Groups of homeomorphisms of one-manifolds, I: actions of nonlinear groups

Benson Farb* and John Franks†

June 20, 20001

Abstract

This self-contained paper is part of a series [FF2, FF3] on actions by diffeomorphisms of infinite groups on compact manifolds. The two main results presented here are:

1. Any homomorphism of (almost any) mapping class group or automorphism group of a free group into $\mathrm{Diff}^r_+(S^1), r \geq 2$ is trivial. For $r = 0$ Nielsen showed that in many cases nontrivial (even faithful) representations exist. Somewhat weaker results are proven for finite index subgroups.
2. We construct a finitely-presented group of real-analytic diffeomorphisms of $\mathbf{R}$ which is not residually finite.

1 Introduction

In this paper we consider infinite groups acting by diffeomorphisms on one-dimensional manifolds. For lattices in higher rank semisimple Lie groups such actions are essentially completely understood:

Theorem 1.1 (Ghys [Gh], Burger-Monod [BM]). *Let Γ be a lattice in a simple Lie group of $\mathbf{R}$-rank at least two. Then any C^0-action of Γ on S^1 has a finite orbit, and any C^1-action of Γ on S^1 must factor through a finite group.*

Theorem 1.1 is the solution in dimension one of Zimmer's program of classifying actions of higher rank lattices in simple Lie groups on compact manifolds. In [La] (see in particular §8), Labourie describes possible extensions of Zimmer's program to other "big" groups. A. Navas [Na] has recently proven that any $C^{1+\alpha}, \alpha > 1/2$ action of a group with Kazhdan's property T factors through a finite group. In this paper we consider three basic examples of nonlinear groups: mapping class groups of surfaces, (outer) automorphism groups of free groups, and Baumslag-Solitar groups.

*Supported in part by the NSF and by the Sloan Foundation.
†Supported in part by NSF grant DMS9803346.

Mapping class groups and automorphism groups of free groups. Let $\mathrm{Mod}(g,k)$ denote the group of isotopy classes of diffeomorphisms of the genus g surface with k punctures. Unlike the case of lattices (Theorem 1.1), the group $\mathrm{Mod}(g,1), g \geq 1$ does have a faithful, C^0-action on S^1 without a global fixed point. This is a classical result of Nielsen. However, imposing a small amout of regularity changes the situation dramatically.

Theorem 1.2 (Mapping class groups). *For $g \geq 3$ and $k = 0, 1$, any C^2 action of $\mathrm{Mod}(g,k)$ on S^1 or on $I = [0,1]$ is trivial.*

The S^1 case of Theorem 1.2 was announced by E. Ghys several years ago (see §8 of [La]), and for real-analytic actions was proved by Farb-Shalen [FS].

Another class of basic examples of "big groups" are automorphism groups of free groups. Let $\mathrm{Aut}(F_n)$ (resp. $\mathrm{Out}(F_n)$) denote the group of automorphisms (resp. outer automorphisms) of the free group F_n of rank n. It is known that $\mathrm{Aut}(F_n), n > 2$ is not linear [FP].

The techniques we develop to prove the results above allow us to prove the following.

Theorem 1.3 (Automorphism groups of free groups). *For $n \geq 6$, any homomorphism from $\mathrm{Aut}(F_n)$ to $\mathrm{Diff}_+^2(S^1)$ factors through $\mathbf{Z}/2\mathbf{Z}$. Any homomorphism from $\mathrm{Aut}(F_n)$ to $\mathrm{Diff}_+^2(I)$ is trivial. The analagous results hold for $\mathrm{Out}(F_n)$.*

M. Bridson and K. Vogtmann [BV] have recently proven a stronger result: any homomorphism from $\mathrm{Aut}(F_n)$ to a group which does not contain a symmetric group must have an image which is at most $Z/2Z$. However, their proof uses torsion elements in an essential way, hence does not extend to finite index subgroups. On the other hand our techniques have implications for finite index subgroups of $\mathrm{Aut}(F_n)$ and $\mathrm{Mod}(g,k)$; such subgroups are typically torsion free.

Theorem 1.4 (Finite index and other subgroups). *Let H be any group of C^2 diffeomorphisms of I or S^1 with the property that no nontrivial element of H has an interval of fixed points (e.g. H is a group of real-analytic diffeomorphisms). Then H does not contain any finite index subgroup of:*

1. $\mathrm{Mod}(g,k)$ *for* $g \geq 3, k \geq 0$.
2. $\mathrm{Aut}(F_n)$ *or* $\mathrm{Out}(F_n)$ *for* $n \geq 6$.
3. *The Torelli group* $T_{g,k}$ *for* $g \geq 3, k \geq 0$.

In [FF3] we construct a C^1 action of $T_{g,k}$ on I and on S^1. Note also that $\mathrm{Out}(F_2)$ has a free subgroup of finite index, which admits a faithful, C^ω action on I and on S^1.

Baumslag-Solitar groups. The *Baumslag-Solitar groups* $\mathrm{BS}(m,n)$ are defined by the presentation

$$\mathrm{BS}(m,n) = < a, b : ab^m a^{-1} = b^n >$$

When $n > m > 1$ the group $\mathrm{BS}(m, n)$ is not residually finite; in particular it is not a subgroup of any linear group (see, e.g. [LS]). We will give a construction which shows that $\mathrm{BS}(m, n)$ is a subgroup of one of the "smallest" infinite-dimensional Lie groups.

Theorem 1.5 (Baumslag-Solitar groups: existence). *The group* $\mathrm{Diff}^{\omega}_{+}(\mathbf{R})$ *of real-analytic diffeomorphisms of* $\mathbf{R}$ *contains a subgroup isomorphic to* $\mathrm{BS}(m, n)$ *for any* $n > m \geq 1$. *The analogous result holds for* $\mathrm{Homeo}_{+}(S^1)$ *and* $\mathrm{Homeo}_{+}(I)$.

It is not difficult to construct pairs of diffeomorphisms $a, b \in \mathrm{Diff}^{\omega}_{+}(\mathbf{R})$ which satisfy the relation $ab^m a^{-1} = b^n$; the difficulty is to prove that (in certain situations) this is the only relation. To do this we use a Schottky type argument.

While Ghys-Sergiescu [GS] showed that $\mathrm{Diff}^{\infty}(S^1)$ contains Thompson's infinite simple (hence non-residually finite) group T, they also showed that T admits no real-analytic action on S^1 (see also [FS]); indeed we do not know of any subgroups of $\mathrm{Diff}^{\omega}(S^1)$ which are not residually finite.

The construction in the proof of Theorem 1.5 gives an abundance analytic actions of $\mathrm{BS}(m, n)$ on $\mathbf{R}$, and C^0 actions of $\mathrm{BS}(m, n)$ on S^1 and on I. The loss of regularity in moving from $\mathbf{R}$ to S^1 is no accident; we will show in contrast to Theorem 1.5 that (for typical m, n) there are no C^2 actions of $\mathrm{BS}(m, n)$ on S^1 or I.

Theorem 1.6 (Baumslag-Solitar groups: non-existence). *No subgroup of* $\mathrm{Diff}^2_{+}(I)$ *is isomorphic to* $\mathrm{BS}(m, n), n > m > 1$. *If further* m *does not divide* n, *then the same holds for* $\mathrm{Diff}^2_{+}(S^1)$.

The hypothesis $m > 1$ in Theorem 1.6 is also necessary since $\mathrm{BS}(1, n), n \geq 1$ is a subgroup of $\mathrm{PSL}(2, \mathbf{R})$, hence of $\mathrm{Diff}^{\omega}_{+}(S^1)$.

In fact $\mathrm{BS}(1, n)$ has many actions on S^1, and it is natural to attempt a classification (up to conjugacy) of all of them. As a first step in this direction we observe (Theorem 5.8), as a corollary of result of M. Shub [Sh] on expanding maps, that the standard, projective action of $\mathrm{BS}(1, n)$ on S^1 is *locally rigid*, i.e. nearby actions are conjugate. An example due to M. Hirsch [H] shows that it is not *globally rigid*, i.e. there are actions which are not conjugate to the standard action (Theorem 5.9 below). It would be interesting to find a numerical invariant which characterizes the standard action, as Ghys [Gh2] has done for Fuchsian groups.

2 Tools

In this section we recall some properties of diffeomorphisms of one-manifolds which will be used throughout the paper.

2.1 Kopell's Lemma and Hölder's Theorem

Our primary tool is the following remarkable result of Nancy Kopell which is Lemma 1 of [K].

Theorem 2.1 (Kopell's Lemma). *Suppose f and g are C^2, orientation-preserving diffeomorphisms of an interval $[a,b)$ such that $fg = gf$. If f has no fixed point in (a,b) and g has a fixed point in (a,b) then $g = id$.*

Another useful result is the following theorem, which is classical.

Theorem 2.2 (Hölder's Theorem). *Suppose a group G of homeomorphisms of $\mathbf{R}$ acts freely and effectively on a closed subset of $\mathbf{R}$. Then G is abelian.*

2.2 The translation number and mean translation number

If $f \in \mathrm{Homeo}_+(S^1)$, i.e, f is an orientation preserving homeomorphism of S^1, then there is a countable collection of lifts of f to orientation preserving homeomorphisms of the line. If F is one such lift then it satisfies $FT = TF$ where $T(x) = x + 1$, and all others are of the form FT^n, $n \in \mathbf{Z}$. Any orientation preserving homeomorphism of $\mathbf{R}$ which commutes with T is a lift of an element of $\mathrm{Homeo}_+(S^1)$. We will denote by $\mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R})$ the group of homeomorphisms of $\mathbf{R}$ which commute with T, or equivalently the group of all lifts of elements of $\mathrm{Homeo}_+(S^1)$.

There are two important and closely related functions from $\mathrm{Homeo}_+(S^1)$ and $\mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R})$ to S^1 and $\mathbf{R}$ respectively, which we now define.

Definition 2.3. *If $F \in \mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R})$ define its* translation number $\tau(F) \in \mathbf{R}$ *by*

$$\tau(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

If $f \in \mathrm{Homeo}_+(S^1)$ define its rotation number, *$\rho(f) \in S^1$, by*

$$\rho(f) = \tau(F) \mod (1),$$

where F is any lift of f.

We summarize some basic properties of the rotation and translation numbers. Proofs of these and additional properties can be found, for example, in [dMvS]

Proposition 2.4 (Properties of rotation number). *If $F \in \mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R})$ the number $\tau(F)$ always exists and is independent of the $x \in \mathbf{R}$ used to define it. It satisfies*

$$\begin{aligned} \tau(F^n) &= n\tau(F) \\ \tau(FT^n) &= \tau(F) + n \\ \tau(F_0 F F_0^{-1}) &= \tau(F) \textit{ for any } F_0 \in \mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R}). \end{aligned}$$

F has a fixed point if and only if $\tau(F) = 0$.

If $f \in \mathrm{Homeo}_+(S^1)$ then $\rho(f)$ is independent of the lift F used to define it. It satisfies

$$\begin{aligned} \rho(f^n) &= n\rho(f) \\ \rho(f_0 f f_0^{-1}) &= \rho(f) \textit{ for any } f_0 \in \mathrm{Homeo}_+(S^1). \end{aligned}$$

f has a fixed point if and only if $\rho(f) = 0$.

