.** *Explicitly construct infinitely many linearly independent cycles in $H_*(\mathcal{I}_g, \mathbf{Q})$ and $H_*(\mathcal{K}_g, \mathbf{Q})$.*

So, we are still at the stage of trying to find explicit nonzero cycles. In a series of papers (see [Jo1] for a summary), Johnson proved the quite nontrivial result:

$$H^1(\mathcal{I}_g, \mathbf{Z}) \approx \frac{\wedge^3 H}{H} \oplus B_2 \tag{16}$$

where B_2 consists of 2-torsion. While the $\wedge^3 H/H$ piece comes from purely algebraic considerations, the B_2 piece is "deeper" in the sense that it is purely topological, and comes from the Rochlin invariant (see [BC] and [Jo1]); indeed the former appears in H_1 of the "Torelli group" in the analogous theory for $\mathrm{Out}(F_n)$, while the latter does not.

Remark on two of Johnson's papers. While Johnson's computation of $H_1(\mathcal{I}_g, \mathbf{Z})$ and his theorem that $\ker(\tau) = \mathcal{K}_g$ are fundamental results in this area, I believe that the details of the proofs of these results are not well-understood. These results are proved in [Jo4] and [Jo3], respectively; the paper [Jo4] is a particularly dense and difficult read. While Johnson's work is always careful and detailed, and so the results should be accepted as true, I think it would be worthwhile to understand [Jo3] and [Jo4], to exposit them in a less dense fashion, and perhaps even to give new proofs of their main results. For [Jo3] this is to some extent accomplished in the thesis [vdB] of van den Berg, where she takes a different approach to computing $H_1(\mathcal{I}_g, \mathbf{Z})$.

Since dimension one is the only dimension $i \geq 1$ for which we actually know the i^{th} cohomology of $\mathcal{I}_g$, and since very general computations seem out of reach at this point, the following seems like a natural next step in understanding the cohomology of $\mathcal{I}_g$.

Problem 5.16. *Determine the subalgebras of $H^*(\mathcal{I}_g, K)$, for $K = \mathbf{Q}$ and $K = \mathbf{F}_2$, generated by $H^1(\mathcal{I}_g, K)$.*

Note that $H^*(\mathcal{I}_g, K)$ is a module over $\mathrm{Sp}(2g, K)$. When $K = \mathbf{Q}$ this problem has been solved in degree 2 by Hain [Ha3] and degree 3 (up to one unknown piece) by Sakasai [Sa]. Symplectic representation theory (over $\mathbf{R}$) is used as a tool in these papers to greatly simplify computations. When $K = \mathbf{F}_2$, the seemingly basic facts one needs about representations are either false or they are beyond the current methods of modular representation theory. Thus computations become more complicated. Some progress in this case is given in [BFa2], where direct geometric computations, evaluating cohomology classes on abelian cycles, shows that each of the images of

$$\sigma^* : H^2(B_3, \mathbf{F}_2) \to H^2(\mathcal{I}_g, \mathbf{F}_2)$$

$$(\sigma|_{\mathcal{K}_g})^* : H^2(B_2, \mathbf{F}_2) \to H^2(\mathcal{K}_g, \mathbf{F}_2)$$

has dimension at least $O(g^4)$.

5.3 Automorphisms and commensurations of the Johnson filtration

The following theorem indicates that all of the algebraic structure of the mapping class group Mod_g is already determined by the infinite index subgroup $\mathcal{I}_g$, and indeed the infinite index subgroup $\mathcal{K}_g$ of $\mathcal{I}_g$. Recall that the *extended mapping class group*, denoted $\mathrm{Mod}_g^{\pm}$, is defined as the group of homotopy classes of *all* homeomorphisms of Σ_g, including the orientation-reversing ones; it contains Mod_g as a subgroup of index 2.

Theorem 5.17. *Let* $g \geq 4$. *Then* $\mathrm{Aut}(\mathcal{I}_g) \approx \mathrm{Mod}_g^{\pm}$ *and* $\mathrm{Aut}(\mathcal{K}_g) \approx \mathrm{Mod}_g$. *In fact* $\mathrm{Comm}(\mathcal{I}_g) \approx \mathrm{Mod}_g^{\pm}$ *and* $\mathrm{Comm}(\mathcal{K}_g) \approx \mathrm{Mod}_g$.

The case of $\mathcal{I}_g, g \geq 5$ was proved by Farb-Ivanov [FI]. Brendle-Margalit [BM] built on [FI] to prove the harder results on $\mathcal{K}_g$. The cases of Aut, where one can use explicit relations, were extended to $g \geq 3$ by McCarthy-Vautaw [MV]. Note too that $\mathrm{Aut}(\mathrm{Mod}_g) = \mathrm{Mod}_g^{\pm}$, as shown by Ivanov (see §8 of [Iv1]).

Question 5.18. *For which* $k \geq 1$ *is it true that* $\mathrm{Aut}(\mathcal{I}_g(k)) = \mathrm{Mod}_g^{\pm}$*? that* $\mathrm{Comm}(\mathcal{I}_g(k)) = \mathrm{Mod}_g^{\pm}$*?*

Theorem 5.17 answers the question for $k = 1, 2$. It would be remarkable if all of Mod_g could be reconstructed from subgroups deeper down in its lower central series.

