A remark about actions of lattices on free groups

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The purpose of this note is record some remarks concerning homomorphisms from higher rank lattices to the outer automorphism group of a free group. These remarks were prompted by a question raised by Steve Gersten during a lecture by the second author at the Banach Centre in Warsaw in August 1997. In that lecture the second author described his theorem with Masur on finiteness of homomorphisms of higher rank lattices to mapping class groups [FM]. Their results deal not only with nonuniform lattices but also with the case of uniform lattices, whereas the results which we shall describe for $\mathrm{Out}(F)$ concern only nonuniform lattices. However we conjecture that the results stated below should also remain valid for uniform lattices.

Theorem 1 *Let G be a semisimple Lie group with finite center and no compact factors. Assume that the real rank of G is at least 2. Let $\Gamma \subset G$ be a nonuniform, irreducible lattice. Then every homomorphism from Γ to the outer automorphism group of a finitely generated free group has finite image.*

$\mathrm{Aut}(F_n)$ can be embedded in $\mathrm{Out}(F_{n+1})$, so Theorem 1 implies that all homomorphisms $\Gamma \to \mathrm{Aut}(F_n)$ have finite image.

There are many homomorphisms from Γ to $\mathrm{Out}(F_n)$ that have finite image. For instance, in the case $\Gamma = \mathrm{SL}_n(\mathbf{Z})$, one can take a congruence quotient $\mathrm{SL}_n(\mathbf{Z}/p\mathbf{Z})$ and consider the action of $\mathrm{SL}_n(\mathbf{Z}/p\mathbf{Z})$ on the 1-skeleton of the flag complex associated to the projective space of dimension n over the finite field $\mathbf{Z}/p\mathbf{Z}$: this flag complex is finite so the fundamental group F of its 1-skeleton is finitely generated; the action of Γ on this 1-skeleton gives a homomorphism $\Gamma \to \mathrm{Out}(F)$.

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Another obvious way of getting homomorphisms with finite image is to take a finite quotient Q of Γ and to consider the induced action of Γ on the fundamental group of the Cayley graph of Q.

In fact all homomorphisms $\Gamma \to \mathrm{Out}(F_n)$ arise from actions of Γ on finite graphs.

Corollary 2 *Let Γ be as above and let $\psi : \Gamma \to Out(F_n)$ be a homomorphism. Then there is a finite graph $\mathcal{G}$ with fundamental group F_n and an action of Γ on $\mathcal{G}$ by automorphisms such that the induced homomorphism $\Gamma \to Out(F_n)$ is ψ.*

This is an immediate consequence of Theorem 1 and Culler's realization theorem [C].

Proof of Theorem 1: Let $\psi : \Gamma \to \mathrm{Out}(F_n)$ be a homomorphism. Because G has rank at least two and Γ is irreducible and nonuniform, Γ has a solvable subgroup S that is not virtually abelian. Bestvina, Feighn and Handel [BFH] have proved that every solvable subgroup of $\mathrm{Out}(F_n)$ is virtually abelian. Hence $\psi(s) = 1$ for some element $s \in S$ of infinite order, and in particular $\ker \psi$ is infinite. But by the Margulis-Kazhdan finiteness theorem (Theorem 8.1.2 in [Zi]), if a normal subgroup of Γ is infinite then it must have finite index in Γ.

2 Fixed Point Theorems

In [FM] it is shown that any action by isometries of an irreducible, higher-rank lattice Γ on a Teichmüller space has a global fixed point. This is a consequence of three theorems: the finiteness of homomorphisms of Γ into mapping class groups (Theorem 1.1 of [FM]), Royden's Theorem [Ro] that the full isometry group of the Teichmüller metric is the mapping class group, and Kerchoff's solution to the Nielsen Realization Conjecture [Ke] that every finite subgroup of a mapping class group has a fixed point in Teichmüller space.

Associated to $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$ one has spaces analogous to Teichmüller space, and it seems natural to ask whether an analogue of the fixed point theorem mentioned above holds in this context. Since we already have Theorem 1 and Culler's realization theorem, the remaining question is whether there is an analogue of Royden's theorem.

In many cases the answer is yes, but this requires a detailed analysis of the local structure of the spaces concerned (well beyond the scope of this brief note). This analysis will be the subject of a future article by Bridson and Vogtmann [BV]. We sketch the proof of one such result to indicate the main ideas involved.

The most studied of the spaces associated to $\mathrm{Out}(F_n)$ is the Outer Space of Culler and Vogtmann [CV]. Just as the points of Teichmüller space can be thought of as marked Riemann surfaces, so Outer Space can be thought of as a space of marked metric graphs; the Outer Space on which $\mathrm{Out}(F_n)$ acts consists of marked graphs of genus n. For each n, there is an equivariant retraction of Outer Space onto a simplicial spine called the Culler-Vogtmann complex K_n.