Suppose G is a subgroup of $\mathrm{Homeo}_+(S^1)$ and $\bar{G}$ is the group of all lifts of elements of G to $\mathbf{R}$. If G preserves a Borel probability measure μ_0 on S^1 then this measure may be lifted to a $\bar{G}$ invariant measure μ on $\mathbf{R}$ which is finite on compact sets and which is preserved by the covering translation $T(x) = x + 1$. This permits us to define the *mean translation number* with respect to μ.

Definition 2.5. *If* $F \in \mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R})$ *define its* mean translation number $\tau_\mu(F) \in \mathbf{R}$ *by*

$$\tau_\mu(F) = \begin{cases} \mu([x, F(x))) & \text{if } F(x) > x, \\ 0 & \text{if } F(x) = x, \\ -\mu([F(x), x)) & \text{if } F(x) < x, \end{cases}$$

where $x \in \mathbf{R}$.

We enumerate some of the well known basic properties of the mean translation number which we will use later.

Proposition 2.6. *Suppose* G *is a subgroup of* $\mathrm{Homeo}_+(S^1)$ *which preserves the Borel probability measure and* $\bar{G}$ *is the group of all lifts of elements of* G. *If* $F \in \bar{G}$ *the mean translation number* $\tau_\mu(F)$ *is independent of the point* x *used to define it. Indeed* $\tau_\mu(F) = \tau(F)$. *Moreover the function* $\tau_\mu : \bar{G} \to \mathbf{R}$ *is a homomorphism (and hence so is* $\tau : \bar{G} \to \mathbf{R}$.*)*

Proof. Consider the function

$$\nu(x, y) = \begin{cases} \mu([x, y))) & \text{if } y > x, \\ 0 & \text{if } y = x, \\ -\mu([y, x)) & \text{if } y < x. \end{cases}$$

It has the property that $\nu(x,y) + \nu(y,z) = \nu(x,z)$.

We also note that $\tau_\mu(F) = \nu(x, F(x)$. To see this is independent of x note for any $y \in \mathbf{R}$, $\nu(x, F(x)) = \nu(x,y) + \nu(y, F(y)) + \nu(F(y), F(x)) = \nu(y, F(y)) + \nu(x,y) - \nu(F(x), F(y))$. But F preserves the measure μ and the orientation of $\mathbf{R}$ so $\nu(x,y) = \nu(F(x), F(y))$. Hence $\nu(x, F(x) = \nu(y, F(y))$.

Since μ is the lift of a probability measure on S^1 we know that $\mu([x, x+k)) = k$ for any $k \in \mathbf{Z}$. So if $F^n(x) \in [x + k, x + k + 1)$ we see that

$$F^n(x) - x - 1 \leq k \leq \nu(x, F^n(x)) \leq k + 1 \leq F^n(x) - x + 1.$$

To see that $\tau_\mu(F) = \tau(F)$ we note

$$\begin{aligned} \tau(F) &= \lim_{n\to\infty} \frac{F^n(x) - x}{n} \\ &= \lim_{n\to\infty} \frac{\nu(x, F^n(x))}{n} \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \nu(F^i(x), F^{i+1}(x)) \\ &= \nu(x, F(x)) \\ &= \tau_\mu(F). \end{aligned}$$

$\diamond$

There are two well known corollaries of this result which we record here for later use.

Corollary 2.7. *Suppose G is a subgroup of* $\mathrm{Homeo}_+(S^1)$ *which preserves the Borel probability measure μ_0 and $\bar{G}$ is the group of all lifts of elements of G. Then each of the functions $\tau_\mu : \bar{G} \to \mathbf{R}$, $\tau : \bar{G} \to \mathbf{R}$ and $\rho : G \to S^1$ is a homomorphism.*

Proof. To see that $\tau_\mu : \bar{G} \to \mathbf{R}$ is a homomorphism we suppose $f, g \in \bar{G}$ and consider ν as defined above. Then $\tau_\mu(fg) = \nu(x, f(g(x))) = \nu(x, g(x)) + \nu(g(x), f(g(x))) == \tau_\mu(g) + \tau_\mu(f)$.

Since $\tau_\mu = \tau$ we know that τ is also a homomorphism. There is a natural homomorphism $\pi : \bar{G} \to G$ which assigns to g its projection on S^1. If $p : \mathbf{R} \to S^1 = \mathbf{R}/\mathbf{Z}$ is the natural projection then for any $f, g \in \bar{G}$,

$$\rho(\pi(fg)) = p(\tau(fg)) = p(\tau(f)) + p(\tau(g)) = \rho(\pi(f)) + \rho(\pi(g)).$$

Hence $\rho : G \to S^1$ is a homomorphism. $\diamond$

Corollary 2.8. *If G is an abelian subgroup of* $\mathrm{Homeo}_+(S^1)$ *and $\bar{G}$ is the group of all lifts to $\mathbf{R}$ of elements of G, then $\bar{G}$ is abelian and both $\tau : \bar{G} \to \mathbf{R}$ and $\rho : G \to S^1$ are homomorphisms.*

Proof. Since G is abelian it is amenable and there is a Borel probability measure μ_0 on S^1 invariant under G. Let μ be the lift of this measure to $\mathbf{R}$. Then Corollary 2.7 implies that $\tau : \bar{G} \to \mathbf{R}$ and $\rho : G \to S^1$ are homomorphisms.

As above let $\pi : \bar{G} \to G$ be the natural projection. The kernel of π is $\{T^n\}_{n \in \mathbf{Z}}$, where $T(x) = x + 1$, i.e., the lifts of the identity. If $f, g \in \bar{G}$ then $[f, g]$ is in the kernel of π so $[f, g] = T^k$, for some $k \in \mathbf{Z}$. But since $\bar{\rho}$ is a homomorphism $\rho([f, g]) = 0$ which implies $k = 0$. Hence $\bar{G}$ is abelian. $\diamond$

The following easy lemma turns out to be very useful.

Lemma 2.9. *Suppose G is a subgroup of* $\mathrm{Homeo}_+(S^1)$ *which preserves the Borel probability measure μ_0 and $f_0, g_0 \in G$ satisfy $\rho(f_0) = \rho(g_0)$. Then $f_0(x) = g_0(x)$ for all x in the support of μ_0.*

Proof. Let μ be the lift of the measure μ_0 to $\mathbf{R}$. Pick lifts f and g of f_0 and g_0 respectively which satisfy $\tau_\mu(f) = \tau_\mu(g)$. Then for any $x \in \mathbf{R}$ we have $\mu([x, f(x))) = \tau_\mu(f) = \tau_\mu(g) = \mu([x, g(x)))$.

Suppose that $g(x) > f(x)$. It then follows that $\mu([f(x), g(x)) = 0$, so for a sufficiently small $\epsilon > 0$, $\mu([f(x), f(x) + \epsilon]) = 0$ and $\mu([g(x) - \epsilon, g(x))) = 0$. Applying gf^{-1} to $[f(x), f(x) + \epsilon)$ we see that $\mu([g(x), g(x) + \epsilon')) = 0$. Hence $\mu([g(x) - \epsilon, g(x) + \epsilon')) = 0$ which is not possible if x and hence $g(x)$ is in the support of μ.

We have shown that $x \in \operatorname{supp}(\mu)$ implies $f(x) \leq g(x)$. The inequality $g(x) \leq f(x)$ is similar. Projecting back to S^1 we conclude that $f_0 = g_0$ on $\operatorname{supp}(\mu_0)$. $\diamond$

One additional well known property we will use is the following.

Proposition 2.10. *If $g \in \mathrm{Homeo}_+(S^1)$ is an irrational rotation then its centralizer $Z(g)$ in $\mathrm{Homeo}_+(S^1)$ is the group of rigid rotations of S^1.*

Proof. Let f be an element of $Z(g)$. Then the group generated by f and g is abelian and hence amenable so there it has an invariant Borel probability measure. But Lebesgue measure is the unique Borel probability measure invariant by the irrational rotation g. Since f preserves orientation and Lebesgue measure, it is a rotation. $\diamond$

3 Mapping Class Groups

In this section we prove Theorem 1.2. Several of the results, in particular those in §3.1, will also be used to prove Theorem 1.3.

3.1 Fully Supported Diffeomorphisms

One of the main techniques of this paper is to use Kopell's Lemma (Theorem 2.1) to understand actions of commuting diffeomorphisms. The key dichotomy that arises is the behavior of diffeomorphisms with an interval of fixed points versus those which we call *fully supported.*

If f is a homeomorphism of a manifold M, we will denote by $\partial \operatorname{Fix}(f)$ the frontier of $\operatorname{Fix}(f)$, i.e. the set $\partial \operatorname{Fix}(f) = \operatorname{Fix}(f) \setminus \operatorname{Int}(\operatorname{Fix}(f))$.

Definition 3.1 (Fully supported homeomorphism). *A homeomorphism f of a manifold M is* fully supported *provided that* $\operatorname{Int}(\operatorname{Fix}(f)) = \emptyset$*, or equivalently* $\partial \operatorname{Fix}(f) = \operatorname{Fix}(f)$. *A subgroup G of* $\mathrm{Homeo}(M)$ *will be called* fully supported *provided that every nontrivial element is fully supported.*

It is a trivial but useful observation that if M is connected then for every nontrivial homeomorphism f of M, the set $\operatorname{Fix}(f) \neq \emptyset$ if and only if $\partial \operatorname{Fix}(f) \neq \emptyset$.

The following lemma indicates a strong consequence of Kopell's Lemma (Theorem 2.1).

Lemma 3.2 (Commuting diffeomorphisms on I). *Suppose f and g are commuting orientation preserving C^2 diffeomorphisms of I. Then f preserves every component of* $\operatorname{Fix}(g)$ *and vice versa. Moreover,* $\partial \operatorname{Fix}(f)) \subset \operatorname{Fix}(g)$ *and and vice versa. In particular if f and g are fully supported then* $\operatorname{Fix}(f) = \operatorname{Fix}(g)$.

Proof. The proof is by contradiction. Assume X is a component of $\operatorname{Fix}(g)$ and $f(X) \neq X$. Since f and g commute $f(\operatorname{Fix}(g)) = \operatorname{Fix}(g)$ so $f(X) \neq X$ implies

$f(X) \cap X = \emptyset$. Let x be an element of X and without loss of generality assume $f(x) < x$. Define

$$a = \lim_{n \to \infty} f^n(x) \text{ and } b = \lim_{n \to -\infty} f^n(x).$$

Then a and b are fixed under both f and g and f has no fixed points in (a, b). Then Kopell's Lemma (Theorem 2.1) implies $g(y) = y$ for all $y \in [a, b]$, contradicting the hypothesis that X is a component of $\mathrm{Fix}(g)$. The observation that $\partial \mathrm{Fix}(f) \subset \mathrm{Fix}(g)$ follows from the fact that $x \in \partial \mathrm{Fix}(f)$ implies that either $\{x\}$ is a component of $\mathrm{Fix}(f)$ or x is the endpoint of an interval which is a component of $\mathrm{Fix}(f)$ so, in either case, $x \in \mathrm{Fix}(g)$. $\diamond$

There is also a version of Lemma 3.2 for the circle. For $g \in \mathrm{Homeo}_+(S^1)$ let $\mathrm{Per}(g)$ denote the set of periodic points of g.