5.4 Graded Lie algebras associated to $\mathcal{I}_g$

Fix a prime $p \geq 2$. For a group Γ let $P_i(\Gamma)$ be defined inductively via $P_1(\Gamma) = \Gamma$ and

$$P_{i+1}(\Gamma) := [\Gamma, P_i(\Gamma)]\Gamma^p \quad \text{for } i \geq 1$$

The sequence $\Gamma \supseteq P_2(\Gamma) \supseteq \cdots$ is called the *lower exponent p central series.* The quotient $\Gamma/P_2(\Gamma)$ has the universal property that any homomorphism from Γ onto an abelian p-group factors through $\Gamma/P_2(\Gamma)$; the group $\Gamma/P_{i+1}(\Gamma)$ has the analgous universal property for homomorphisms from Γ onto class i nilpotent p-groups. We can form the direct sum of vector spaces over the field $\mathbf{F}_p$:

$$\mathcal{L}_p(\Gamma) := \bigoplus_{i=1}^{\infty} \frac{P_i(\Gamma)}{P_{i+1}(\Gamma)}$$

The group commutator on Γ induces a bracket on $\mathcal{L}_p(\Gamma)$ under which it becomes a graded Lie algebra over $\mathbf{F}_p$. See, e.g. [Se] for the basic theory of Lie algebras over $\mathbf{F}_p$.

When $p = 0$, that is when $P_{i+1}(\Gamma) = [\Gamma, P_i(\Gamma)]$, we obtain a graded Lie algebra $\mathcal{L}(\Gamma) := \mathcal{L}_0(\Gamma)$ over $\mathbf{Z}$. The Lie algebra $\mathcal{L}(\Gamma) \otimes \mathbf{R}$ is isomorphic to the associated graded Lie algebra of the Malcev Lie algebra associated to Γ. In [Ha3] Hain found a presentation for the infinite-dimensional Lie algebra $\mathcal{L}(\mathcal{I}_g) \otimes \mathbf{R}$: it is (at least for $g \geq 6$) the quotient of the free Lie algebra on $H_1(\mathcal{I}_g, \mathbf{R}) = \wedge^3 H/H, H := H_1(\Sigma_g, \mathbf{R})$, modulo a finite set of *quadratic relations*, i.e. modulo an ideal generated by certain elements lying in $[P_2(\mathcal{I}_g)/P_3(\mathcal{I}_g)] \otimes \mathbf{R}$. Each of these relations can already be seen in the Malcev Lie algebra of the pure braid group.

The main ingredient in Hain's determination of $\mathcal{L}(\mathcal{I}_g) \otimes \mathbf{R}$ is to apply Deligne's *mixed Hodge theory.* This theory is a refinement and extension of the classical Hodge decomposition. For each complex algebraic variety V it produces, in a functorial way, various filtrations with special properties on $H^*(V, \mathbf{Q})$ and its complexification. This induces a remarkably rich structure on many associated invariants of V. A starting point for all of this is the fact that $\mathcal{M}_g$ is a complex algebraic variety. Since, at the end of the day, Hain's presentation of $\mathcal{L}(\mathcal{I}_g) \otimes \mathbf{R}$ is rather simple, it is natural to pose the following.

Problem 5.19. *Give an elementary, purely combinatorial-topological and group-theoretic, proof of Hain's theorem.*

It seems that a solution to Problem 5.19 will likely require us to advance our understanding of $\mathcal{I}_g$ in new ways. It may also give a hint towards attacking the following problem, where mixed Hodge theory does not apply.

Problem 5.20 (Hain for $\mathrm{Aut}(F_n)$)**.** *Give an explicit finite presentation for the Malcev Lie Algebra $\mathcal{L}(\mathrm{IA}_n)$, where IA_n is the group of automorphisms of the free group F_n acting trivially on $H_1(F_n, \mathbf{Z})$.*

There is a great deal of interesting information at the prime 2 which Hain's theorem does not address, and indeed which remains largely unexplored. While Hain's theorem tells us that reduction mod 2 gives us a large subalgebra of $\mathcal{L}_2(\mathcal{I}_{g,1})$ coming from $\mathcal{L}_0(\mathcal{I}_{g,1})$, the Lie algbera $\mathcal{L}_2(\mathcal{I}_{g,1})$ over $\mathbf{F}_2$ is much bigger. This can already be seen from (16). As noted above, the 2-torsion B_2 exists for "deeper" reasons than the other piece of $H_1(\mathcal{I}_{g,1}, \mathbf{Z})$, as it comes from the Rochlin invariant as opposed to pure algebra. Indeed, for the analogous "Torelli group" IA_n for $\mathrm{Aut}(F_n)$, the corresponding "Johnson homomorphism" gives all the first cohomology. Thus the 2-torsion in $H^1(\mathcal{I}_g, \mathbf{Z})$ is truly coming from 3-manifold theory.