Theorem 3 *If G and Γ are as above, then every simplicial action of Γ on the Culler-Vogtmann complex K_3 has a global fixed point.*

It follows from Culler's realization theorem that in the natural action of $\mathrm{Out}(F_3)$ on K_3 every finite subgroup has a fixed point. Thus, in the light of Theorem 1, it suffices to prove that $\mathrm{Out}(F_3)$ is the full simplicial automorphism group of K_3. As we remarked above, this requires a detailed analysis of the local structure of K_3, beginning with the detailed description of the links of vertices given in [Br]. We provide a brief outline of this analysis; a detailed proof is given in [BV]. Note that Ivanov [Iv] has proven the analogue of this question for mapping class groups (and in fact used it to give another proof of Royden's Theorem).

The first important fact (proved in [CV]) is that K_3 is the union of the stars of marked graphs which have only one 0-cell, i.e. *roses*, and $\mathrm{Out}(F_3)$ acts transitively on the set of roses. One can show that the set of vertices of K_3 corresponding to roses and trivalent graphs is distinguished by the fact that the link of any such vertex cannot be expressed as the join of two non-empty simplicial complexes; thus any simplicial isomorphism of K_3 must preserve this set and the subcomplex of the 1-skeleton that it spans. Moreover, in this subcomplex the vertices corresponding to roses have valence 72 whereas the other vertices have valence at most 16. Thus any simplicial isomorphism of K_3 must preserve the set of roses.

A detailed examination of the link of a vertex corresponding to a rose reveals that its stabilizer in $\mathrm{Out}(F_3)$ contains all of the symmetries of the link. Thus, given an arbitrary simplicial isomorphism of K_3, by composing it with an element of $\mathrm{Out}(F_3)$ we can suppose that it fixes the star of a given rose ρ_0 pointwise. This reduces us to the task of showing that the only simplicial isomorphism of K_3 which can fix the star of a rose pointwise is the identity.

The final stage of the proof exploits the fact that it is possible to move from ρ_0 to any other rose in K_3 along an edge-path given by a finite sequence of Nielsen moves. A further detailed analysis of links reveals that if a simplicial automorphism fixes the star of a rose pointwise then it must also fix the stars of all Nielsen-adjacent roses pointwise (see [BV]).

3 Comments

We conjecture that the theorems stated above also hold for *uniform* lattices. The analogue of Theorem 1 in the case where the target is a mapping class group [FM], makes essential use of the theory of Kaimanovich-Masur [KM] concerning random walks on mapping class groups. This prompts the question of whether there is a similar theory for random walks on $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$.

In [FS] Theorem 1 is used to prove a rigidity result for lattice actions on compact 3-manifolds. If Theorem 1 were proved for uniform lattices, then the results of [FS] could be strengthened accordingly.

References

[BFH] M. Bestvina, M. Feighn and M. Handel, Solvable subgroups of $Out(F_n)$ are virtually abelian, preprint, June 1997.

[Br] M.R. Bridson, Geodesics and curvature in metric simplicial complexes, in *Group Theory From a Geometrical Viewpoint*, E. Ghys, A. Haefliger, A. Verjovsky, ed., World Sci., Singapore (1991), pp.373-463.

[BV] M.R. Bridson and K. Vogtmann, The symmetries of outer space, in preparation.

[C] M. Culler, Finite groups of automorphisms of free groups, in *Contributions to Group Theory*, pp.197–207, Contemp. Math., 33 (1984), Amer. Math. Soc., Providence, R.I.

[CV] M. Culler and K. Vogtmann, Moduli of graphs and automorphisms of free groups, *Invent. Math.* 84 (1986), pp.91–119.

[FM] B. Farb and H. Masur, Superrigidity and mapping class groups, to appear in *Topology.*

[FS] B. Farb and P. Shalen, Lattice actions, 3-manifolds, and homology, March 1998 preprint.

[Iv] N. Ivanov, Automorphism of complexes of curves and of Teichmüller spaces, *Internat. Math. Res. Notices*, no. 14 (1997), p.651-666.

[Ke] S. Kerchoff, The Nielsen Realization Problem, *Ann. of Math.* 117 (1983), pp.235-265.

[KM] V. Kaimanovich and H. Masur, The poisson boundary of the mapping class group, *Invent. Math.* 125 (1996), No.2, pp.221-264.

[Ro] H.L. Royden, Automorphisms and isometries of Teichmüller spaces, in *Advances in the theory of Riemann surfaces*, Ann. of Math. Studies 66 (1971), pp.369-383.

[Zi] R. Zimmer, *Ergodic Theory and Semisimple Groups*, Monographs in Math., Vol. 81, Birkhauser, 1984.

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