Lemma 3.3 (Commuting diffeomorphisms on S^1). *Suppose f and g are commuting orientation preserving C^2 diffeomorphisms of S^1. Then f preserves every component of* $\mathrm{Per}(g)$ *and vice versa. Moreover,* $\partial \mathrm{Per}(f) \subset \mathrm{Per}(g)$ *and vice versa. In particular if neither* $\mathrm{Per}(f)$ *or* $\mathrm{Per}(g)$ *has interior then* $\mathrm{Per}(f) = \mathrm{Per}(g)$.

Proof. First consider the case that $\mathrm{Per}(g) = \emptyset$. The assertion that f preserves components of $\mathrm{Per}(g)$ is then trivial. Also, g is topologically conjugate to an irrational rotation which by Proposition 2.10 implies that f is topologically conjugate to a rotation and hence $\mathrm{Per}(f) = \emptyset$ or $\mathrm{Per}(f) = S^1$. In either case we have the desired result.

Thus we may assume both f and g have periodic points. Since for any circle homeomorphism all periodic points must have the same period, we can let p be the least common multiple of the periods and observe that $\mathrm{Per}(f) = \mathrm{Fix}(f^p)$ and $\mathrm{Per}(g) = \mathrm{Fix}(g^p)$. Since f and g commute and both f^p and g^p have fixed points, if $x \in \mathrm{Fix}(f^p)$ then $y = \lim_{n \to \infty} g^{np}(x)$ will exist and be a common fixed point for f^p and g^p. If we split the circle at y we obtain two commuting diffeomorphisms, f^p and g^p, of an interval to which we may apply Lemma 3.2 and obtain the desired result. $\diamond$

Lemma 3.4. *Let $g_0 \in \mathrm{Diff}^2_+(S^1)$ and let $Z(g_0)$ denote its centralizer. Then the rotation number $\rho : Z(g_0) \to S^1$ is a homomorphism.*

Proof. We first observe the result is easy if g_0 has no periodic points. In this case g_0 is conjugate in $\mathrm{Homeo}_+(S^1)$ to an irrational rotation by Denjoy's Theorem, and the centralizer of an irrational rotation is abelian by Proposition 2.10. So the centralizer of g in $\mathrm{Diff}^1_+(S^1)$ is abelian. But then ρ restricted to an abelian subgroup of $\mathrm{Homeo}_+(S^1)$ is a homomorphism by Proposition 2.8.

Thus we may assume g_0 has a periodic point, say of period p. Then $h_0 = g_0^p$ has a fixed point. Let h be a lift of h_0 to $\mathbf{R}$ which has a fixed point. Let G denote the group of all lifts to $\mathbf{R}$ of all elements of $Z(g_0)$. Note that every element of this group commutes with h because by Corollary 2.8 any lifts to $\mathbf{R}$ of commuting homeomorphisms of S^1 commute.

Let $X = \partial \operatorname{Fix}(h)$ and observe that $g(X) = X$ for all $g \in G$, so G acts on the unbounded closed set X and G acts on $X_0 = \partial \operatorname{Fix}(h_0)$ which is the image of X under the covering projection to S^1 . We will show that if $\operatorname{Fix}(g) \cap X \neq \emptyset$ then $X \subset \operatorname{Fix}(g)$. Indeed applying Lemma 3.3 to h_0 and map g_1 of S^1 which g covers, we observe that $\partial \operatorname{Fix}(h) \subset \operatorname{Fix}(g_1)$ which implies $X \subset \partial \operatorname{Fix}(g)$ if g has has a fixed point.

It follows that if $H = \{g \in G \mid g(x) = x \text{ for all } x \in X\}$ is the stabilizer of X then G/H acts freely on X and hence is abelian by Hölder's theorem. Also G/H acts on X_0 and hence there is a measure μ_0 supported on X_0 and invariant under G/H. Clearly this measure is also invariant under the action of $Z(g_0)$ on S^1. The measure μ_0 lifts to a G-invariant measure μ.

It follows from Corollary 2.7 that the translation number $\tau : G \to \mathbf{R}$ and the rotation number $\rho : Z(g_0) \to S^1$ are homomorphisms. $\diamond$

The *commutativity graph* of a set S of generators of a group is the graph consisting of one vertex for each $s \in S$ and an edge connecting elements of S which commute.

Theorem 3.5 (abelian criterion for I). *Let $\{g_1, g_2, \dots, g_k\}$ be a set of fully supported elements of $\operatorname{Diff}^2_+(I)$ and let G be the group they generate. Suppose that the commutativity graph of this generating set is connected. Then G is abelian.*

Proof. By Lemma 3.2 we may conclude that for each j_1, j_2 we have $\operatorname{Fix}(g_{j_1}) = \operatorname{Fix}(g_{j_2})$. Call this set of fixed points F. Clearly F is the set of global fixed points of G.

Fix a value of i and consider $Z(g_i)$. Restricting to any component U of the complement of F we consider the possiblity that there is an $h \in Z(g_i)$ with a fixed point in U. Kopell's Lemma (Theorem 2.1), applied to the closure of the open inteval U, tells us that such an h is the identity on U. Thus the restriction of $Z(g_i)$ to U is free and hence abelian by Hölder's Theorem. But U was an arbitrary component of the complement of F and obviously elements of $Z(g_i)$ commute on their common fixed set F. So we conclude that $Z(g_i)$ is abelian.

We have hence shown that if g_j and g_k are joined by a path of length two in the commutativity graph, they are joined by a path of length one. A straightforward induction shows that any two generators are joined by a path of length one, i.e. any two commute. $\diamond$

There is also a version of Theorem 3.5 for the circle.

Corollary 3.6 (Abelian criterion for S^1). *Let $\{g_1, g_2, \dots, g_k\}$ be a set of fully supported elements of $\operatorname{Diff}^2_+(S^1)$, each of which has a fixed point, and let G be the group they generate. Suppose that the commutativity graph of these generators is connected. Then G is abelian.*

Proof. By Lemma 3.3 we may conclude that for each j, k we have $\mathrm{Per}(g_j) = \mathrm{Per}(g_k)$. But since each g_i has a fixed point $\mathrm{Per}(g_i) = \mathrm{Fix}(g_i)$. Hence there is a common fixed point for all of the generators. Splitting S^1 at a common fixed point we see G is isomorphic to a subgroup of $\mathrm{Diff}^2_+(I)$ satisfying the hypothesis of Theorem 3.5. It follows that G is abelian. $\diamond$

Question. Let G be a subgroup of $\mathrm{Homeo}_+(S^1)$. If the commutativity graph for a set of generators of G is connected, does this imply that rotation number is a homomorphism when restricted to G?

We will need to understand relations between fixed sets not just of commuting diffeomorphisms, but also of diffeomorphisms with another basic relation which occurs in mapping class groups as well as in automorphism groups of free groups.

Lemma 3.7 (*aba* lemma). *Suppose a and b are elements of $\mathrm{Homeo}_+(I)$ or elements of $\mathrm{Homeo}_+(S^1)$ which have fixed points. Suppose also that a and b satisfy the relation $a^{n_1}b^{m_3}a^{n_2} = b^{m_1}a^{n_3}b^{m_2}$ with $m_1 + m_2 \neq m_3$. If J is a nontrivial interval in $\mathrm{Fix}(a)$ then either $J \subset \mathrm{Fix}(b)$ or $J \cap \mathrm{Fix}(b) = \emptyset$.*

Proof. Suppose $J \subset \mathrm{Fix}(a)$ and suppose $z \in J \cap \mathrm{Fix}(b)$. We will show that this implies that $J \subset \mathrm{Fix}(b)$.

Let $x_0 \in J$. Observe that $a^{n_1}b^{m_3}a^{n_2} = b^{m_1}a^{n_3}b^{m_2}$ implies $a^{-n_2}b^{-m_3}a^{-n_1} = b^{-m_2}a^{-n_3}b^{-m_1}$. Hence we may assume without loss of generality that $b^{m_3}(x_0)$ is in the subinterval of J with endpoints x_0 and z (otherwise replace a and b by their inverses). In particular $b^{m_3}(x_0) \in J$. Note that

$$b^{m_3}(x_0) = a^{n_1}b^{m_3}(x_0) = a^{n_1}b^{m_3}a^{n_2}(x_0) = b^{m_1}a^{n_3}b^{m_2}(x_0) = b^{m_1}b^{m_2}(x_0) = b^{m_1+m_2}(x_0)$$

This implies $b^{m_1+m_2-m_3}(x_0) = x_0$ and since $m_1 + m_2 - m_3 \neq 0$ we can conclude that $b(x_0) = x_0$. Since $x_0 \in J$ was arbitrary we see $J \subset \mathrm{Fix}(b)$. $\diamond$

3.2 Groups of type $MC(n)$

We will need to abstract some of the properties of the standard generating set for the mapping class group of a surface in order to apply them in other circumstances.

Definition 3.8 (Groups of type $MC(n)$). *We will say that a group G is* of type $MC(n)$ *provided it is nonabelian and has a set of generators $\{a_i, b_i\}_{i=1}^n \cup \{c_j\}_{j=1}^{n-1}$ with the properties that*

1. $a_ia_j = a_ja_i,\ b_ib_j = b_jb_i,\ c_ic_j = c_jc_i,\ a_ic_j = c_ja_i,$ *for all* i, j,
2. $a_ib_j = b_ja_i,$ *if* $i \neq j$,
3. $b_ic_j = c_jb_i,$ *if* $j \neq i, i-1$,

4. $a_ib_ia_i = b_ia_ib_i$ *for* $1 \leq i \leq n$, *and* $b_ic_jb_i = c_jb_ic_j$ *whenever* $j = i, i-1$.

A useful consequence of the $MC(n)$ condition is the following.

Lemma 3.9. *If* G *is a group of type* $MC(n)$ *and* $\{a_i, b_i\}_{i=1}^n \cup \{c_j\}_{j=1}^{n-1}$ *are the generators guaranteed by the definition then any two of these generators are conjugate in* G.

Proof. The relation $aba = bab$ implies $a = (ab)^{-1}b(ab)$, so a and b are conjugate. Therefore property (4) of the definition implies a_i is conjugate to b_i and b_i is conjugate to c_i and c_{i-1}. This proves the result. $\diamond$

The paradigm of a group of type $MC(n)$ is the mapping class group $\mathrm{Mod}(n, 0)$.

Proposition 3.10 ($\mathbf{Mod}(n, 0)$ and $\mathbf{Mod}(n, 1)$). *For* $n > 2$ *and* $k = 0, 1$ *the group* $\mathrm{Mod}(n, k)$ *contains a set of elements*

$$\mathcal{S} = \{a_i, b_i\}_{i=1}^n \cup \{c_j\}_{j=1}^{n-1}$$

with the following properties:

1. *For* $k = 0$ *the set* $\mathcal{S}$ *generates all of* $\mathrm{Mod}(n, 0)$.
2. *The group generated by* $\mathcal{S}$ *is of type* $MC(n)$.
3. *There is an element of* $g \in \mathrm{Mod}(n, k)$ *such that* $g^{-1}a_1g = a_n$, $g^{-1}b_1g = b_n$, $g^{-1}b_2g = b_{n-1}$, *and* $g^{-1}c_1g = c_{n-1}$.
4. *If* G_0 *is the subgroup of* $\mathrm{Mod}(n, k)$ *generated by the subset*

$$\{a_i, b_i\}_{i=1}^{n-1} \cup \{c_j\}_{j=1}^{n-2}$$

then G_0 *is of type* $MC(n-1)$.