Problem 5.21 (Malcev mod 2)**.** *Give an explicit finite presentation for the $\mathbf{F}_2$-Lie algebra $\mathcal{L}_2(\mathcal{I}_{g,1})$.*

We can also build a Lie algebra using the Johnson filtration. Let

$$\mathfrak{h}_g := \bigoplus_{k=1}^{\infty} \frac{\mathcal{I}_g(k)}{\mathcal{I}_g(k+1)} \otimes \mathbf{R}$$

Then $\mathfrak{h}$ is a real Lie algebra. In §14 of [Ha3], Hain proves that the Johnson filtration is not cofinal with the lower central series of $\mathcal{I}_g$. He also relates $\mathfrak{h}_g$ to $\mathfrak{t}_g$. The following basic question remains open.

Question 5.22 (Lie algebra for the Johnson filtration)**.** *Is $\mathfrak{h}_g$ a finitely presented Lie algebra? If so, give an explicit finite presentation for it.*

5.5 Low-dimensional homology of principal congruence subgroups

Recall that the *level L congruence subgroup* $\mathrm{Mod}_g[L]$ is defined to be the subgroup of Mod_g which acts trivially on $H_1(\Sigma_g, \mathbf{Z}/L\mathbf{Z})$. This normal subgroup has finite index; indeed the quotient of Mod_g by $\mathrm{Mod}_g[L]$ is the finite symplectic group $\mathrm{Sp}(2g, \mathbf{Z}/L\mathbf{Z})$. When $L \geq 3$ the group $\mathrm{Mod}_g[L]$ is torsion free, and so the corresponding cover of moduli space is actually a manifold. These manifolds arise in algebraic geometry as they parametrize so-called "genus g curves with level L structure"; see [Ha2].

Problem 5.23. *Compute $H_1(\mathrm{Mod}_g[L]; \mathbf{Z})$.*

McCarthy and (independently) Hain proved that $H_1(\mathrm{Mod}_g[L], \mathbf{Z})$ is finite for $g \geq 3$; see, e.g. Proposition 5.2 of [Ha2][7]. As discussed in §5 of [Ha2], the following conjecture would imply that the (orbifold) Picard group for the moduli spaces of level L structures has rank one; this group is finitely-generated by the Hain and McCarthy result just mentioned.

Conjecture 5.24 (Picard number one conjecture for level L structures)**.** *Prove that $H_2(\mathrm{Mod}_g[L]; \mathbf{Q}) = \mathbf{Q}$ when $g \geq 3$. More generally, compute $H_2(\mathrm{Mod}_g[L]; \mathbf{Z})$ for all $g \geq 3, L \geq 2$.*

Harer [Har2] proved this conjecture in the case $L = 1$. This generalization was stated (for Picard groups) as Question 7.12 in [HL]. The case $L = 2$ was claimed in [Fo], but there is apparently an error in the proof. At this point even the $(g, L) = (3, 2)$ case is open.

Here is a possible approach to Conjecture 5.24 for $g \geq 4$. First note that, since $\mathrm{Mod}_g[L]$ is a finite index subgroup of the finitely presented group Mod_g, it is finitely presented. As we have a lot of explicit information about the finite group $\mathrm{Sp}(2g, \mathbf{Z}/L\mathbf{Z})$, it seems possible in principle to answer the following, which is also a test of our understanding of $\mathrm{Mod}_g[L]$.

Problem 5.25 (Presentation for level L structures)**.** *Give an explicit finite presentation for $\mathrm{Mod}_g[L]$.*

Once one has such a presentation, it seems likely that it would fit well into the framework of Pitsch's proof [Pi] that $\mathrm{rank}(H_2(\mathrm{Mod}_{g,1}, \mathbf{Z})) \leq 1$ for $g \geq 4$. Note that Pitsch's proof was extended to punctured and bordered case by Korkmaz-Stipsicsz; see [Ko]. What Pitsch does is to begin with an explicit, finite presentation of $\mathrm{Mod}_{g,1}$, and then to apply Hopf's formula for groups Γ presented as the quotient of a free group F by the normal closure R of the relators:

$$H_2(\Gamma, \mathbf{Z}) = \frac{R \cap [F, F]}{[F, R]} \tag{17}$$

In other words, elements of $H_2(\Gamma, \mathbf{Z})$ come precisely from commutators which are relators, except for the trivial ones. Amazingly, one needs only write the form of an arbitrary element of the

[7] Actually, Hain proves a much stronger result, computing $H^1(\mathrm{Mod}_g[L], V)$ for V any finite-dimensional symplectic representation.

numerator in (17), and a few tricks reduces the computation of (an upper bound for) space of solutions to computing the rank of an integer matrix. In our case this approach seems feasible, especially with computer computation, at least for small L. Of course one hopes to find a general pattern.

6 Linear and nonlinear representations of Mod_g

While for $g > 2$ it is not known whether or not Mod_g admits a faithful, finite-dimensional linear representation, there are a number of known linear and nonlinear representations of Mod_g which are quite useful, have a rich internal structure, and connect to other problems. In this section we pose a few problems about some of these.