Proof. Let Σ denote the surface of genus n with k punctures. Let $i(\alpha, \beta)$ denote the geometric intersection number of the (isotopy classes of) closed curves α and β on Σ. Let a_i, b_i, c_j with $1 \leq i \leq n, 1 \leq j \leq n$ be Dehn twists about essential, simple closed curves $\alpha_i, \beta_i, \gamma_j$ with the following properties:

- $i(\alpha_i, \alpha_j) = i(\beta_i, \beta_j) = i(\gamma_i, \gamma_j) = 0$ for $i \neq j$.
- $i(\alpha_i, \beta_i) = 1$ for each $1 \leq i \leq n$.
- $i(\gamma_j, \beta_j) = 1 = i(\gamma_j, \beta_{j+1})$ for $1 \leq j \leq n-1$.

It is known (see, e.g.[Iv]) that there exist such a_i, b_i, c_j which generate $\mathrm{Mod}(n, 0)$; we choose such elements. These are also nontrivial in $\mathrm{Mod}(n, 1)$. Since Dehn twists about simple closed curves with intersection number zero commute, and since Dehn twists a, b about essential, simple closed curves with intersection number one satisy $aba = bab$ (see, e.g. Lemma 4.1.F of [Iv]), properties (1)-(4) in Definition 3.8 follow. We note that the proofs of these relations do not depend on the location of the puncture, as long as it is chosen off the curves $\alpha_i, \beta_i, \gamma_j$.

To prove item (2) of the lemma, consider the surface of genus $g-2$ and with 2 boundary components obtained by cutting Σ along α_1 and γ_1. The loop β_1 becomes a pair of arcs connecting 2 pairs of boundary components, and β_2 becomes an arc connecting 2 boundary components. The genus, boundary components, and combinatorics of arcs is the exact same when cutting Σ along α_n and γ_{n-1}. Hence by the classification of surfaces it follows that there is a homeomorphism between resulting surfaces, inducing a homeomorphism $h : \Sigma \to \Sigma$ with $h(\alpha_1) = \alpha_n, h(\gamma_1) = \gamma_{n-1}$ and $h(\beta_i) = \beta_{n-i+1}, i = 1, 2$. The homotopy class of h gives the required element g.

To prove item (3), let τ be an essential, separating, simple closed curve on Σ such that one of the components Σ' of $\Sigma - \tau$ has genus one and contains a_n and b_n. Then there is a homeomorphism $h \in \mathrm{Homeo}_+(\Sigma)$ taking any element of $\{a_i, b_i\}_{i=1}^{n-1} \cup \{c_j\}_{j=1}^{n-2}$ to any other element, and which is the identity on Σ'. Then the isotopy class of h, as an element of $\mathrm{Mod}(n,0)$, is the required conjugate, since it lies in the subgroup of $\mathrm{Mod}(0,n)$ of diffeomorphisms supported on $\Sigma - \Sigma'$, which is isomorphic to $\mathrm{Mod}(n-1,1)$ and equals G_0. $\diamond$

3.3 Actions on the interval

In this subsetion we consider actions on the interval $I = [0,1]$.

Theorem 3.11 ($MC(n)$ actions on I). *Any C^2 action of a group G of type $MC(n)$ for $n \geq 2$ on an interval I is abelian.*

Proof. We may choose a set of generators $\{a_i, b_i\}_{i=1}^{n} \cup \{c_j\}_{j=1}^{n-1}$ for G with the properties listed in Definition 3.8.

If G has global fixed points other than the endpoints of I we wish to consider the restriction of the action to the closure of a component of the complement of these global fixed points. If all of these restricted actions are abelian then the original action was abelian. Hence it suffices to prove that these restrictions are abelian. Thus we may consider an action of G on a closed interval I_0 with no interior global fixed points. None of the generators above can act trivially on I_0 since the fact that they are all conjugate would mean they all act trivially.

We first consider the case that one generator has a nontrivial interval of fixed points in I_0 and show this leads to a contradiction. Since they are all conjugate, each of the generators has a nontrivial interval of fixed points.

Choose an interval J of fixed points which is maximal among all intervals of fixed points for all of the a_i. That is, J is a nontrivial interval of fixed points for one of the a_i, which we assume (without loss of generality) is a_2, and there is no J' which properly contains J and which is pointwise fixed by some a_j.

Suppose $J = [x_0, x_1]$. At least one of x_0 and x_1 is not an endpoint of I_0 since a_2 is not the identity. Suppose it is x_0 which is in $\mathrm{Int}(I_0)$. Then $x_0 \in \partial\,\mathrm{Fix}(a_2)$, so by Lemma 3.2 we know that x_0 is fixed by all of the generators except possibly b_2, since b_2 is the only generator with which a_2 does not commute. The point

x_0 cannot be fixed by b_2 since otherwise it would be an interior point of I_0 fixed by all the generators, but there are no global fixed points in I_0 other than the endpoints. The identity $a_2b_2a_2 = b_2a_2b_2$ together with Lemma 3.7 then tells us that $b_2(J) \cap J = \emptyset$.

Now assume without loss of generality that $b_2(x_0) > x_0$ (otherwise replace all generators by their inverses). Define

$$y_0 = \lim_{n\to-\infty} b_2^n(x_0) \text{ and } y_1 = \lim_{n\to\infty} b_2^n(x_0).$$

Since $b_2(J) \cap J = \emptyset$ we have $b_2^{-1}(J) \cap J = \emptyset$ and hence $y_0 < x_0 < x_1 < y_1$. But x_0 is a fixed point of a_1 (as well as the other generators which commute with a_2) and a_1 commutes with b_2. Hence y_0 and y_1 are fixed by a_1 (since b_2 preserves $\mathrm{Fix}(a_1)$). We can now apply Kopell's Lemma (Theorem 2.1) to conclude that a_1 is the identity on $[y_0, y_1]$ which contradicts the fact that $J = [x_0, x_1]$ is a maximal interval of fixed points among all the a_i.

We have thus contradicted the supposition that one of the generators has a nontrivial interval of fixed points in I_0, so we may assume that each of the generators, when restricted to I_0, has fixed point set with empty interior. That is, each generator is fully supported on I_0.

We now note that given any two of the generators above, there is another generator h with which they commute. Thus we may conclude from Lemma 3.5 that the action of G on I_0 abelian. $\diamond$

Proof of Theorem 1.2 for I: By Proposition 3.10, the mapping class group $\mathrm{Mod}(g,k), g \geq 2, k = 0,1$ is a group of type $MC(g)$. Since every abelian quotient of $\mathrm{Mod}(g,k), g \geq 2$ is finite (see, e.g. [Iv]), in fact trivial for $g \geq 3$, and since finite groups must act trivially on I, the statement of Theorem 1.2 for I follows.

3.4 Actions on the circle

We are now prepared to prove Theorem 1.2 in the case of the circle S^1.

Theorem 1.2 (Circle case). *Any C^2 action of the mapping class group $M(g,k)$ of a surface of genus $g \geq 3$ with $k = 0,1$ punctures on S^1 must be trivial.*

As mentioned in the introduction, the C^2 hypothesis is necessary.

Proof. **Closed case.** We first consider the group $G = \mathrm{Mod}(g,0)$, and choose generators $\{a_i, b_i\}_{i=1}^n \cup \{c_j\}_{j=1}^{n-1}$ for G with the properties listed in Definition 3.8 and Proposition 3.10. All of these elements are conjugate by Lemma 3.9, so they all have the same rotation number.

We wish to consider a subgroup G_0 of type $MC(n-1)$. We let $u = a_n^{-1}$ and for $1 \leq i \leq n-1$ we set $A_i = a_i u$ and $B_i = b_i u$. For $1 \leq i \leq n-2$ we let $C_i = c_i u$. Since u commutes with any element of $\{a_i, b_i\}_{i=1}^{n-1} \cup \{c_j\}_{j=1}^{n-2}$ we have the relations

1. $A_iA_j = A_jA_i,\ B_iB_j = B_jB_i,\ C_iC_j = C_jC_i,\ A_iC_j = C_jA_i$, for all i, j,
2. $A_iB_j = B_jA_i$, if $i \neq j$,
3. $B_iC_j = C_jB_i$, if $j \neq i, i+1$,
4. $A_iB_iA_i = B_iA_iB_i$ and $B_iC_jB_i = C_jB_iC_j$ whenever $j = i, i-1$.

We now define G_0 to be the subgroup of G generated by $\{A_i, B_i\}_{i=1}^{n-1} \cup \{C_j\}_{j=1}^{n-2}$. The fact that u commutes with each of $\{a_i, b_i\}_{i=1}^{n-1} \cup \{c_j\}_{j=1}^{n-2}$ and has the opposite rotation number implies that the rotation number of every element of $\{A_i, B_i\}_{i=1}^{n-1} \cup \{C_j\}_{j=1}^{n-2}$ is 0. Thus each of these elements has a fixed point.

We note that given any two of these generators of G_0 there is another generator with which they commute. If all of these generators have fixed point sets with empty interior then we may conclude from Lemma 3.2 that any two of them have equal fixed point sets, i.e. that $\mathrm{Fix}(A_i) = \mathrm{Fix}(B_j) = \mathrm{Fix}(C_k)$ for all i, j, k. So in this case we have found a common fixed point for all generators of G_0. If we split S^1 at this point we get an action of G_0 on an interval I which must be abelian by Theorem 3.11. It follows that all of the original generators $\{a_i, b_i\}_{i=1}^{n} \cup \{c_j\}_{j=1}^{n-1}$ commute with each other except possibly c_{n-1} may not commute with b_{n-1} and b_n, and a_n and b_n may not commute.

Let $\psi : G \to \mathrm{Diff}^2_+(S^1)$ be the putative action, and consider the element g guaranteed by Proposition 3.10. We have that

$$[\psi(a_n), \psi(b_n)] = \psi(g)^{-1}[\psi(a_1), \psi(b_1)]\psi(g) = id$$

and

$$[\psi(c_{n-1}), \psi(b_n)] = \psi(g)^{-1}[\psi(c_1), \psi(b_1)]\psi(g) = id$$

and

$$[\psi(c_{n-1}), \psi(b_{n-1})] = \psi(g)^{-1}[\psi(c_1), \psi(b_2)]\psi(g) = id$$

Hence G is abelian, hence trivial since (see [Iv]) abelian quotients of $\mathrm{Mod}(g, k), g \geq 3$ are trivial.

Thus we are left with the case that one generator of G_0 has a nontrivial interval of fixed points in S^1. Since they are all conjugate, each of the generators of G_0 has has a nontrivial interval of fixed points. Also we may assume no generator fixes every point of S^1 since if one did the fact that they are all conjugate would imply they all act trivially.

Choose a maximal interval of fixed points J for any of the subset of generators $\{A_i\}$. That is, J is a nontrivial interval of fixed points for one of the $\{A_i\}$, which we assume (without loss of generality) is A_2, and there is no J' which properly contains J and which is pointwise fixed by some A_j.

Suppose x_0 and x_1 are the endpoints of J. Then $x_0, x_1 \in \partial\,\mathrm{Fix}(A_2)$ so by Lemma 3.3, x_0 and x_1 are fixed points for all of the generators except B_2, since B_2 is the only one with which A_2 does not commute. If the point $x_0 \in \mathrm{Fix}(B_2)$ we have found a common fixed point for all the generators of G_0 and we can split S^1 at this point obtaining an action of G_0 on an interval which implies G_0 is abelian by Theorem 3.11. So suppose $x_0 \notin \mathrm{Fix}(B_2)$.