6.1 Low-dimensional linear representations

It would be interesting to classify all irreducible complex representations $\psi : \mathrm{Mod}_g \to \mathrm{GL}(m, \mathbf{C})$ for m sufficiently small compared to g. This was done for representations of the n-strand braid group for $m \leq n - 1$ by Formanek [For]. There are a number of special tricks using torsion in Mod_g and so, as with many questions of this type, one really wants to understand low-dimensional irreducible representations of the (typically torsion-free) finite index subgroups of Mod_g. It is proven in [FLMi] that no such *faithful* representations exist for $n < 2\sqrt{g-1}$.

One question is to determine if the standard representation on homology $\mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbf{Z})$ is minimal in some sense. Lubotzky has found finite index subgroups $\Gamma < \mathrm{Mod}_g$ and surjections $\Gamma \to \mathrm{Sp}(2g - 2, \mathbf{Z})$. I believe that it should be possible to prove that representations of such Γ in low degrees must have traces which are algebraic integers. This problem, and various related statements providing constraints on representations, reduce via now-standard methods to the problem of understanding representations $\rho : \mathrm{Mod}_g \to \mathrm{GL}(n, K)$, where K is a discretely valued field such as the p-adic rationals. The group $\mathrm{GL}(n, K)$ can be realized as a group of isometries of the corresponding Bruhat-Tits affine building; this is a nonpositively curved (in the CAT(0) sense), $(n - 1)$-dimensional simplicial complex. The general problem then becomes:

Problem 6.1 (Actions on buildings)**.** *Determine all isometric actions $\psi : \mathrm{Mod}_g \to \mathrm{Isom}(X^n)$, where X^n is an n-dimensional Euclidean building, and n is sufficiently small compared to g.*

For example, one would like conditions under which ψ has a *global fixed point*, that is, a point $x \in X^n$ such that $\psi(\mathrm{Mod}_g) \cdot x = x$. One method to attack this problem is the so-called "Helly technique" introduced in [Fa4]. Using standard CAT(0) methods, one can show that each Dehn twist T_α in Mod_g has a nontrivial fixed set F_α under the ψ-action; F_α is necessarily convex. Considering the nerve of the collection $\{F_\alpha\}$ gives a map $\mathcal{C}_g \to X^n$ from the complex of curves to X^n. Now $\mathcal{C}_g$ has the homotopy type of a wedge of spheres (see, e.g., [Iv1]), while X^n is contractible.

Hence the spheres in the nerve must be filled in, which gives that many more elements $\psi(T_\alpha)$ have common fixed points. The problem now is to understand in an explicit way the spheres inside $\mathcal{C}_g$.

6.2 Actions on the circle

It was essentially known to Nielsen that $\mathrm{Mod}_{g,1}$ acts faithfully by orientation-preserving homeomorphisms on the circle. Here is how this works: for $g \geq 2$ any homeomorphism $f \in \mathrm{Homeo}^+(\Sigma_g)$ lifts to a quasi-isometry $\widetilde{f}$ of the hyperbolic plane $\mathbf{H}^2$. Any quasi-isometry of $\mathbf{H}^2$ takes geodesic rays to a uniformly bounded distance from geodesic rays, thus inducing a map $\partial\widetilde{f} : S^1 \to S^1$ on the boundary of infinity of $\mathbf{H}^2$, which is easily checked to be a homeomorphism, indeed a quasi-symmetric homeomorphism. If $h \in \mathrm{Homeo}^+(\Sigma_g)$ is homotopic to f, then since homotopies are compact one sees directly that $\widetilde{h}$ is homotopic to $\widetilde{f}$, and so these maps are a bounded distance from each other in the sup norm. In particular $\partial\widetilde{h} = \partial\widetilde{f}$; that is, $\partial\widetilde{f}$ depends only the homotopy class of f. It is classical that quasi-isometries are determined by their boundary values, hence $\partial\widetilde{f} = Id$ only when f is homotopically trivial. Now there are $\pi_1\Sigma_g$ choices for lifting any such f, so the group $\Gamma_g \subset \mathrm{Homeo}^+(S^1)$ of all lifts of all homotopy classes of $f \in \mathrm{Homeo}^+(\Sigma_g)$ gives an exact sequence

$$1 \to \pi_1\Sigma_g \to \Gamma_g \to \mathrm{Mod}_{g,1} \to 1 \tag{18}$$

Since each element of $\mathrm{Mod}_{g,1}$ fixes a common marked point on Σ_g, there is a canonical way to choose a lift of each f; that is, we obtain a section $\mathrm{Mod}_{g,1} \to \Gamma_g$ splitting (18). In particular we have an injection

$$\mathrm{Mod}_{g,1} \hookrightarrow \mathrm{Homeo}^+(S^1) \tag{19}$$

This inclusion provides a so-called (left) circular ordering on $\mathrm{Mod}_{g,1}$ - see [Cal]. Note that no such inclusion as in (19) exists for Mod_g since any finite subgroup of $\mathrm{Homeo}^+(S^1)$ must be cyclic[8], but Mod_g has noncyclic finite subgroups.