Recall that the diffeomorphism B_2 has rotation number 0, so it fixes some point. Thus the identity $A_2B_2A_2 = B_2A_2B_2$, together with the fact that B_2 cannot fix x_0, implies by Lemma 3.7 that $B_2(J) \cap J = \emptyset$ and $J \cap B_2^{-1}(J) = \emptyset$. Define

$$y_0 = \lim_{n \to -\infty} B_2^n(x_0) \text{ and } y_1 = \lim_{n \to \infty} B_2^n(x_0).$$

We will denote by K the interval in S^1 with endpoints y_0 and y_1 which contains x_0. Note that K properly contains J and that $B_2(K) = K$.

Since $x_0 \in \mathrm{Fix}(A_1)$ and A_1 commutes with B_2 we conclude y_0 and y_1 are fixed by A_1. We can now apply Kopell's Lemma (Theorem 2.1) to the interval K with diffeomorphisms B_2 and A_1 to conclude that A_1 is the identity on K. But J is a proper subinterval of K which contradicts the fact that J was a maximal interval of fixed points for any one of the A_i. Thus in this case too we have arrived at a contradiction.

Punctured case. We now consider the group $\mathrm{Mod}(g, 1)$, and choose elements $\{a_i, b_i\}_{i=1}^n \cup \{c_j\}_{j=1}^{n-1}$ as above. The argument above shows that the subgroup G of $\mathrm{Mod}(g, 1)$ generated by these elements [1] acts trivially on S^1. We claim that the normal closure of G in $\mathrm{Mod}(g, 1)$ is all of $\mathrm{Mod}(g, 1)$, from which it follows that $\mathrm{Mod}(g, 1)$ acts trivially, finishing the proof.

To prove the claim, let S_g denote the closed surface of genus g and recall the exact sequence (see, e.g. [Iv]):

$$1 \to \pi_1(S_g) \to \mathrm{Mod}(g, 1) \to \mathrm{Mod}(g, 0) \to 1$$

where $\pi_1(Sigma_g)$ is generated by finitely many "pushing the point" homeomorphisms p_τ around generating loops τ in $\pi_1(S_g)$ with basepoint the puncture. Each generating loop τ has intersection number one with exactly one of the loops, say β_i, corresponding to one of the twist generators of G. Conjugating b_i by p_τ gives a twist about a loop β_i' which together with β_i bounds an annulus containing the puncture. The twist about β_i composed with a negative twist about β_i' gives the isotopy class of p_τ. In this way we see that the normal closure of G in $\mathrm{Mod}(g, 1)$ contains G together with each of the generators of the kernel of the above exact sequence, proving the claim.

$\diamond$

4 $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$ and other subgroups

In §4.1 we prove Theorem 1.3. Note that the result of Bridson-Vogtmann mentioned in the introduction implies this theorem since it is easy to see that any finite subgroup of $\mathrm{Homeo}_+(S^1)$ is abelian so their result implies ours. We give our proof here because it is short, straightforward and provides a good illustration of the use of the techniques developed above. While several aspects of the

[1] We do not know wether or not G actually equals $\mathrm{Mod}(g, 1)$; there seems to be some confusion about this point in the literature.

proof are similar to the case of $\mathrm{Mod}(g, 0)$, we have not been able to find a single theorem from which both results follow.

In §4.2 we prove theorem 1.4, extending the application to finite index subgroups of $\mathrm{Mod}(g, k)$ and $\mathrm{Aut}(F_n)$.

4.1 Actions of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$

We begin with a statement of a few standard facts about generators and relations of $\mathrm{Aut}(F_n)$.

Lemma 4.1 (Generators for $\mathrm{Aut}(F_n)$). *The group $\mathrm{Aut}(F_n)$ has a subgroup of index two which has a set of generators $\{A_{ij}, B_{ij}\}$ with $i \neq j,\ 1 \leq i \leq n$ and $1 \leq j \leq n$. These generators satisfy the relations*

$$A_{ij}A_{kl} = A_{kl}A_{ij} \text{ and } B_{ij}B_{kl} = B_{kl}B_{ij} \text{ if } \{i,j\} \cap \{k,l\} = \emptyset \tag{i}$$

$$[A_{ij}, A_{jk}] = A_{ik}^{-1} \text{ and } [A_{ij}, A_{jk}^{-1}] = A_{ik} \tag{ii}$$

$$[B_{ij}, B_{jk}] = B_{ik}^{-1} \text{ and } [B_{ij}, B_{jk}^{-1}] = B_{ik} \tag{iii}$$

$$A_{ij}A_{ji}^{-1}A_{ij} = A_{ji}^{-1}A_{ij}A_{ji}^{-1} \text{ and } B_{ij}B_{ji}^{-1}B_{ij} = B_{ji}^{-1}B_{ij}B_{ji}^{-1} \tag{iv}$$

Proof. Let $\{e_i\}_{i=1}^n$ be the generators of F_n and define automorphisms A_{ij} and B_{ij} by

$$A_{ij}(e_k) = \begin{cases} e_i e_j, & \text{if } i = k, \\ e_k, & \text{otherwise, and} \end{cases}$$

$$B_{ij}(e_k) = \begin{cases} e_j e_i, & \text{if } i = k, \\ e_k, & \text{otherwise.} \end{cases}$$

Then $\{A_{ij}, B_{ij}\}$ with $i \neq j$, generate the index two subgroup of $\mathrm{Aut}(F_n)$ given by those automorphisms which induce on the abelianization $\mathbf{Z}^n$ of F_n an automorphism of determinant one (see, e.g. [LS]). A straightforward but tedious computation shows that relations (i) – (iv) are satisified. $\diamond$

We next find some fixed points.

Lemma 4.2. *Suppose $n > 4$ and $\phi : \mathrm{Aut}(F_n) \to \mathrm{Diff}^2_+(S^1)$ is a homomorphism with $a_{ij} = \phi(A_{ij})$ and $b_{ij} = \phi(B_{ij})$ where A_{ij} and B_{ij} are the generators of Lemma 4.1. Then each of the diffeomorphisms a_{ij} and b_{ij} has a fixed point.*

Proof. We fix i, j and show a_{ij} has a fixed point. Since $n > 4$ there is an a_{kl} with $\{i,j\} \cap \{k,l\} = \emptyset$. Let $Z(a_{kl})$ denote the centralizer of a_{kl}, so $a_{ij} \in Z(a_{kl})$. Also, since $n > 4$ there is $1 \leq q \leq n$ which is distinct from i, j, k, l, so $a_{iq}, a_{qj} \in Z(a_{kl})$.

By Lemma 3.4 we know that the rotation number $\rho : Z(a_{kl}) \to \mathbf{R}$ is a homomorphism so $\rho(a_{ij}) = \rho([a_{iq}, a_{qj}]) = 0$. This implies that a_{ij} has a fixed point. $\diamond$

We can now prove the main result of this section

Theorem 1.3. *For $n \geq 6$, any homomorphisms from* $\mathrm{Aut}(F_n)$ *or or* $\mathrm{Out}(F_n)$ *to* $\mathrm{Diff}^2_+(I)$ *or* $\mathrm{Diff}^2_+(S^1)$ *factors through* $\mathbf{Z}/2\mathbf{Z}$.

Proof. Let H be the index two subgroup of $\mathrm{Aut}(F_n)$ from Lemma 4.1. Let ϕ be a homomorphism H to $\mathrm{Diff}^2_+(I)$ or $\mathrm{Diff}^2_+(S^1)$ and let $a_{ij} = \phi(A_{ij})$ and $b_{ij} = \phi(B_{ij})$ where A_{ij} and B_{ij} are the generators from Lemma 4.1. Then $\{a_{ij}, b_{ij}\}$ are generators for $Image(\phi)$. We will show in fact that the subgroup G_a generated by $\{a_{ij}\}$ is trivial, as is the subgroup G_b generated by $\{b_{ij}\}$. The arguments are identical so we consider only the set $\{a_{ij}\}$. Since $n \geq 6$, we have by Lemma 4.1 that the commutavity graph for the generators $\{a_{ij}\}$ of G_a is connected. Hence by Corollary 3.6, if these generators are fully supported then G_a is abelian. But the relations $[a_{ij}, a_{jk}^{-1}] = a_{ik}$ then imply that G_a is trivial.

Hence we may assume at least one of these generators is not fully supported. Let J be an interval which is a maximal component of fixed point sets for any of the a_i. More precisely, we choose J so that there is a_{pq} with J a component of $\mathrm{Fix}(a_{pq})$ and so that there is no a_{kl} such that $\mathrm{Fix}(a_{kl})$ properly contains J.

We wish to show that J is fixed pointwise by each a_{ij}. In case $\{p, q\} \cap \{i, j\} = \emptyset$ we note that at least the endpoints of J are fixed by a_{ij} because a_{pq} and a_{ij} commute and the endpoints of J are in $\partial\,\mathrm{Fix}(a_{pq})$. So Lemma 3.2 or Lemma 3.3 implies these endpoints are fixed by a_{ij}.

For the general case we first show $a_{ij}(J) \cap J \neq \emptyset$. In fact there is no a_{ij} with the property that $a_{ij}(J) \cap J = \emptyset$ because if there were then the interval J' defined to be the smallest interval containing $\{a_{ij}^n(J)\}_{n \in \mathbf{Z}}$ is an a_{ij}-invariant interval with no interior fixed points for a_{ij}. (In case $G_a \subset \mathrm{Diff}^2_+(S^1)$ we need the fact that a_{ij} has a fixed point, which follows from Lemma 4.2, in order to know this interval exists.) But since $n \geq 6$ there is some a_{kl} which commutes with both a_{ij} and a_{pq}, and it leaves both J and J' invariant. Applying Kopell's Lemma (Theorem 2.1) to a_{kl} and a_{ij} we conclude that $J' \subset \mathrm{Fix}(a_{kl})$, which contradicts the maximality of J. Hence we have shown that $a_{ij}(J) \cap J \neq \emptyset$ for all i, j.

Now by Lemma 4.1 the relation $a_{pq} a_{qp}^{-1} a_{pq} = a_{qp}^{-1} a_{pq} a_{qp}^{-1}$, holds so we may apply Lemma 3.7 to conclude that J is fixed pointwise by a_{qp}.

We next consider the generator a_{pk}. If x_0 is an endpoint of J then since $a_{qk}(J) \cap J \neq \emptyset$ at least one of $a_{qk}(x_0)$ and $a_{qk}^{-1}(x_0)$ must be in J. Hence the relations $[a_{pq}, a_{qk}] = a_{pk}^{-1}$ and $[a_{pq}, a_{qk}^{-1}] = a_{pk}$ imply that $a_{pk}(x_0) = x_0$. This holds for the other endpoint of J as well, so J is invariant under a_{pk} for all k. The same argument shows J is invariant under a_{qk} for all k. But since a_{pq} is the identity on J we conclude from $[a_{pq}, a_{qk}] = a_{pk}^{-1}$ that a_{pk} is the identity on J. Similarly a_{qk} is the identity on J.

Next the relation $a_{qk} a_{kq}^{-1} a_{qk} = a_{kq}^{-1} a_{qk} a_{kq}^{-1}$, together with Lemma 3.7 implies that a_{kq} is the identity on J. A similar argument gives the same result for a_{kp}. Finally, the relation $[a_{ip}, a_{pj}^{-1}] = a_{ij}$ implies that a_{ij} is the identity on J.

Thus we have shown that any subgroup of $\mathrm{Diff}^2_+(I)$ or $\mathrm{Diff}^2_+(S^1)$ which is a homomorphic image of H, the index two subgroup of $\mathrm{Aut}(F_n)$, has an interval of global fixed points. In the case of $\mathrm{Diff}^2_+(S^1)$ we can split at a global fixed point to get a subgroup of $\mathrm{Diff}^2_+(I)$ which is a homomorphic image of H. In the I case we can restrict the action to a subinterval on which the action has no global fixed point. But our result then says that for the restricted action there is an interval of global fixed points, which is a contradiction. We conclude that the subgroup of G_a generated by $\{a_{ij}\}$ is trivial and the same argument applies to the subgroup G_b generated by $\{b_{ij}\}$. So $\phi(H)$ is trivial. Since $\mathrm{Aut}(F_n)/H$ has order two any homomorphism ϕ from $\mathrm{Aut}(F_n)$ to $\mathrm{Diff}^2_+(I)$ or $\mathrm{Diff}^2_+(S^1)$ has an image whose order is at most two. The fact that $\mathrm{Diff}^2_+(I)$ has no elements of finite order implies that in this case ϕ is trivial.

Since there is a natural homomorphism from $\mathrm{Aut}(F_n)$ onto the group $\mathrm{Out}(F_n)$, the statement of Theorem 1.3 for $\mathrm{Out}(F_n)$ holds. $\diamond$

4.2 Finite index and other subgroups

In this section we prove Theorem 1.4. Recall that the Torelli group $T_{g,k}$ is the subgroup of $\mathrm{Mod}(g,k)$ consisting of diffeomorphisms of the k-punctured, genus g surface $\Sigma_{g,k}$ which act trivially on $H^1(\Sigma_{g,k};\mathbf{Z})$. Of course any finite index subgroup of $\mathrm{Mod}(g,k)$ contains a finite index subgroup of $T_{g,k}$, so we need only prove the thoerem for the latter group; that is, (3) implies (1).

So, let H be any finite index subgroup of $T_{g,k}, g \geq 3, k \geq 0$ or of $\mathrm{Aut}(F_n)$ or $\mathrm{Out}(F_n)$ for $n \geq 6$. Suppose that H acts faithfully as a group of C^2 diffeomorphisms of I or S^1, so that no nontrivial element of H has an interval of fixed points (i.e. the action is fully supported, in the terminology of §3.1). In each case we claim that H contains infinite order elements a, b, c with the property that, for each $r \geq 2$, the elements a^r, b^r generate an infinite, non-abelian group which commutes with c^s for all s.

To prove the claim for $T_{g,k}$ one takes a, b, c to be Dehn twists about homologically trivial, simple closed curves with a and b having positive intersection number and c disjoint from both. For r sufficiently large the elements a^r, b^r, c^r clearly lie in H. As powers of Dehn twists commute if and only if the curves have zero intersection number, the claim is proved for T_g, k. A direct calculation using appropriate generators A_{ij} as in Lemma 4.1 gives the same claim for $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ for $n > 2$.

Theorem 1.4 in the case of I now follows immediately from Theorem 3.5 and the fact that the group $< a^r, b^r, c^r >$ has connected commuting graph but is non-abelian. For the case of S^1, note that if c^r has a fixed point for any $r > 0$ so do a^r, b^r by Lemma 3.2, so after taking powers we are done by the same argument, this time applying Corollary 3.6.

If no positive power of c has a fixed point on S^1, then c must have irrational rotation number, in which case the centralizer of c in $\mathrm{Homeo}(S^1)$ is abelian by 2.10. But $< a, b >$ lie in this centralizer and do not commute, a contradiction.

The proof of Theorem 1.4 for $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ is similar to the above, so we leave it to the reader.

5 Baumslag-Solitar groups

In this section we study actions of the groups

$$\mathrm{BS}(m,n) =< a, b : ab^m a^{-1} = b^n >$$

For $n > m > 1$ these groups are not residually finite, hence are not linear (see, e.g. [LS]).

5.1 An analytic action of $BS(m,n)$ on $\mathbf{R}$

In this subsection we construct an example of an analytic action of $BS(m,n)$, $1 < m < n$ on $\mathbf{R}$. It is straightforward to find diffeomorphisms of $\mathbf{R}$ which satisfy the Baumslag-Solitar relation.

Proposition 5.1 (Diffeomorphisms satisfying the Baumslag-Solitar relation). *Let $f_n : S^1 \to S^1$ be any degree n covering and let $f_m : S^1 \to S^1$ be any degree m covering. Let g_n and g_m be lifts of f_n and f_m respectively to the universal cover $\mathbf{R}$. Then the group of homeomorphisms of $\mathbf{R}$ generated by $g = g_m g_n^{-1}$ and the covering translation $h(x) = x + 1$ satisfies the relation $gh^m g^{-1} = h^n$.*

Proof. We have

$$\begin{aligned} g_n^{-1}(h^n(x)) &= g_n^{-1}(x+n) \\ &= g_n^{-1}(x) + 1 \\ &= h(g_n^{-1}(x)), \end{aligned}$$

and

$$\begin{aligned} g_m(h(x)) &= g_m(x+1) \\ &= g_m(x) + m \\ &= h^m(g_m(x)). \end{aligned}$$

So

$$\begin{aligned} g(h^n(x)) &= g_m(g_n^{-1}(h^n(x))) \\ &= g_m(h(g_n^{-1}(x))) \\ &= h^m(g_m(g_n^{-1}(x))) \\ &= h^m(g(x)). \end{aligned}$$

◇

We will now show how to choose the diffeomorphisms constructed in Proposition 5.1 more carefully so that we will be able to use a Schottky type argument to show that the diffeomorphisms will satisfy no other relations. Part of the difficulty will be that the "Schottky sets" will have to have infinitely many components.

We first construct, for each $j \geq 1$, a C^∞ diffeomorphism Θ_j of $\mathbf{R}$ which is the lift of a map of degree j on S^1. Let $a = 1/10$. We choose a C^∞ function $\Theta_j : (-a/2, 1 + a/2) \to \mathbf{R}$ with the following properties:

$$\begin{aligned}
&\Theta_j(x) = \frac{a}{2}x \text{ for } x \in (-a/2, a/2), \\
&\Theta_j(x) = j + \frac{a}{2}(x-1) \text{ for } \in (a, 1 + a/2), \\
&\Theta_j'(x) > 0.
\end{aligned}$$

Note that for $x \in (-a/2, a/2)$ we have $\Theta_j(x+1) = \Theta_j(x) + j$. It follows that we can extend Θ_j to all of $\mathbf{R}$ by the rule $\Theta_j(x+k) = \Theta_j(x) + jk$. Thus the extended Θ_j is the lift of a map of degree j on S^1. This map is in fact a covering map since Θ_j is a diffeomorphism.

We shall be particularly interested in the intervals $A = [-a, a]$, $A^s = [-a, 0]$, $A^u = [0, a]$, and $C = [a, 1-a]$. Note that $[0,1] = A^u \cup C \cup (A^s + 1)$. Also by construction $\Theta_j(A^s) \subset A^s$ and $\Theta_j(C) \subset (A^s + j)$. More importantly we have the following inclusions concerning the union of integer translates of these sets. For a set $X \subset \mathbf{R}$ and a fixed i_0, we will denote by $X + i_0$ the set $\{x + i_0 | \; x \in X\}$, by $X + \mathbf{Z}$ the set $\{x + i | \; x \in X, \; i \in \mathbf{Z}\}$, and by $X + m\mathbf{Z}$ the set $\{x + mi | \; x \in X, \; i \in m\mathbf{Z}\}$.

Suppose k is not congruent to 0 modulo j. Then we have

$$\begin{aligned}
\Theta_j(A^s + \mathbf{Z}) &\subset A^s + j\mathbf{Z}, \\
\Theta_j(C + \mathbf{Z}) &\subset A^s + j\mathbf{Z}, \\
\Theta_j(A^u + j\mathbf{Z} + k) &\subset A^s + j\mathbf{Z} \\
\Theta_j^{-1}(C + \mathbf{Z}) &\subset A^u + \mathbf{Z}.
\end{aligned}$$

We now define two diffeomorphisms of $\mathbf{R}$. Let $g_n(x) = \Theta_n(x)$ and let $g_m(x) = \Theta_m(x - 1/2) + 1/2$. We define $B = A + 1/2$, $B^s = A^s + 1/2$, $B^u = A^u + 1/2$, and $C_2 = C + 1/2$. Then from the equations above we have

$$\begin{aligned}
g_n(C + \mathbf{Z}) &\subset A^s + n\mathbf{Z} \\
g_n^{-1}(C + \mathbf{Z}) &\subset A^u + \mathbf{Z} \\
g_n^{-1}(A + k) &\subset A^u + \mathbf{Z} \text{ if } k \notin n\mathbf{Z} && (1) \\
g_m(C_2 + \mathbf{Z}) &\subset B^s + m\mathbf{Z} \\
g_m^{-1}(C_2 + \mathbf{Z}) &\subset B^u + \mathbf{Z} && (2) \\
g_m^{-1}(B + k) &\subset B^u + \mathbf{Z} \text{ if } k \notin m\mathbf{Z} && (3)
\end{aligned}$$

The diffeomorphisms g_m and g_n can be approximated by analytic diffeomorphisms which still satisfy the equations above and are still lifts of covering maps on S^1. More precisely, the function $\phi(x) = g_n(x) - nx$ is a periodic function on $\mathbf{R}$ and may be C^1 approximated by a periodic analytic function. If we replace

g_n by $\phi(x) + nx$ then the new analytic g_n will have postive derivative and be a lift of a degree n map on S^1. If the approximation is sufficiently close, it will also still satisfy the equations above. In a similar fashion we may perturb g_m to be analytic while retaining its properties.

Note that $B + \mathbf{Z} \subset C + \mathbf{Z}$ and $A + \mathbf{Z} \subset C_2 + \mathbf{Z}$, so we have the following key properties:

$$g_n(B + \mathbf{Z}) \subset A^s + n\mathbf{Z}, \tag{4}$$

$$g_n^{-1}(B + \mathbf{Z}) \subset A^u + \mathbf{Z} \tag{5}$$

$$g_m(A + \mathbf{Z}) \subset B^s + m\mathbf{Z} \tag{6}$$

$$g_m^{-1}(A + \mathbf{Z}) \subset B^u + \mathbf{Z} \tag{7}$$

Define $g(x) = g_n(g_m^{-1}(x))$ and $h(x) = x+1$, and let G be the group generated by g and h. Propostion 5.1 implies that $gh^m g^{-1} = h^n$. Hence there is a surjective homomorphism $\Phi : \mathrm{BS}(m, n) \to G$ sending b to g and a to h. We need only show that Φ is injective.

Lemma 5.2 (normal forms). *In the group $BS(m, n)$ with generators b and a satisfying the relation $ba^m b^{-1} = a^n$ every nontrivial element can be written in the form*

$$a^{r_n} b^{e_n} a^{r_{n-1}} b^{e_{n-1}} \dots a^{r_1} b^{e_1} a^{r_0}$$

where $e_i = \pm 1$ and $r_i \in \mathbf{Z}$ have the property that whenever $e_i = 1$ and $e_{i-1} = -1$ we have $r_i \notin m\mathbf{Z}$, and whenever $e_i = -1$ and $e_{i-1} = 1$ we have $r_i \notin n\mathbf{Z}$.