The action given by (19) gives a dynamical description of $\mathrm{Mod}_{g,1}$ via its action on S^1. For example, each pseudo-Anosov in $\mathrm{Mod}_{g,1}$ acts on S^1 with finitely many fixed points, with alternating sources and sinks as one moves around the circle (see, e.g., Theorem 5.5 of [CB]). There is an intrinsic non-smoothness to this action, and indeed in [FF] it is proven that any homomorphism $\rho : \mathrm{Mod}_{g,1} \to \mathrm{Diff}^2(S^1)$ has trivial image; what one really wants to prove is that no finite index subgroup of Mod_g admits a faithful C^2 action on S^1. It would be quite interesting to prove that the action (19) is canonical, in the following sense.

Question 6.2 (Rigidity of the $\mathrm{Mod}_{g,1}$ action on S^1)**.** *Is any faithful action $\rho : \mathrm{Mod}_{g,1} \to \mathrm{Homeo}^+(S^1)$ conjugate in $\mathrm{Homeo}^+(S^1)$ to the standard action, given in (19)? What about the same question for finite index subgroups of $\mathrm{Mod}_{g,1}$?*

[8] One can see this by averaging any Riemannian metric on S^1 by the group action.

Perhaps there is a vastly stronger, topological dynamics characterization of $\mathrm{Mod}_{g,1}$ inside $\mathrm{Homeo}^+(S^1)$, in the style of the Convergence Groups Conjecture (theorem of Tukia, Casson-Jungreis and Gabai), with "asymptotically source – sink" being replaced here by "asymptotically source – sink – $\cdots$ – source – sink", or some refinement/variation of this.

Now, the group of lifts of elements of $\mathrm{Homeo}^+(S^1)$ to homeomorphisms of $\mathbf{R}$ gives a central extension

$$1 \to \mathbf{Z} \to \widetilde{\mathrm{Homeo}}(S^1) \to \mathrm{Homeo}^+(S^1) \to 1$$

which restricts via (19) to a central extension

$$1 \to \mathbf{Z} \to \widetilde{\mathrm{Mod}_{g,1}} \to \mathrm{Mod}_{g,1} \to 1 \tag{20}$$

Note that $\mathrm{Mod}_{g,1}$ has torsion. Since $\widetilde{\mathrm{Homeo}}(S^1) \subset \mathrm{Homeo}^+(\mathbf{R})$ which clearly has no torsion, it follows that (20) does not split. In particular the extension (20) gives a nonvanishing class $\xi \in H^2(\mathrm{Mod}_{g,1}, \mathbf{Z})$. Actually, it is not hard to see that ξ is simply the "euler cocycle", which assigns to any pointed map $\Sigma_h \to \mathcal{M}_g$ the euler class of the pullback bundle of the "universal circle bundle" over $\mathcal{M}_g$.

The torsion in $\mathrm{Mod}_{g,1}$ and in Mod_g preclude each from having a left-ordering, or acting faithfully on $\mathbf{R}$. As far as we know this is the only obstruction; it disappears when one passes to appropriate finite index subgroups.

Question 6.3 (orderability)**.** *Does* $\mathrm{Mod}_g, g \geq 2$ *have some finite index subgroup which acts faithfully by homeomorphisms on* S^1*? Does either* Mod_g *or* $\mathrm{Mod}_{g,1}$ *have a finite index subgroup which acts faithfully by homeomorphisms on* $\mathbf{R}$*?*

Note that Thurston proved that braid groups are orderable. Since $\mathcal{I}_g$ and $\mathcal{I}_{g,1}$ are residually torsion-free nilpotent, they are isomorphic to a subgroup of $\mathrm{Homeo}^+(\mathbf{R})$; in fact one can show that (20) splits when restricted to $\mathcal{I}_{g,1}$. On the other hand, Witte [Wi] proved that no finite index subgroup of $\mathrm{Sp}(2g, \mathbf{Z})$ acts faithfully by homoeomorphisms on S^1 or on $\mathbf{R}$.

Non-residual finiteness of the universal central extension. The Lie group $\mathrm{Sp}(2g, \mathbf{R})$ has infinite cyclic fundamental group. Its universal cover $\widetilde{\mathrm{Sp}(2g, \mathbf{R})}$ gives a central extension

$$1 \to \mathbf{Z} \to \widetilde{\mathrm{Sp}(2g, \mathbf{R})} \to \mathrm{Sp}(2g, \mathbf{R}) \to 1$$

which restricts to a central extension

$$1 \to \mathbf{Z} \to \widetilde{\mathrm{Sp}(2g, \mathbf{Z})} \to \mathrm{Sp}(2g, \mathbf{Z}) \to 1 \tag{21}$$

The cocycle $\zeta \in H^2(\mathrm{Sp}(2g, \mathbf{Z}), \mathbf{Z})$ defining the extension (21) is nontrivial and bounded; this comes from the fact that it is proportional to the Kahler class on the corresponding locally symmetric quotient (which is a $K(\pi, 1)$ space). Deligne proved in [De] that $\widetilde{\mathrm{Sp}(2g, \mathbf{Z})}$ is *not residually finite.* Since there is an obvious surjection of exact sequences from (19) to (21), and since both central extensions give a bounded cocycle, one begins to wonder about the following.