While Lemma 5.2 follows immediately from the normal form theorem for HNN extensions, we include a proof here since it is so simple in this case.

Proof. If we put no restrictions on the integers r_i and, in particular allow them to be 0, then it is trivially true that any element can be put in the form above. If for any i, $e_i = 1$, $e_{i-1} = -1$ and $r_i = mk \in m\mathbf{Z}$ then the fact that $ba^{mk}b^{-1} = a^{nk}$ allows us to substitute and obtain another expression in the form above, representing the same element of $\mathrm{BS}(m, n)$, but with fewer occurences of the terms b and b^{-1}. Similarly if $e_i = -1$, $e_{i-1} = 1$ and $r_i = nk \in n\mathbf{Z}$ then $b^{-1}a^{nk}b$ can be replaced with a^{mk} further reducing the occurences of the terms b and b^{-1}.

These substitutions can be repeated at most a finite number of times after which we have the desired form. $\diamond$

We now show that $\mathrm{BS}(m, n)$ is a subgroup of $\mathrm{Diff}_+^\omega(\mathbf{R})$. Note that $\mathrm{Diff}_+^\omega(\mathbf{R}) \subset \mathrm{Homeo}_+(I) \subset \mathrm{Homeo}_+(S^1)$, the first inclusion being induced by the one-point compactification of $\mathbf{R}$. Hence the following implies Theorem 1.5.

Proposition 5.3. *The homomorphism $\Phi : BS(m, n) \to G$ defined by $\Phi(b) = g$ and $\Phi(a) = h$ is an isomorphism.*

Proof. By construction Φ is surjective; we now prove injectivity. To this end, consider an arbitrary nontrivial element written in the form of Lemma 5.2. After conjugating we may assume that $r_n = 0$ (replacing r_0 by $r_0 - r_n$). So we must show that (Φ applied to) the element

$$\alpha = g^{e_n} h^{r_{n-1}} g^{e_{n-1}} \dots h^{r_1} g^{e_1} h^{r_0}$$

acts nontrivially on $\mathbf{R}$. If $n = 0$, i.e. if $\alpha = h^{r_0}$, then α clearly acts nontrivially if $r_0 \neq 0$, so we may assume $n \geq 1$.

Let x be an element of the interior of $C \cap C_2$. We will prove by induction on n that $\alpha(x) \in (A^s + n\mathbf{Z}) \cup (B^s + m\mathbf{Z})$. This clearly implies $\alpha(x) \neq x$.

Let $n = 1$. Then assuming $e_1 = 1$, we have $\alpha(x) = g(h^{r_0}(x)) = g_n(g_m^{-1}(h^{r_0}(x)))$. But $x \in C_2 + \mathbf{Z}$ implies $h^{r_0}(x) = x + r_0 \in C_2 + \mathbf{Z}$. So equation (2) implies $g_m^{-1}(x + r_0) \in B^u + \mathbf{Z}$, and then equation (4) implies $g_n(g_m^{-1}(x + r_0)) \in A^s + n\mathbf{Z}$. Thus $e_1 = 1$ implies $\alpha(x) \in A^s + n\mathbf{Z}$. One shows similarly if $e_1 = -1$ then $\alpha(x) = g(h^{r_0}(x)) \in B^s + m\mathbf{Z}$.

Now as induction hypothesis assume that

$$y = g^{e_k} h^{r_{k-1}} g^{e_{k-1}} \dots h^{r_1} g^{e_1} h^{r_0}(x)$$

and that either

$$\begin{aligned} &e_k = 1 \text{ and } y \in A^s + n\mathbf{Z}, \text{ or} \\ &e_k = -1 \text{ and } y \in B^s + m\mathbf{Z}. \end{aligned}$$

We wish to establish the induction hypothesis for $k + 1$. There are four cases corresponding to the values of ± 1 for each of e_k and e_{k+1}. If $e_k = 1$ and $e_{k+1} = 1$ then $y \in A^s + n\mathbf{Z}$ so $h^{r_i}(y) = y + r_i \in A^s + \mathbf{Z}$ and equation (7) implies that $g_m^{-1}(y + r_i) \in B^u + \mathbf{Z}$. Hence by equation (4), $g(h^{r_i}(y)) = g_n(g_m^{-1}(y + r_i)) \in A^s + n\mathbf{Z}$ as desired. On the other hand if $e_k = 1$ and $e_{k+1} = -1$ then $r_k \notin n\mathbf{Z}$. Hence $y \in A^s + n\mathbf{Z}$ implies $h^{r_i}(y) = y + r_i \in A^s + p$ with $p \notin n\mathbf{Z}$. Consequently $y' = g_n^{-1}(y + r_i) \in A^u + \mathbf{Z}$ by equation (1). So equation (6) implies $g^{-1}(h^{r_i}(y)) = g_m(y') \in B^s + m\mathbf{Z}$. Thus we have verified the induction hypothesis for $k + 1$ in the case $e_k = 1$.

If $e_k = -1$ and $e_{k+1} = -1$ then $y \in B^s + m\mathbf{Z}$ so $h^{r_i}(y) = y + r_i \in B^s + \mathbf{Z}$ and equation (5) implies that $g_n^{-1}(y + r_i) \in A^u + \mathbf{Z}$. Hence by equation (6) $g(h^{r_i}(y)) = g_m(g_n^{-1}(y + r_i)) \in B^s + m\mathbf{Z}$ as desired. Finally for the case $e_k = -1$ and $e_{k+1} = 1$ we have $r_k \notin m\mathbf{Z}$. Hence $y \in B^s + m\mathbf{Z}$ implies $h^{r_i}(y) = y + r_i \in B^s + p$ with $p \notin m\mathbf{Z}$. Consequently $y' = g_m^{-1}(y + r_i) \in B^u + \mathbf{Z}$ by equation (3). So equation (4) implies $g(h^{r_i}(y)) = g_n(y') \in A^s + n\mathbf{Z}$. Thus we have verified the induction hypothesis for $k + 1$ in the case $e_k = -1$ also. $\diamond$

Remark. We showed in the proof of Proposition 5.3 that any element of the form

$$g^{e_n} h^{r_{n-1}} g^{e_{n-1}} \dots h^{r_1} g^{e_1} h^{r_0}$$

acts nontrivially on $\mathbf{R}$ provided $e_i = \pm 1$, $r_i \in \mathbf{Z}$, whenever $e_i = 1$ and $e_{i-1} = -1$ we have $r_i \notin m\mathbf{Z}$, and whenever $e_i = -1$ and $e_{i-1} = 1$ we have $r_i \notin n\mathbf{Z}$. As a corollary we obtain the following lemma which we will need later.

Lemma 5.4. *In the group $BS(m,n)$ with generators b and a satisfying the relation $ba^m b^{-1} = a^n$ every element of the form*

$$b^{e_n} a^{r_{n-1}} b^{e_{n-1}} \dots a^{r_1} b^{e_1} a^{r_0},$$

is nontrivial provided $e_i = \pm 1$, $r_i \in \mathbf{Z}$, and whenever $e_i = 1$, $e_{i-1} = -1$ we have $r_i \notin m\mathbf{Z}$ and whenever $e_i = -1$, $e_{i-1} = 1$ we have $r_i \notin n\mathbf{Z}$.

Of course this lemma can also be obtained from the normal form theorem for HNN extensions.

5.2 General properties of $BS(m,n)$ actions

In this short subsection we note two simple implications of the Baumslag-Solitar relation.

Lemma 5.5. *If $BS(m,n)$ acts by homeomorphisms on S^1 with generators g, h satisfying $gh^m g^{-1} = h^n$ then h has a periodic point whose period is a divisor of $|n-m|$.*

Proof. Consider the group of all lifts to $\mathbf{R}$ of elements of $BS(m,n)$ acting on S^1. Choose lifts H and G of h and g respectively and let $T(x) = x+1$ denote the covering translation. We have

$$GH^m G^{-1} = H^n T^p$$

for some integer p. Each lift of an element of $BS(m,n)$ has a well defined translation number in $\mathbf{R}$ and these are topological conjugacy invariants. Hence

$$m\tau(H) = \tau(H^m) = \tau(GH^m G^{-1}) = \tau(H^n T^p) = p + n\tau(H).$$

Solving we conclude

$$\tau(H) = \frac{p}{m-n}.$$

Since H has a rational rotation number h has a point whose period is a divisor of $|n-m|$. $\diamond$

Lemma 5.6. *Suppose g and h are homeomorphisms of $\mathbf{R}$ and satisfy $gh^m g^{-1} = h^n$. Then if h is fixed point free, g has a fixed point.*

Proof. Assume, without loss of generality, that $h(x) = x+1$ and $n > m$ and $g(0) \in [p, p+1]$. Then

$$h^{nk}(g(0)) = (g(0) + nk) \in [p+nk, p+nk+1].$$

But

$$\begin{aligned} h^n g &= gh^m, \text{ so} \\ h^{nk} g &= gh^{mk}, \text{ and} \\ h^{nk}(g(0)) &= g(h^{mk}(0)) = g(mk), \text{ so} \\ g(mk) &\in [p+nk, p+nk+1]. \end{aligned}$$

For k sufficiently large this implies $g(mk) > mk$ and for k sufficiently negative that $g(mk) < mk$. The intermediate value theorem implies g has a fixed point. $\diamond$

5.3 C^2 actions of $BS(m,n)$

In contrast to the analytic actions of $\mathrm{BS}(m,n)$ on $\mathbf{R}$, we will show that there are no such actions, even C^2 actions, on either I or S^1.

Lemma 5.7. *Suppose g and h are orientation-preserving C^2 diffeomorphisms of I satisfying $gh^m g^{-1} = h^n$. Then h and ghg^{-1} commute.*

Proof. If $x \in \mathrm{Fix}(h)$ then $g(x) = g(h^m(x)) = h^n(g(x))$ so $g(x)$ is a periodic point for h. But the only periodic points of h are fixed. We conclude that $g(\mathrm{Fix}(h)) = \mathrm{Fix}(h)$.

Let (a,b) be any component of the complement of $\mathrm{Fix}(h)$ in $[0,1]$. Since $g(\mathrm{Fix}(h)) = \mathrm{Fix}(h)$ we have that $ghg^{-1}([a,b]) = [a,b]$.

Let H be the centralizer of $h^n = gh^m g^{-1}$. Note that h and ghg^{-1} are both in H. Let f be any element of H. Then if f has a fixed point in (a,b), since it commutes with h^n we know by Kopell's Lemma (Theorem 2.1) that $f = id$. In other words H acts freely (though perhaps not effectively) on (a,b). Hence the restrictions of any elements of H to $[a,b]$ commute by Hölder's Theorem. In particular the restrictions of h and ghg^{-1} to $[a,b]$ commute. But (a,b) was an arbitrary component of the complement of $\mathrm{Fix}(h)$. Since the restrictions of h and ghg^{-1} to $\mathrm{Fix}(h)$ are both the identity we conclude that h and ghg^{-1} commute on all of I. $\diamond$

As a corollary we have:

Theorem 1.6 for I: *If m and n are greater than 1 then no subgroup of $\mathrm{Diff}^2_+(I)$ is isomorphic to $BS(m,n)$.*

Proof. If there is such a subgroup, it has generators a and b satisfying $a^n = ba^m b^{-1}$. Lemma 5.7 asserts that a and $c = bab^{-1}$ commute. But the commutator $[c,a]$ is $ba^{-1}b^{-1}abab^{-1}a$ and this element is nontrivial by Lemma 5.4. This contradicts the existence of such a subgroup. $\diamond$

Showing that $\mathrm{BS}(m,n)$ has no C^2 action on the circle is a bit harder.