Question 6.4 ((Non)residual finiteness). *Is the (universal) central extension* $\widetilde{\mathrm{Mod}_{g,1}}$ *of* $\mathrm{Mod}_{g,1}$ *residually finite, or not?*

Note that an old result of Grossman states that Mod_g and $\mathrm{Mod}_{g,1}$ are both residually finite. The group $\mathrm{Sp}(2g, \mathbf{Z})$ is easily seen to be residually finite; indeed the intersection of all congruence subgroups of $\mathrm{Sp}(2g, \mathbf{Z})$ is trivial.

6.3 The sections problem

Consider the natural projection $\pi : \mathrm{Homeo}^+(\Sigma_g) \to \mathrm{Mod}_g$, and let H be a subgroup of Mod_g. We say that π *has a section over* H if there exists a homomorphism $\sigma : \mathrm{Mod}_g \to \mathrm{Homeo}^+(\Sigma_g)$ so that $\pi \circ \sigma = \mathrm{Id}$. This means precisely that H has a section precisely when it can be realized as a group of homeomorphisms, not just a group of homotopy classes of homeomorphisms. The general problem is then the following.

Problem 6.5 (The sections problem). *Determine those subgroups* $H \leq \mathrm{Mod}_g$ *for which* π *has a section over* H*. Do this as well with* $\mathrm{Homeo}^+(\Sigma_g)$ *replaced by various subgroups, such as* $\mathrm{Diff}^r(S)$ *with* $r = 1, 2, \ldots, \infty, \omega$*; similarly for the group of area-preserving diffeomorphisms, quasiconformal homeomorphisms, etc..*

Answers to Problem 6.5 are known in a number of cases.

1. When H is free then sections clearly always exist over H.
2. Sections to π exist over free abelian H, even when restricted to $\mathrm{Diff}^\infty(\Sigma_g)$. This is not hard to prove, given the classification by Birman-Lubotzky-McCarthy [BLM] of abelian subgroups of Mod_g.
3. Sections exist over any finite group $H < \mathrm{Mod}_g$, even when restricted to $\mathrm{Diff}^\omega(\Sigma_g)$. This follows from the Nielsen Realization Conjecture, proved by Kerckhoff [Ke], which states that any such H acts as a group of automorphisms of some genus g Riemann surface.
4. In contrast, Morita showed (see, e.g., [Mo5]) that π does not have a section with image in $\mathrm{Diff}^2(\Sigma_2)$ over all of Mod_g when $g \geq 5$. The C^2 assumption is used in a crucial way since Morita uses a putative section to build a codimension 2 foliation on the universal curve over $\mathcal{M}_g$, to whose normal bundle he applies the Bott vanishing theorem, contradicting nonvanishing of a certain (nontrivial!) Miller-Morita-Mumford class. It seems like Morita's proof can be extended to finite index subgroups of Mod_g.
5. Markovic [Mar] has recently proven that $H = \mathrm{Mod}_g$ does not even have a section into $\mathrm{Homeo}(\Sigma_g)$, answering a well-known question of Thurston.

As is usual when one studies representations of a discrete group Γ, one really desires a theorem about all finite index subgroups of Γ. One reason for this is that the existence of torsion and special relations in a group Γ often highly constrains its possible representations. Markovic's proof in [Mar] uses both torsion and the braid relations in what seems to be an essential way; these both disappear in most finite index subgroups of Mod_g. Thus it seems that a new idea is needed to answer the following.

Question 6.6 (Sections over finite index subgroups)**.** *Does the natural map* $\mathrm{Homeo}^+(\Sigma_g) \to \mathrm{Mod}_g$ *have a section over a finite index subgroup of* Mod_g*, or not?*

Of course the ideas in [Mar] are likely to be pertinent. Answers to Problem 6.5 even for specific subgroups (e.g. for $\mathcal{I}_g$ or more generally $\mathcal{I}_g(k)$) would be interesting. It also seems reasonable to believe that the existence of sections is affected greatly by the degree of smoothness one requires.

Instead of asking for *sections* in the above questions, one can ask more generally whether there are any actions of Mod_g on Σ_g.

Question 6.7. *Does* Mod_g *or any of its finite index subgroups have any faithful action by homeomorphisms on* Σ_g*?*

7 Pseudo-Anosov theory

Many of the problems in this section come out of joint work with Chris Leininger and Dan Margalit, especially that in the paper [FLM].

7.1 Small dilatations

Every pseudo-Anosov mapping class $f \in \mathrm{Mod}_g$ has a *dilatation* $\lambda(f) \in \mathbf{R}$. This number is an algebraic integer which records the exponential growth rate of lengths of curves under iteration of f, in any fixed metric on S; see [Th]. The number $\log(\lambda(f))$ equals the minimal topological entropy of any element in the homotopy class f; this minimum is realized by a pseudo-Anosov homeomorphism representing f (see [FLP, Exposé 10]). $\log(\lambda(f))$ is also the translation length of f as an isometry of the *Teichmüller space of* S equipped with the Teichmüller metric. Penner considered the set

$$\mathrm{spec}(\mathrm{Mod}_g) = \{\log(\lambda(f)) : f \in \mathrm{Mod}_g \text{ is pseudo-Anosov}\} \subset \mathbf{R}$$

This set can be thought of as the *length spectrum* of $\mathcal{M}_g$. We can also consider, for various subgroups $H < \mathrm{Mod}_g$, the subset $\mathrm{spec}(H) \subset \mathrm{spec}(\mathrm{Mod}_g)$ obtained by restricting to pseudo-Anosov elements of H. Arnoux–Yoccoz [AY] and Ivanov [Iv2] proved that $\mathrm{spec}(\mathrm{Mod}_g)$ is discrete as a subset of $\mathbf{R}$. It follows that for any subgroup $H < \mathrm{Mod}_g$, the set $\mathrm{spec}(H)$ has a least element

$L(H)$. Penner proved in [Pe] that

$$\frac{\log 2}{12g-12} \leq L(\mathrm{Mod}_g) \leq \frac{\log 11}{g}$$

In particular, as one increases the genus, there are pseudo-Anosovs with stretch factors arbitrarily close to one. In contrast to our understanding of the asymptotics of $L(\mathrm{Mod}_g)$, we still do not know the answer to the following question, posed by McMullen.

Question 7.1. *Does* $\lim_{g\to\infty} gL(\mathrm{Mod}_g)$ *exist?*

Another basic open question is the following.

Question 7.2. *Is the sequence* $\{L(\mathrm{Mod}_g)\}$ *monotone decreasing? strictly so?*

Explicit values of $L(\mathrm{Mod}_g)$ are known only when $g = 1$. In this case one is simply asking for the minimum value of the largest root of a polynomial as one varies over all integral polynomials $x^2 - bx + 1$ with $b \geq 3$. This is easily seen to occur when $b = 3$. For $g = 2$ Zhirov [Zh] found the smallest dilatation for pseudo-Anosovs with orientable foliation. It is not clear if this should equal $L(\mathrm{Mod}_2)$ or not.

Problem 7.3. *Compute* $L(\mathrm{Mod}_g)$ *explicitly for small* $g \geq 2$.

In principle $L(\mathrm{Mod}_g)$ can be computed for any given g. The point is that one can first bound the degree of $L(\mathrm{Mod}_g)$, then give bounds on the smallest possible value $\lambda(\alpha)$, where α ranges over all algebraic integers of a fixed range of degrees, and $\lambda(\alpha)$ denotes the largest root of the minimal polynomial of α. One then finds all train tracks on Σ_g, and starts to list out all pseudo-Anosovs. It is possible to give bounds for when the dilatations of these become large. Now one tries to match up the two lists just created, to find the minimal dilatation pseudo-Anosov on Σ_g. Of course actually following out this procedure, even for small g, seems to be impracticable.

Question 7.4. *Is there a unique (up to conjugacy) minimal dilation pseudo-Anosov in* Mod_g*?*

Note that this is true for $g = 1$; the unique minimum is realized by the conjugacy class of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Here is a natural refinement of the problem of finding $L(\mathrm{Mod}_g)$. Fix a genus g. Fix a possible r-tuple $(k_1, \ldots, k_r)$ of *singularity data* for Σ_g. By this we mean to consider possible foliations with r singularities with $k_1, \ldots, k_r$ prongs, respectively. For a fixed g, there are only finitely many possible tuples, as governed by the Poincare-Hopf index theorem. Masur-Smillie [MS] proved that, for every admissible tuple, there is some pseudo-Anosov on Σ_g with stable foliation having the given singularity data. Hence the following makes sense.

Problem 7.5 (Shortest Teichmüller loop in a stratum)**.** *For each fixed $g \geq 2$, and for each r-tuple as above, give upper and lower bounds for*

$$\lambda_g(k_1, \ldots, k_r) := \inf\{\log \lambda(f) : f \in \mathrm{Mod}_g \textit{ whose stable foliation has data } (k_1, \ldots, k_r)\}$$

This problem is asking for bounds for the shortest Teichmüller loop lying in a given substratum in moduli space (i.e. the projection in $\mathcal{M}_g$ of the corresponding substratum in the cotangent bundle).

$$L(\mathrm{Mod}_g) = \min\{\lambda_g(k_1, \ldots, k_r)\}$$

where the min is taken of all possible singularity data.

7.2 Multiplicities

The set spec(Mod_g) has unbounded multiplicity; that is, given any $N > 0$, there exists $r \in$ spec(Mod_g) such that there are at least n conjugacy classes $f_1, \ldots f_n$ of pseudo-Anosovs in Mod_g having $\log(\lambda(f_i)) = r$. The reason for this is that $\mathcal{M}_g$ contains isometrically embedded finite volume hyperbolic 2-manifolds X, e.g. the so-called *Veech curves*, and any such X has (hyperbolic) length spectrum of unbounded mulitplicity.