Theorem 1.6 for S^1: *If m is not a divisor of n then no subgroup of $\mathrm{Diff}^2_+(S^1)$ is isomorphic to $BS(m,n)$.*

Proof. Suppose there is such a subgroup, and we have generators g, h satisfying $gh^m g^{-1} = h^n$. By Lemma 5.5, h has a periodic point. Let $Per(h)$ be the closed set of periodic points of h (all of which have the same period, say p). If $x \in \operatorname{Per}(h)$ then $g(x) = gh^{mp} = h^{np}g(x)$ which shows $g(\operatorname{Per}(h)) = \operatorname{Per}(h)$.

The omega limit set $\omega(x, g)$ of a point x under g is equal to a subset of the set of periodic points of g if g has periodic points, and is independent of x if g has no periodic points. Since g is C^2 we know by Denjoy's theorem that $\omega(g, x)$ is either all of S^1 or is a subset of $\operatorname{Per}(g)$ (see [dMvS]). In the case at hand $\omega(g, x)$ cannot be all of S^1 since $x \in \operatorname{Per}(h)$ implies $\omega(x, g) \subset \operatorname{Per}(h)$ and this would imply $h^p = id$. Hence there exists $x_0 \in \operatorname{Per}(g) \cap \operatorname{Per}(h)$. Then $x_0 \in \operatorname{Fix}(h^p) \cap \operatorname{Fix}(g^q)$ where q is the period of points in $\operatorname{Per}(g)$.

The fact that x_0 has period p for h implies that its rotation number under h (well defined as an element of $\mathbf{R}/\mathbf{Z}$) is $k/p + \mathbf{Z}$ for some k which is relatively prime to p. Using the fact that $\rho(h^m) = \rho(gh^m g^{-1})$, we conclude that

$$m(k/p) + \mathbf{Z} = \rho(h^m) = \rho(gh^m g^{-1}) = \rho(h^n) = n(k/p) + \mathbf{Z}$$

Consequently $mk \equiv nk \mod p$ and since k and p are relatively prime $m \equiv n \mod p$. The fact that $1 < m < n$ and $n - m = rp$ for some $r \in \mathbf{Z}$ tells us that $p \notin m\mathbf{Z}$ because if it were then n would be a multiple of m. Similarly $p \notin n\mathbf{Z}$.

Let G be the group generated by $h_0 = h^p$ and $g_0 = g^q$. Now G has a global fixed point x_0, so we can split S^1 at x_0 to obtain a C^2 action of G on I. Note that $g_0 h_0^{m^q} g_0^{-1} = g^q h^{pm^q} g^{-q} = h^{pm^q} = h_0^{n^q}$.

We can apply Lemma 5.7 to conclude that h_0 and $g_0 h_0 g_0^{-1}$ commute. Thus h^p and $c = g^q h^p g^{-q}$ commute. But their commutator

$$[c, h^p] = g^q h^{-p} g^{-q} h^{-p} g^q h^p g^{-q} h^p.$$

Since $p \notin m\mathbf{Z}$ and $p \notin n\mathbf{Z}$ this element is nontrivial by Lemma 5.4. This contradicts the existence of such a subgroup. $\diamond$

5.4 Local rigidity of the standard action of $BS(1, n)$

We consider the action of $BS(1, n)$ on $\mathbf{R}$ given by elements of the form $x \to n^k x + b$ where $k \in \mathbf{Z}$ and $b \in \mathbf{Z}[1/n]$. It is generated by the two elements $g_0(x) = nx$ and $h_0(x) = x + 1$.

The following result is essentially the theorem of Shub from [Sh]. We give the proof since framing his result in our context is nearly as long.

Theorem 5.8. *There are neighborhoods of g_0 and h_0 in the uniform C^1 topology such that whenever g and h are chosen from these respective neighborhoods, and when the correspondences $g_0 \to g$ and $h_0 \to h$ induce an isomorphism of $BS(1, n)$ to the group generated by g and h, then the perturbed action is topologically conjugate to the original action.*

Proof. Since g is uniformly C^1 close to $g_0(x) = nx$, its inverse is a contraction of $\mathbf{R}$ and has a unique fixed point, which after a change of coordinates we may assume is 0. Since h is uniformly close to $h_0(x) = x+1$ it is fixed point free and after a further change of co-ordinates we may assume $h^n(0) = n$. Consider the space H of C^0 maps $\phi : \mathbf{R} \to \mathbf{R}$ such that $h_0\phi = \phi h$. and $h(0) = 0$. Any $\phi \in H$ satisfies $\phi(n) = n$ for all $n \in \mathbf{Z}$. Moreover, such a ϕ is completely determined by its values on the interval $[0,1]$, since $\phi(x+n) = \phi(x)+n$. Indeed any C^0 map $\phi : [0,1] \to \mathbf{R}$ is the restriction of some element of H. Clearly, a sequence in H will converge if and only if it converges on $[0,1]$. Hence H can be considered a closed subset of the complete metric space of continuous functions on $[0,1]$ with the C^0 *sup* norm.

Consider the map F on H given by $G(\phi) = g_0^{-1}\phi g$. Then $G(\phi)h = g_0^{-1}\phi g h = g_0^{-1}\phi h^n g = g_0^{-1} h_0^n \phi g = h_0 g_0^{-1}\phi g = h_0 F(\phi)$. It follows $F : H \to H$. Moreover in the C^0 *sup* norm this map is easily seen to be a contraction. It follows that there is a unique fixed point ϕ_0 for F in H.

The map $\phi_0 : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies

$$h_0^n\phi_0 = \phi_0 h^n, \text{ and}$$
$$g_0^n\phi_0 = \phi_0 g^n,$$

for all $n \in \mathbf{Z}$. The first of these equations implies ϕ_0 has image which is unbounded above and below, so ϕ_0 is surjective. This equation also implies that there is a uniform bound on the size of $\phi_0^{-1}(x)$ for $x \in \mathbf{R}$. Indeed if M is an integer and $M > sup|\phi_0(0) - \phi_0(x)|$ for $x \in [0,1]$ then $y > M+1$ or $y < -M$ implies $\phi_0(y) \neq \phi_0(x)$. So the sets $\phi_0^{-1}(z)$ have diameters with an upper bound independent of $z \in \mathbf{R}$. In particular, ϕ_0 is proper.

We can now see that ϕ_0 is injective. Since $\phi_0(y) = \phi_0(x)$ implies $g_0^n(\phi_0(y)) = g_0^n(\phi_0(x))$ which in turn implies $\phi_0(g^n(y)) = \phi_0(g^n(x))$ and g is is a uniform expansion, it follows that if ϕ_0 fails to be injective then the sets $\phi_0^{-1}(z)$ do not have a diameter with an upper bound independent of z. This contradiction implies ϕ_0 must be injective. The fact that ϕ_0 is proper implies ϕ_0^{-1} is continuous. Hence ϕ_0 is a topological conjugacy from the standard affine action of $BS(1,n)$ on $\mathbf{R}$ to the action of the group generated by g and h. $\diamond$

In contrast to local rigidity of the standard action, Hirsch [H] has found real-analytic actions of $\mathrm{BS}(1,n)$ on the line which are not topologically conjugate to the standard action.

Theorem 5.9 (Hirsch [H]). *There is an analytic action of $BS(1,n)$ on $\mathbf{R}$ which is not topologically conjugate to the standard affine action.*

The construction is similar to our construction of an analytic $BS(m,n)$ action on $\mathbf{R}$. The relation $ghg^{-1} = h^n$ is satisfied by analytic diffeomorphisms g and h and g has a fixed attracting fixed point. Hence g cannot be topologically conjugate to the affine function $g_0(x) = nx$.

References

[BV] M. Bridson and K. Vogtmann, Quotients of automorphism groups of free groups, preprint.

[BM] M. Burger and N. Monod, Bounded cohomology of lattices in higher rank Lie groups, *J. Eur. Math. Soc.*1 (1999), no. 2, 199–235.

[dMvS] Welington de Melo and Sebastien van Strien *One-dimensional Dynamics* Springer Verlag, Berlin, (1991)

[FF2] B. Farb and J. Franks, Group actions on one-manifolds, II: Extensions of Hölder's Theorem, preprint.

[FF3] B. Farb and J. Franks, Groups of homeomorphisms of one-manifolds, III: Nilpotent subgroups, in preparation.

[FS] B. Farb and P. Shalen, Groups of real-analytic diffeomorphisms of the circle, to appear in *Ergodic Theory and Dynam. Syst.*

[FP] E. Formanek and C. Procesi, The automorphism group of a free group is not linear, *J. Algebra* 149 (1992), no. 2, 494–499.

[Gh] E. Ghys, Actions de réseaux sur le cercle, *Inventiones Math.* 137 (1999), no. 1, 199–231.

[Gh2] E. Ghys, Rigidité différentiable des groupes fuchsiens, *Inst. Hautes tudes Sci. Publ. Math.*, No. 78 (1993), 163–185.

[GS] E. Ghys and V. Sergiescu, Sur un groupe remarquable de difféomorphismes du cercle, *Comm. Math. Helv.*, Vol. 62 (1987), p.185-239.

[H] M. Hirsch, A stable analytic foliation with only exceptional minimal set, in *Lecture Notes in Math.*, Vol. 468. Springer-Verlag, 1975.

[Iv] N. Ivanov, *Mapping Class Groups*, preprint.

[K] N. Kopell, Commuting Diffeomorphisms in *Global Analysis, Proceedings of the Symposium on Pure Mathematics* **XIV**, Amer. Math. Soc., Providence, RI (1970), 165–184.

[dM] W. de Melo, *Lectures on One-dimensional Dynamics* Intstituto de Matematica Pura e Aplicada do CNPq, Rio de Janeiro, (1989)

[La] F. Labourie, Large groups actions on manifolds, Proc. I.C.M. (Berlin, 1998), *Doc. Math.* 1998, Extra Vol. II, 371–380.

[LS] R.C. Lyndon and P. Schupp, *Combinatorial Group Theory*, Springer-Verlag, 1977.

[Na] A. Navas, Actions de groupes de Kazhdan sure le cercle, April 2001 preprint.

[Sh] M. Shub, Expanding Maps in *Global Analysis, Proceedings of the Symposium on Pure Mathematics* **XIV**, Amer. Math. Soc., Providence, RI (1970), 273–276.

[Zi] R. Zimmer, Actions of semisimple groups and discrete subgroups, *Proc. I.C.M.*, Berkeley 1986, p.1247-1258.

Benson Farb:
Dept. of Mathematics, University of Chicago
5734 University Ave.
Chicago, Il 60637
E-mail: farb@math.uchicago.edu

John Franks:
Dept. of Mathematics, Northwestern University
Evanston, IL 60208
E-mail: john@math.northwestern.edu