A related mechanism which produces length spectra with unbounded multiplicities is the *Thurston representation.* This gives, for a pair of curves a, b on Σ_g whose union fills Σ_g, an injective representation $\rho :< T_a, T_b >\to \mathrm{PSL}(2, \mathbf{R})$ with the following properties: image(ρ) is discrete; each element of image(ρ) is either pseudo-Anosov or is a power of $\rho(T_a)$ or $\rho(T_b)$; and spec($< T_a, T_b >$) is essentially the length spectrum of the quotient of $\mathbf{H}^2$ by image(ρ). Again it follows that spec($< T_a, T_b >$) has unbounded multiplicity. Since one can find a, b as above, each of which is in addition separating, it follows that spec($\mathcal{I}_g$) and even spec($\mathcal{I}_g(2)$) have unbounded multiplicity.

Question 7.6. *Does* spec($\mathcal{I}_g(k)$) *have bounded multiplicity for $k \geq 3$?*

One way to get around unbounded multiplicities is to look at the *simple* length spectrum, which is the subset of spec(Mod_g) coming from pseudo-Anosovs represented by simple (i.e. non-self-intersecting) geodesic loops in $\mathcal{M}_g$.

Question 7.7 (Simple length spectrum)**.** *Does the simple length spectrum of $\mathcal{M}_g$, endowed with the Teichmüller metric, have bounded multiplicity? If so, how does the bound depend on g?*

Of course this question contains the corresponding question for (many) hyperbolic surfaces, which itself is still open. These questions also inspire the following.

Problem 7.8. *Give an algorithm which tells whether or not any given pseudo-Anosov is represented by a simple closed Teichmüller geodesic, and also whether or not this geodesic lies on a Veech curve.*

Note that the analogue of Question 7.7 is not known for hyperbolic surfaces, although it is true for a generic set of surfaces in $\mathcal{M}_g$.

7.3 Special subgroups

In [FLM] we provide evidence for the principle that algebraic complexity implies dynamical complexity. A paradigm for this is the following theorem.

Theorem 7.9 ([FLM])**.** *For $g \geq 2$, we have*

$$.197 < L(\mathcal{I}_g) < 4.127$$

The point is that $L(\mathcal{I}_g)$ is universally bounded above and below, independently of g. We extend this kind of universality to every term of the Johnson filtration, as follows.

Theorem 7.10 ([FLM])**.** *Given $k \geq 1$, there exist $M(k)$ and $m(k)$, where $m(k) \to \infty$ as $k \to \infty$, so that*

$$m(k) < L(\mathcal{I}_g(k)) < M(k)$$

for every $g \geq 2$.

Question 7.11. *Give upper and lower bounds for $L(\mathcal{I}_g(k))$ for all $k \geq 2$ which are of the same order of magnitude.*

In [FLM] bounds on $L(H)$ are given for various special classes of subgroups $H < \mathrm{Mod}_g$. It seems like there is much more to explore in this direction. One can also combine these types of questions with problems such as Problem 7.5.

7.4 Densities in the set of dilatations

Let P be a property which pseudo-Anosov homeomorphisms might or might not have. For example, P might be the property of lying in a fixed subgroup of $H < \mathrm{Mod}_g$, such as $H = \mathcal{I}_g$; or P might be the property of having dilatation which is an algebraic integer of a fixed degree. One can then ask the natural question: how commonly do the dilatations of elements with P arise in $\mathrm{spec}(\mathrm{Mod}_g)$?

To formalize this, recall that the *(upper) density* $d^*(A)$ of a subset A of the natural numbers $\mathbf{N}$ is defined as

$$d^*(A) := \limsup_{N\to\infty} \frac{\#A \cap [0,n]}{n}$$

This notion can clearly be extended from $\mathbf{N}$ to any countable ordered set $\mathcal{S}$ once an order-preserving bijection $\mathcal{S} \to \mathbf{N}$ is fixed.

Now fix $g \geq 2$, and recall that $\mathrm{spec}(\mathrm{Mod}_g) \subset \mathbf{R}^+$ is defined to be the set of (logs of) dilatations of pseudo-Anosov homeomorphisms of Mod_g. The set $\mathrm{spec}(\mathrm{Mod}_g)$ comes with a natural order $\lambda_1 < \lambda_2 < \cdots$, which determines a fixed bijection $\mathrm{spec}(\mathrm{Mod}_g) \to \mathbf{N}$. If we wish to keep track of all pseudo-Anosovs, and not just their dilatations, we can simply consider the (total) ordering on the set of all pseudo-Anosovs $\mathcal{P}_g$ given by $f \leq g$ if $\lambda(f) \leq \lambda(g)$.

Question 7.12. *For various subgroups $H < \mathrm{Mod}_g$, compute the density of* $\mathrm{spec}(H)$ *in* $\mathrm{spec}(\mathrm{Mod}_g)$ *and the density of $H \cap \mathcal{P}_g$ in $\mathcal{P}_g$. In particular, what is the density of* $\mathrm{spec}(\mathrm{Mod}_g[L])$ *in* $\mathrm{spec}(\mathrm{Mod}_g)$*? What about $H = \mathcal{I}_g(k)$ with $k \geq 1$?*

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Dept. of Mathematics, University of Chicago
5734 University Ave.
Chicago, Il 60637
E-mail: farb@math.uchicago.edu